Lecture 6 (4/7/00)

Vectors and Matrices

The purpose of the computer (for our purposes) is to provide a fast, efficient way of calculating complicated and time consuming things. Thus we need a convenient way of representing data. Vectors and matrices are ideal for this as they are the "filing" cabinets of the math world.

Define the following vectors

$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \qquad x_i - \text{components of } \vec{x}$$

$$\vec{y} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} \qquad y_i - \text{components of } \vec{y}$$

- $\vec{x} = \vec{y} \rightarrow$ only if $x_i = y_i \quad i = 1, 2, 3, \ldots, n$
- $\vec{x} + \vec{y} = \begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \\ \vdots \\ x_n + y_n \end{pmatrix}$
- $\vec{x} - \vec{y} = \begin{pmatrix} x_1 - y_1 \\ x_2 - y_2 \\ \vdots \\ x_n - y_n \end{pmatrix}$
- $c\vec{x} = \begin{pmatrix} cx_1 \\ cx_2 \\ \vdots \\ cx_n \end{pmatrix} \Rightarrow$ stretching
- $c\vec{x} + d\vec{y} = \begin{pmatrix} cx_1 + dy_1 \\ cx_2 + dy_2 \\ \vdots \\ cx_n + dy_n \end{pmatrix}$

- $\vec{x} \cdot \vec{y} = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n$ (inner product)
- $\| \vec{x} \| = (\vec{x} \cdot \vec{x})^{1/2} = (x_1^2 + x_2^2 + \cdots + x_n^2)^{1/2}$ (norm)
- $\| c\vec{x} \| = |c| \, \| \vec{x} \|$ (c = stretching factor)

<u>Example</u> : $\vec{x} = \begin{pmatrix} 1 \\ 5 \\ 0 \end{pmatrix}$, $\vec{y} = \begin{pmatrix} 2 \\ 8 \\ 3 \end{pmatrix}$

$\vec{x} + \vec{y} = \begin{pmatrix} 3 \\ 13 \\ 3 \end{pmatrix}$ $\quad$ $\vec{x} - \vec{y} = \begin{pmatrix} -1 \\ -3 \\ -3 \end{pmatrix}$

$\vec{x} \cdot \vec{y} = 2 + 40 + 0 = 42$ $\quad$ $\| \vec{x} \| = \sqrt{1+25+0} = \sqrt{26}$ $\quad$ $\| \vec{y} \| = \sqrt{4+64+9} = \sqrt{77}$

<u>properties</u> :

$$\begin{aligned}
\vec{x} + \vec{y} &= \vec{y} + \vec{x} \\
\vec{0} + \vec{x} &= \vec{x} + \vec{0} \\
\vec{x} - \vec{x} &= \vec{x} + (-\vec{x}) = \vec{0} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \\
(\vec{x} + \vec{y}) + \vec{z} &= \vec{x} + (\vec{y} + \vec{z}) \\
(a+b)\vec{x} &= a\vec{x} + b\vec{x} \\
a(\vec{x} + \vec{y}) &= a\vec{x} + a\vec{y} \\
a(b\vec{x}) &= (ab)\vec{x}
\end{aligned}$$

<u>Matrices</u> : Define the following

$$A = [a_{ij}]_{M \times N} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1N} \\ a_{21} & a_{22} & \cdots & a_{2N} \\ \vdots & & & \\ a_{M1} & a_{M2} & \cdots & a_{MN} \end{pmatrix} \qquad M \times N \text{ matrix}$$

i-row
j-column

Example

$$A = \begin{pmatrix} -2 & 4 & 9 \\ 5 & -7 & 1 \\ 0 & -3 & 8 \\ 4 & 6 & -5 \end{pmatrix}$$

row vectors:

$v_1 = (-2 \quad 4 \quad 9)$
$v_2 = (5 \quad -7 \quad 1)$
$v_3 = (0 \quad -3 \quad 8)$
$v_4 = (4 \quad 6 \quad -5)$

column vectors: $c_1 = \begin{pmatrix} -2 \\ 5 \\ 0 \\ 4 \end{pmatrix}$ $c_2 = \begin{pmatrix} 4 \\ -7 \\ -3 \\ 6 \end{pmatrix}$ $c_3 = \begin{pmatrix} 9 \\ 1 \\ 8 \\ -5 \end{pmatrix}$

- $A = B \rightarrow$ only if $a_{ij} = b_{ij}$ for all i, j
- $A + B = [a_{ij}]_{m \times n} + [b_{ij}]_{m \times n} = [a_{ij} + b_{ij}]_{m \times n}$
- $A - B = [a_{ij}]_{m \times n} - [b_{ij}]_{m \times n} = [a_{ij} - b_{ij}]_{m \times n}$
- $cA = [c a_{ij}]_{m \times n}$
- $pA + qB = [p a_{ij} + q b_{ij}]_{m \times n}$
- $0 = [0]_{m \times n} = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & & & \vdots \\ 0 & \cdots & \cdots & 0 \end{pmatrix}$
- $I = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & & & & \vdots \\ 0 & \cdots & \cdots & 0 & 1 \end{pmatrix} = [\delta_{ij}]_{m \times n}$ $\qquad \delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$

Example $\quad A = \begin{pmatrix} -1 & 2 \\ 7 & 5 \\ 3 & -4 \end{pmatrix} \qquad B = \begin{pmatrix} -2 & 3 \\ 1 & -4 \\ -9 & 7 \end{pmatrix}$

$$2A - 3B = 2\begin{pmatrix} -1 & 2 \\ 7 & 5 \\ 3 & -4 \end{pmatrix} - 3\begin{pmatrix} -2 & 3 \\ 1 & -4 \\ -9 & 7 \end{pmatrix} = \begin{pmatrix} -2 & 4 \\ 14 & 10 \\ 6 & -8 \end{pmatrix} - \begin{pmatrix} -6 & 9 \\ 3 & -12 \\ -27 & 21 \end{pmatrix} = \begin{pmatrix} 4 & -5 \\ 11 & 22 \\ 33 & -29 \end{pmatrix}$$

<u>properties</u>

$$A + B = B + A$$
$$0 + A = A + 0$$
$$A - A = 0$$
$$(A+B)+C = A + (B+C)$$
$$(p+q)A = pA + qA$$
$$p(A+B) = pA + pB$$
$$p(qA) = (pq)A$$

$$AB \neq BA \quad \leftarrow \text{very important}$$

<u>Transpose</u>

$\vec{x}^T$ or A^T

- $\vec{x} = \begin{pmatrix} 2 \\ 3 \\ 5 \end{pmatrix}$ $\qquad \vec{x}^T = \begin{pmatrix} 2 & 3 & 5 \end{pmatrix}$

- $A = [a_{ij}]_{M \times N}$ $\qquad A^T = [a_{ji}]_{N \times M}$

$$A = \begin{pmatrix} -2 & 5 & 12 \\ 1 & 4 & -1 \\ 7 & 0 & 6 \\ 11 & -3 & 8 \end{pmatrix} \qquad A^T = \begin{pmatrix} -2 & 1 & 7 & 11 \\ 5 & 4 & 0 & -3 \\ 12 & -1 & 6 & 8 \end{pmatrix}$$

4x3 $\qquad$ 3x4

- Symmetric Matrix (Hermitian or Self-Adjoint)

$$A = A^T$$

$$A = \begin{pmatrix} 1 & -7 & 4 \\ -7 & 2 & 0 \\ 4 & 0 & 3 \end{pmatrix} \qquad A^T = \begin{pmatrix} 1 & -7 & 4 \\ -7 & 2 & 0 \\ 4 & 0 & 3 \end{pmatrix} = A$$

matrix multiplication

Consider the two matrices

$$A = [a_{ik}]_{m \times n}$$

$$B = [b_{kj}]_{n \times p}$$

then

$$AB = C = [C_{ij}]_{m \times p} \Rightarrow$$ columns of A must equal rows of B

Example

$$A = \begin{pmatrix} 2 & 3 \\ -1 & 4 \end{pmatrix} \qquad B = \begin{pmatrix} 5 & -2 & 1 \\ 3 & 8 & -6 \end{pmatrix}$$

$$AB = \begin{pmatrix} 2 & 3 \\ -1 & 4 \end{pmatrix} \begin{pmatrix} 5 & -2 & 1 \\ 3 & 8 & -6 \end{pmatrix}$$

$$= \begin{pmatrix} 5 \cdot 2 + 3 \cdot 3 & -2 \cdot 2 + 3 \cdot 8 & 2 \cdot 1 + 3 \cdot -6 \\ -1 \cdot 5 + 3 \cdot 4 & -1 \cdot -2 + 4 \cdot 8 & -1 \cdot 1 + 4 \cdot -6 \end{pmatrix}$$

$$= \begin{pmatrix} 10 + 9 & -4 + 24 & 2 - 18 \\ -5 + 12 & 2 + 32 & -1 - 24 \end{pmatrix}$$

$$= \begin{pmatrix} 19 & 20 & -16 \\ 7 & 34 & -25 \end{pmatrix}$$

Note: $AB \neq BA$ in general

if $AB = BA \Rightarrow$ then A and B commute