Sample Solutions for Assignment 1.

Reading: Ch 1. Secs. 2.1-2.10.

1. Use MATLAB to evaluate the second order accurate approximation

\[ u''(x) \approx \frac{u(x + h) + u(x - h) - 2u(x)}{h^2} \]

for \( u(x) = \sin x \) and \( x = \pi/6 \). Try \( h = 10^{-1}, 10^{-2}, \ldots, 10^{-16} \), and make a table of values of \( h \), the computed finite difference quotient, and the error. Explain your results.

<table>
<thead>
<tr>
<th>h</th>
<th>Abs. Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.0e-01</td>
<td>4.2e-04</td>
</tr>
<tr>
<td>1.0e-02</td>
<td>4.2e-06</td>
</tr>
<tr>
<td>1.0e-03</td>
<td>4.2e-08</td>
</tr>
<tr>
<td>1.0e-04</td>
<td>3.0e-09</td>
</tr>
<tr>
<td>1.0e-05</td>
<td>6.0e-07</td>
</tr>
<tr>
<td>1.0e-06</td>
<td>6.7e-05</td>
</tr>
<tr>
<td>1.0e-07</td>
<td>6.0e-03</td>
</tr>
<tr>
<td>1.0e-08</td>
<td>6.1e-01</td>
</tr>
<tr>
<td>1.0e-09</td>
<td>5.6e+01</td>
</tr>
<tr>
<td>1.0e-10</td>
<td>5.0e-01</td>
</tr>
<tr>
<td>1.0e-11</td>
<td>5.0e-01</td>
</tr>
<tr>
<td>1.0e-12</td>
<td>5.0e-01</td>
</tr>
<tr>
<td>1.0e-13</td>
<td>5.6e+09</td>
</tr>
<tr>
<td>1.0e-14</td>
<td>5.6e+11</td>
</tr>
<tr>
<td>1.0e-15</td>
<td>5.0e-01</td>
</tr>
<tr>
<td>1.0e-16</td>
<td>5.6e+15</td>
</tr>
</tbody>
</table>

It can be seen that the approximation is second-order accurate, since for moderate size values of \( h \), reducing \( h \) by a factor of 10 reduces the error by a factor of 100. For smaller values of \( h \), roundoff comes into play. Since the truncation error is \( O(h^2) \) and the roundoff is about \( (1e - 16)/h^2 \), the two will be equal and the total error will be smallest when \( h \) is about the fourth root of \( 1e - 16 \), or, \( h = 1e - 4 \). This is seen in the above results. For smaller values of \( h \), roundoff dominates and the error increases and does strange things. Therefore, always use \( h \) greater than about \( 1e - 4 \).

2. Use the formula in the previous exercise with \( h = 0.2, h = 0.1, \) and \( h = 0.05 \) to approximate \( u''(x) \), where \( u(x) = \sin x \) and \( x = \pi/6 \). Use one step of Richardson extrapolation, combining the results from \( h = 0.2 \) and \( h = 0.1 \), to obtain a higher order accurate approximation. Do the same with the results from \( h = 0.1 \) and \( h = 0.05 \).
Finally do a second step of Richardson extrapolation, combining the two previously extrapolated values, to obtain a still higher order accurate approximation. Make a table of the computed results and their errors. What do you think is the order of accuracy after one step of Richardson extrapolation? How about after two?

<table>
<thead>
<tr>
<th>h</th>
<th>Abs. Error</th>
<th>one-step Richardson</th>
<th>two-step Richardson</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.0e-01</td>
<td>1.7e-03</td>
<td>5.6e-07</td>
<td>2.5e-11</td>
</tr>
<tr>
<td>1.0e-01</td>
<td>4.2e-04</td>
<td></td>
<td></td>
</tr>
<tr>
<td>5.0e-02</td>
<td>1.0e-04</td>
<td>3.5e-08</td>
<td>2.5e-11</td>
</tr>
</tbody>
</table>

The error after one step of Richardson extrapolation appears to be $O(h^4)$, since reducing $h$ by a factor of 2 reduces the error by about a factor of $2^4 = 16$. One can’t tell from the above results but one can see by using a Taylor series expansion that the error after two steps of Richardson extrapolation is $O(h^6)$.

3. Using Taylor series, derive the error term for the approximation

$$u'(x) \approx \frac{1}{2h}[-3u(x) + 4u(x + h) - u(x + 2h)].$$

Expanding $u(x + h)$ and $u(x + 2h)$ in Taylor series about $x$, we find:

$$u(x + h) = u(x) + hu'(x) + \frac{h^2}{2!}u''(x) + \frac{h^3}{3!}u'''(x) + O(h^4),$$

$$u(x + 2h) = u(x) + 2hu'(x) + \frac{(2h)^2}{2!}u''(x) + \frac{(2h)^3}{3!}u'''(x) + O(h^4).$$

Combining these expressions,

$$\frac{1}{2h}[-3u(x) + 4u(x + h) - u(x + 2h)] = u'(x) + \frac{1}{2h}\frac{h^3}{3!}(4 - 8)u'''(x) + \frac{1}{2h}O(h^4)$$

$$= u'(x) - \frac{h^2}{3}u'''(x) + O(h^3) = u'(x) + O(h^2).$$

Thus, the error is $\frac{h^2}{3}u'''(x) + O(h^3) = O(h^2)$.

4. Consider a forward difference approximation for the second derivative of the form

$$u''(x) \approx Au(x) + Bu(x + h) + Cu(x + 2h).$$

Use Taylor’s theorem to determine the coefficients $A$, $B$, and $C$ that give the maximal order of accuracy and determine what this order is.

Using the expansions from the previous problem for $u(x + h)$ and $u(x + 2h)$, we find

$$Au(x) + Bu(x + h) + Cu(x + 2h) = (A + B + C)u(x) + h(B + 2C)u'(x) + \frac{h^2}{2!}(B + 4C)u''(x) + \frac{h^3}{3!}(4 - 8)u'''(x) + O(h^4).$$
\[
\frac{h^3}{3!}(B + 8C)u'''(x) + BO(h^4) + CO(h^4).
\]

In order for this to approximate \( u''(x) \), we need

\[
A + B + C = 0, \quad B + 2C = 0, \quad \frac{h^2}{2!}(B + 4C) = 1,
\]

and, if possible, \( B + 8C = 0 \). Solving the above three equations for \( A, \) \( B, \) and \( C, \) we find

\[
A = \frac{1}{h^2}, \quad B = -\frac{2}{h^2}, \quad C = \frac{1}{h^2},
\]

and then the equation \( B + 8C = 0 \) is not satisfied. Therefore these coefficients give the highest order of accuracy and that order is \( O(h) \), since \( \frac{h^3}{3!}(B + 8C)u'''(x) = O(h) \).

5. Consider the two-point boundary value problem

\[
u'' + 2xu' - x^2u = x^2, \quad u(0) = 1, \quad u(1) = 0.
\]

Let \( h = 1/4 \) and explicitly write out the difference equations, using centered differences for all derivatives.

Letting \( x_j = j/4, \) \( j = 0, 1, \ldots, 4, \) the equations are

\[
\frac{u_{j+1} + u_{j-1} - 2u_j}{h^2} + 2x_j \frac{u_{j+1} - u_{j-1}}{2h} - x_j^2 u_j = x_j^2, \quad j = 1, 2, 3.
\]

Plugging in all of the numbers, these equations become:

\[
- \left(32 + \frac{1}{16}\right) u_1 + 17u_2 = \frac{1}{16} - 15
\]

\[
- \left(32 + \frac{1}{4}\right) u_2 + 18u_3 + 14u_1 = \frac{1}{4}
\]

\[
- \left(32 + \frac{9}{16}\right) u_3 + 13u_2 = \frac{9}{16},
\]

or, in matrix form

\[
\begin{pmatrix}
\frac{-513}{16} & \frac{17}{4} & 0 \\
14 & -\frac{192}{4} & 18 \\
0 & 13 & -\frac{521}{16}
\end{pmatrix}
\begin{pmatrix}
u_1 \\
u_2 \\
u_3
\end{pmatrix}
=
\begin{pmatrix}
\frac{-239}{16} \\
\frac{4}{3} \\
\frac{9}{16}
\end{pmatrix}.
\]

6. A rod of length 1 meter has a heat source applied to it and it eventually reaches a steady-state where the temperature is not changing. The conductivity of the rod is a function of position \( x \) and is given by \( c(x) = 1 + x^2 \). The left end of the rod is held at a constant temperature of 1 degree. The right end of the rod is insulated so that
no heat flows in or out from that end of the rod. This problem is described by the boundary value problem:

\[
\frac{d}{dx} \left( (1 + x^2) \frac{du}{dx} \right) = f(x), \quad 0 \leq x \leq 1,
\]

\[u(0) = 1, \quad u'(1) = 0.\]

(a) Write down a set of difference equations for this problem. Be sure to show how you do the differencing at the endpoints. [Note: It is better not to rewrite \(\frac{d}{dx}((1 + x^2)\frac{du}{dx})\) as \((1 + x^2)u''(x) + 2xu'(x)\); leave the equation in the form above.]

Let \(x_{j \pm 1/2} = x_j \pm h/2\). At the interior nodes, \(j = 1, \ldots, m\), we can write

\[-\frac{(1 + x_{j+1/2}^2) + (1 + x_{j-1/2}^2)}{h^2} u_j + \frac{(1 + x_{j+1/2}^2)u_{j+1} + (1 + x_{j-1/2}^2)u_{j-1}}{h^2} = f(x_j).\]

We know the value of \(u_0\), and we can add an \((m+1)st\) equation for \(u_{m+1}\) by writing a second order accurate approximation to \(u'(1)\) and setting that equal to 0:

\[
\frac{3u_{m+1} - 4u_m + u_{m-1}}{2h} = 0.
\]

In matrix form, these equations take the form \(Au = f\), where

\[
A = \frac{1}{h^2} \begin{pmatrix}
    a_1 & b_1 & \\
    b_1 & a_2 & \\
        & \ddots & \\
    & b_{m-1} & a_m & b_m \\
    b_{m-1} & & b_m & \frac{1}{2}h & -2h & \frac{3}{2}h
\end{pmatrix}, \quad \begin{pmatrix}
    u_1 \\
    u_2 \\
    \vdots \\
    u_m \\
    u_{m+1}
\end{pmatrix},
\]

\[
f = \begin{pmatrix}
    f(x_1) - (1 + (h/2)^2)/h^2 \\
    f(x_2) \\
    \vdots \\
    f(x_m) \\
    0
\end{pmatrix}.
\]

and

\[a_j = -(2 + x_{j+1/2}^2 + x_{j-1/2}^2), \quad b_j = (1 + x_{j+1/2}^2),\]

(b) Write a MATLAB code to solve the difference equations. You can test your code on a problem where you know the solution by choosing a function \(u(x)\) that satisfies the boundary conditions and determining what \(f(x)\) must be in order for \(u(x)\) to solve the problem. Try \(u(x) = (1 - x)^2\). Then \(f(x) = 2(3x^2 - 2x + 1)\).

See \texttt{prob6.m}.

(c) Try several different values for the mesh size \(h\). Based on your results, what would you say is the order of accuracy of your method?
The method is second order accurate in the $L_2$-norm as illustrated below: When $h$ is reduced by a factor of 2, the error goes down by about a factor of 4.

<table>
<thead>
<tr>
<th>h</th>
<th>$L_2$-norm of error</th>
<th>error(2h)/error(h)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\frac{1}{16}$</td>
<td>1.7565e-03</td>
<td></td>
</tr>
<tr>
<td>$\frac{1}{8}$</td>
<td>4.2625e-04</td>
<td>4.1207e+00</td>
</tr>
<tr>
<td>$\frac{1}{4}$</td>
<td>1.0522e-04</td>
<td>4.0512e+00</td>
</tr>
<tr>
<td>$\frac{1}{2}$</td>
<td>2.6150e-05</td>
<td>4.0236e+00</td>
</tr>
<tr>
<td>$\frac{1}{1}$</td>
<td>6.5190e-06</td>
<td>4.0113e+00</td>
</tr>
</tbody>
</table>