University of Washington
AMATH 301 Spring 2017
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Lecture Notes
Week 10

Frequency Domain Analysis (1-D)
[For 2-D cases, watch the third video of Week 10]

Discrete Fourier Transform

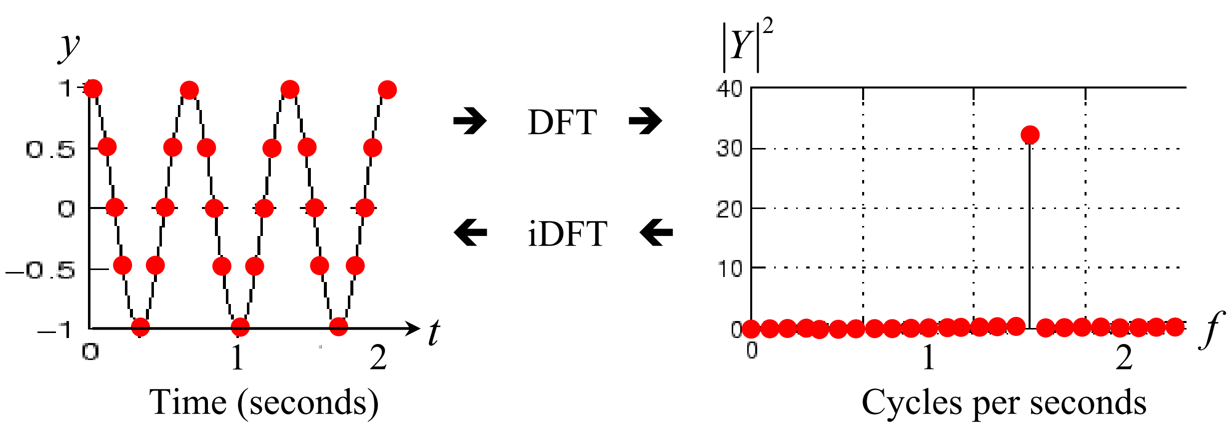
Given a discrete finite time series `y=[y1,y2,y3,…,yN]` at uniformly sampled time points `t=[t1,t2,t3,…,tN]`, `y` can be expressed as a summation series of `N` sinusoidal waves:

$$y_j = \frac{1}{N}\sum_{k=1}^{N} Y_k \exp\left[i\frac{(k-1)(j-1)2\pi}{N}\right], \tag{1}$$

where `Y=[Y1,Y2,Y3,…,YN]` (complex) are the Discrete Fourier Transform (DFT) at frequencies `f=[f1,f2,f3,…,fN]`, where $f_k = \frac{k-1}{N\Delta t}$. Mathematically, Y_k is obtained by

$$Y_k = \sum_{j=1}^{N} y_j \exp\left[-i\frac{(k-1)(j-1)2\pi}{N}\right]. \tag{2}$$

$|Y_k|^2$ are estimates of spectral powers at frequencies f_k. Conversely, y is the inverse Discrete Fourier Transform of Y.



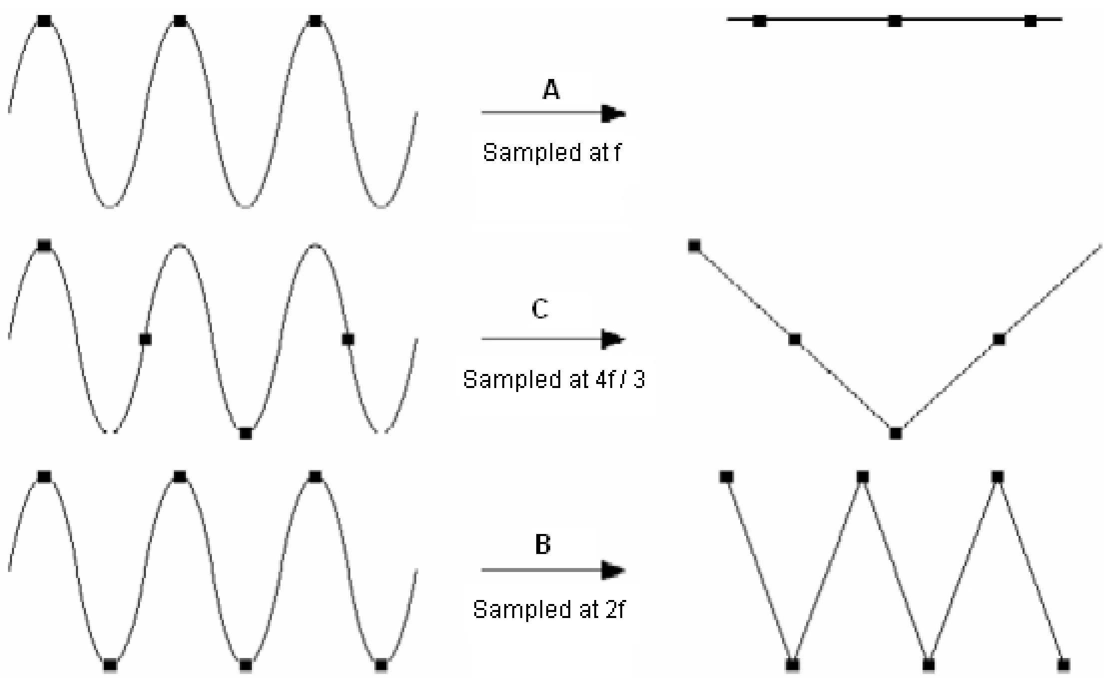
For real data y, half of Y are redundant: $Y_{N-k+2} = Y_k^*$ (see Appendix).

1. $f_k = \dfrac{k-1}{N\Delta t}$ is physically meaningful only for $k = 1, \ldots, \dfrac{N}{2}+1$. Other frequencies are negative frequencies and are equivalent to the positive frequencies.

2. $f_{\frac{N}{2}+1} = \dfrac{1}{2\Delta t}$ is the highest frequency that DFT can "see"; called Nyquist frequency.

3. Nyquist frequency due to aliasing: Oscillations with frequencies higher than $\dfrac{1}{2\Delta t}$ cannot be distinguished from oscillations with frequencies lower than $\dfrac{1}{2\Delta t}$.

Nyquist Theorem:



Only Case B (sampling twice over each cycle) correctly measures the signal.

Fast Fourier Transfrom

Eqs (1) and (2) are too slow to calculate: order of complexity = $O(N^2)$. J. W. Cooley and J. W. Tukey simplified the algorithm and called it Fast Fourier Transform (FFT), which works best if $N = 2^p$ and the order of complexity is O(Nln$N$).

Matlab: `Y=fft(y)` and `y=ifft(Y)`.
`abs(Y).^2` are spectral powers.

If $N \neq 2^p$, pad zeros:

```
Y    = fft(y, 2.^nextpow2(length(y)) );
ynew = ifft(Y);
y    = ynew(1:length(y));
```

Appendix

For real y,

$$
\begin{aligned}
y &= y^* \\
&= \frac{1}{N}\sum_{k=1}^{N} Y_k^* \exp\left[-i\frac{(k-1)(j-1)2\pi}{N}\right] \\
&= \frac{1}{N}\sum_{k=1}^{N} Y_k^* \exp\left[-i\frac{(k-1)(j-1)2\pi}{N} + i(j-1)2\pi\right] && \because \exp\left[i(j-1)2\pi\right]=1 \\
&= \frac{1}{N}\sum_{k=1}^{N} Y_k^* \exp\left[i\frac{(N-k+1)(j-1)2\pi}{N}\right] \\
&= \frac{1}{N}\sum_{s=1}^{N} Y_{N-s+1}^* \exp\left[i\frac{s(j-1)2\pi}{N}\right] && \text{by replacing } s = N-k+1
\end{aligned}
$$

$$
\begin{aligned}
s_{\max} &= N - k_{\min} + 1 = N \\
s_{\min} &= N - k_{\max} + 1 = 1
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{N}\sum_{s=1}^{N-1} Y_{N-s+1}^* \exp\left[i\frac{s(j-1)2\pi}{N}\right] + \frac{1}{N}Y_1^* \exp\left[i\frac{N(j-1)2\pi}{N}\right] \\
&= \frac{1}{N}\sum_{s=1}^{N-1} Y_{N-s+1}^* \exp\left[i\frac{s(j-1)2\pi}{N}\right] + \frac{1}{N}Y_1^* && \because \exp\left[i\frac{N(j-1)2\pi}{N}\right]=1 \\
&= \frac{1}{N}\sum_{s=0}^{N-1} Y_{N-s+1}^* \exp\left[i\frac{s(j-1)2\pi}{N}\right] && \because 1 = \exp\left[i\frac{0\times(j-1)2\pi}{N}\right] \\
&= \frac{1}{N}\sum_{k=1}^{N} Y_{N-k+2}^* \exp\left[i\frac{(k-1)(j-1)2\pi}{N}\right] && \text{by replacing } s = k-1
\end{aligned}
$$

Therefore, $Y_k^* = Y_{N-k+2}$.