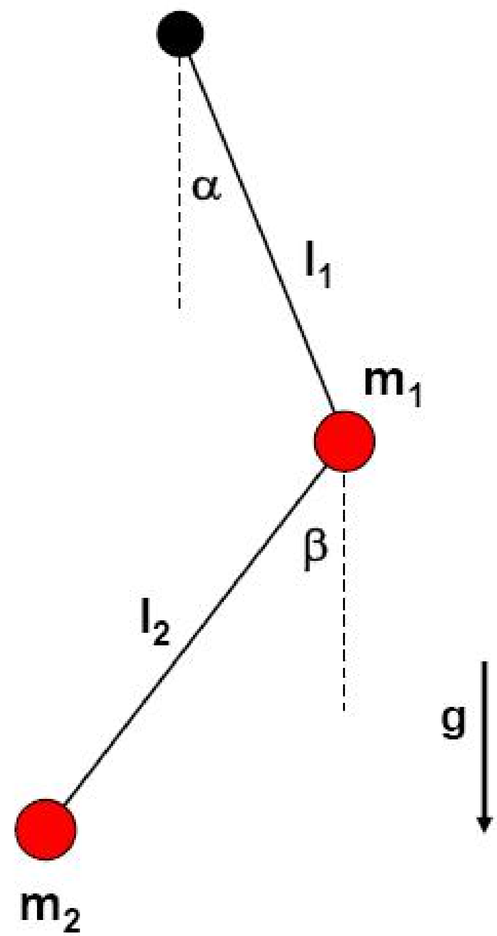
Comparison of Integrators

in the Double Planar Pendulum



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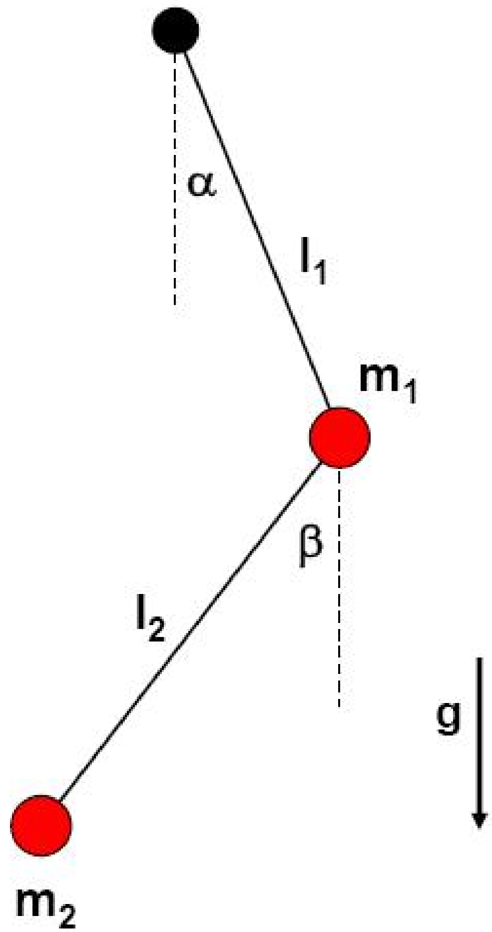
Integrators

- Matlab's ODE45
  - **Runge-Kutta 45**
  - **I keep timestep fixed**
- Variational Integrator
  - **Discretized Euler-Lagrange Equations**
  - **I use trapezoidal approximation**
- Benchmark
  - **Adaptive step Runge-Kutta-Fehlberg 78**
  - **Nearly symplectic & preserves energy**
- Goals
  - **Conservation of Energy**
  - **Fast Runtime**
  - **Ease of Implementation**

Lagrangian: $L = T - V$



$$T = \frac{1}{2}(m_1 + m_2) l_1 \dot{\alpha}^2 + \frac{1}{2} m_2 l_2^2 \dot{\beta}^2 + m_2 l_1 l_2 \dot{\alpha} \dot{\beta} \cos(\alpha - \beta)$$

$$V = (m_1 + m_2) l_1 g (1 - \cos(\alpha)) + m_2 l_2 g (1 - \cos(\beta))$$

Euler-Lagrange Equations

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = 0$$

alternately,

$$M\ddot{q} + \dot{M}\dot{q} = -\nabla V(q, \dot{q})$$

where

$$L = \frac{1}{2}\dot{q}^T M(q)\dot{q} - V(q)$$

- □ Solve for $\ddot{\alpha}$ and $\ddot{\beta}$
- □ Plug into RK45 or RK78 integrator

Variational Integration

- Approximate

$$L_d(q_k, q_{k+1}, h) \approx \int_{kh}^{(k+1)h} L(q, \dot{q}, t)dt$$

and replace

$$\delta \int_0^T L(q, \dot{q}, t)dt = 0$$

with

$$\delta \sum_{k=0}^{N-1} L_d(q_k, q_{k+1}, h) = 0$$

Discrete Euler-Lagrange Eq's

$$\delta \sum_{k=0}^{N-1} L_d(q_k, q_{k+1}, h) = 0$$

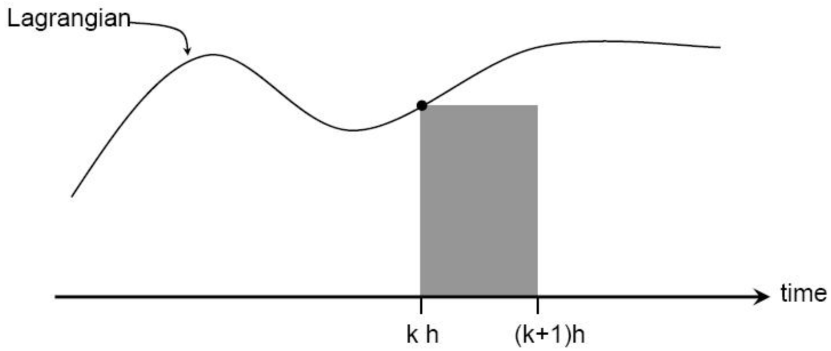
$$\Longrightarrow \sum_{k=0}^{N-1} [D_1 L_d(q_k, q_{k+1}, h) \cdot \delta q_k + D_2 L_d(q_k, q_{k+1}, h) \cdot \delta k_{k+1}]$$

$$= \sum_{k=1}^{N-1} [D_2 L_d(q_{k-1}, q_k, h) + D_1 L_d(q_k, q_{k+1}, h)] \cdot \delta q_k = 0$$

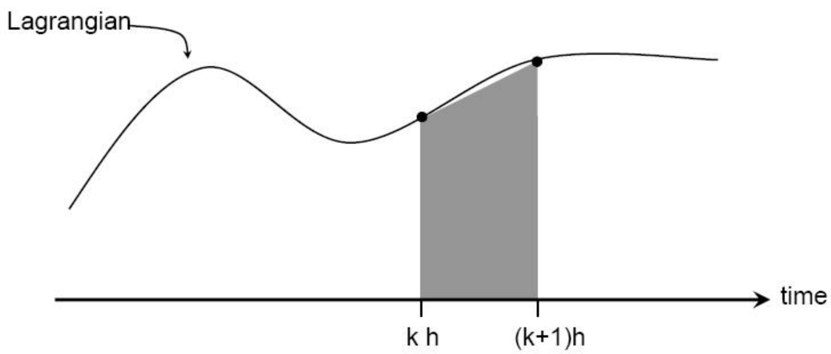
$$\Longrightarrow D_2 L_d(q_{k-1}, q_k, h) + D_1 L_d(q_k, q_{k+1}, h) = 0$$

- D_1 and D_2 are slot derivatives

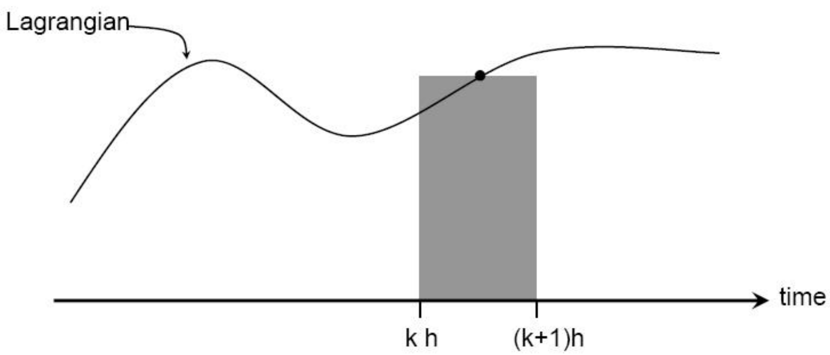
Approximating $\int_{kh}^{(k+1)h} L dt$



☐ Rectangle rule



☐ Trapezoid rule



☐ Midpoint rule

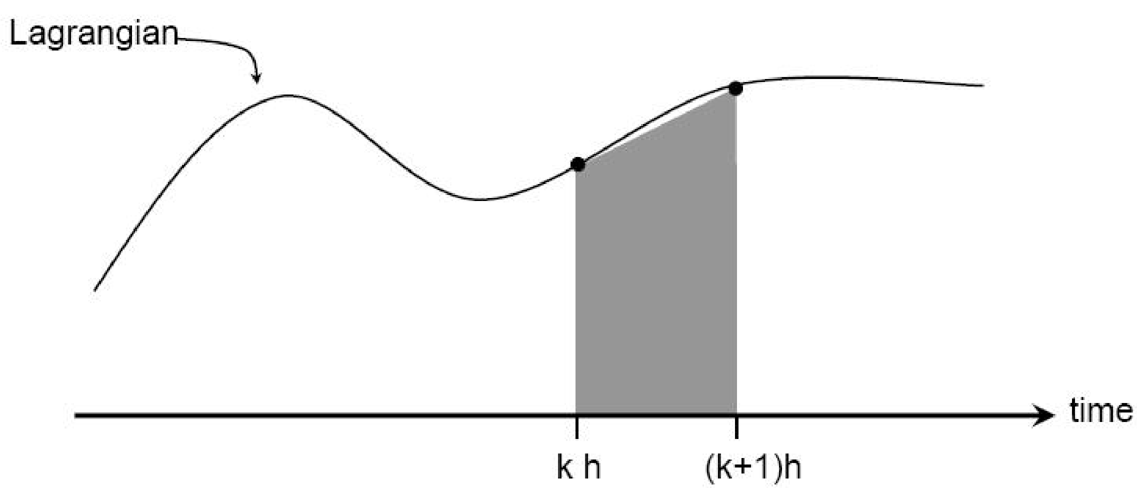
Trapezoid Rule Approximation

$$L_d^{trap} = \frac{h}{2}\left[L\left(q_k, \frac{q_{k+1}-q_k}{h}\right) + L\left(q_{k+1}, \frac{q_{k+1}-q_k}{h}\right)\right]$$

uses

$$\dot{q} = \frac{q_{k+1}-q_k}{h}$$

- Compute $D_2L_d(q_{k-1}, q_k, h)$ and $D_1L_d(q_k, q_{k+1}, h)$



Update Algorithm

$$p^k = \frac{\partial}{\partial q_k} L_d(q_{k-1}, q_k, h) = -\frac{\partial}{\partial q_k} L_d(q_k, q_{k+1}, h)$$

$$p^{k+1} = \frac{\partial}{\partial q_{k+1}} L_d(q_k, q_{k+1}, h)$$

- ***Step 0: Start with*** q_0, p^0
- ***Step 1: Solve first equation for*** q_{k+1}
- ***Step 2: Solve second equation for*** p^{k+1}
- ***Step 3: Repeat Steps 1 & 2***

Newton-Raphson Method

- Iterative method to find root of $f(q)$

$$q_{k+1} = q_k - J^{-1}(k_k) \cdot f(q_k)$$

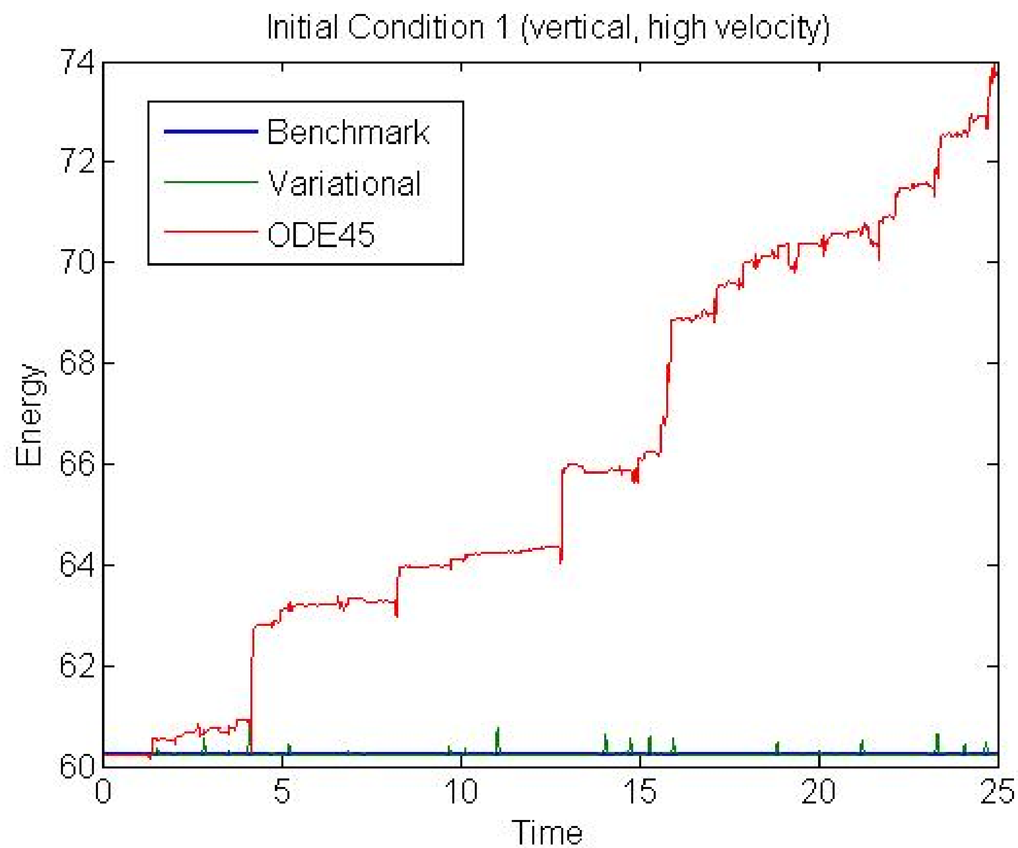
where

$J(q_k) = Df(q_k)$ is the Jacobian

- Converges very quickly for close guess q_0

Energy Conservation

- Both bobs start vertical w/ some angular velocity



- Variational integrator conserves energy
- ODE45 wrecks energy

Efficiency & Accuracy

- Variational integrator with $h = .01$ is comparable to ODE45 with $h = .0001$
- VI is not optimized, but is faster than ODE45
- VI accuracy and speed are comparable to optimized RKF78 algorithm
- VI equations are much more difficult than EL to derive and implement (downside)

Conclusions

- Positives
  - **Variational integrator outperforms ODE45**
  - **VI competes with RKF78 algorithm**
  - **VI conserves energy well, even with long integration times**
- Negatives
  - **VI is considerably harder to implement**

Future Directions

- Optimize variational integrator
- Investigate other parameters m_1, m_2, l_1, l_2, g
- VI with rectangle and midpoint approximations

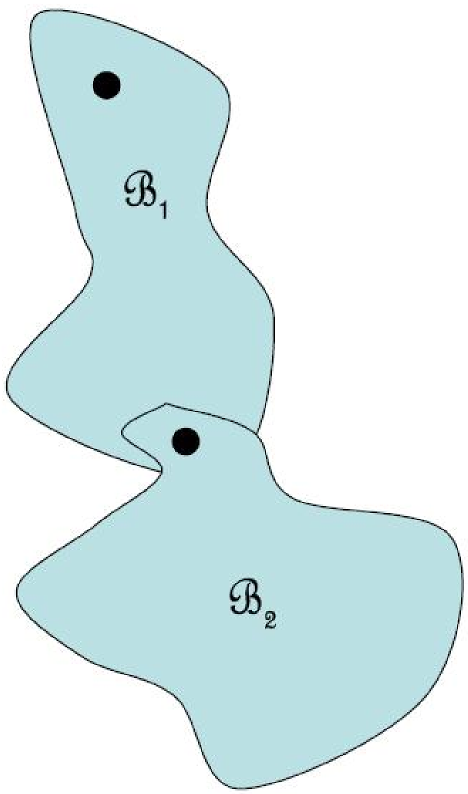
- High resolution Poincaré sections
- Chaotic structure: horseshoes, symbolic dynamics
- Almost invariant sets

- Physically construct double pendulum
- Forced double pendulum
- Investigate control issues

Arbitrarily Shaped Pendula

$$T = \frac{1}{2}\dot{\alpha}^2(I_1^2 + L_1^2 m_2) + \frac{1}{2}\dot{\beta}^2 I_2^2 + \dot{\alpha}\dot{\beta}\cos(\alpha - \beta) L_1 I_2^1$$

$$V = (1 - \cos(\alpha))(R_1 m_1 + L_1 m_2) g + (1 - \cos(\beta)) R_2 m_2 g$$



- $I_1^2 = \int_{B_1} \rho_1 x^2 dx$
- $I_2^2 = \int_{B_2} \rho_2 y^2 dy$
- $I_2^1 = \int_{B_2} \rho_2 y dy$

References

- **Gawlik, E., J.E. Marsden and P. Du Toit, Variational integrators and the three-body problem. *Caltech SURF*, Final Report, [2006].**
- **Levien, R.R. and S.M. Tan, Double pendulum: An experiment in chaos. *Am. J. Phys.*, 61 (11),1038-1044, [1993].**
- **Marsden, J.E. and M. West, Discrete mechanics and variational integrators. *Acta Numerica*, 357-514, [2001].**

The End

Questions...

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