1 Linear Algebra

trace: $\text{tr}(\mathbf{A}) = \sum_i a_{ii}$

$\text{tr}(\mathbf{A} + \mathbf{B}) = \text{tr}(\mathbf{A}) + \text{tr}(\mathbf{B})$

$\text{tr}(\mathbf{AB}) = \text{tr}(\mathbf{BA})$

rank=# linearly independent rows = # linearly independent columns

$\text{rank}(\mathbf{AB}) \leq \text{rank}(\mathbf{A})$

$\text{rank}(\mathbf{AB}) \leq \text{rank}(\mathbf{B})$

$\text{rank}(\mathbf{A}) = \text{rank}(\mathbf{A}') = \text{rank}(\mathbf{A}'\mathbf{A}) = \text{rank}(\mathbf{AA}')$

Inverses of square matrices with full rank $\mathbf{A}^{-1}\mathbf{A} = \mathbf{AA}^{-1} = I$

Generalized inverses: $\mathbf{A}^-$ is a generalized inverse of $\mathbf{A}$ if $\mathbf{AA}^-\mathbf{A} = \mathbf{A}$. Generalized inverses are not unique except for square matrices of full rank.

Inner product of vectors $\mathbf{a}, \mathbf{b}$: $\mathbf{a}'\mathbf{b}$

Vector norm: $||\mathbf{a}|| = \sqrt{(\mathbf{a}'\mathbf{a})}$.

Orthogonal vectors: $\mathbf{a}'\mathbf{b} = 0$

$\mathbf{A}$ is an orthogonal matrix if $\mathbf{A}^{-1} = \mathbf{A}'$

Eigenvalues and eigenvectors: If $\mathbf{Ax} = \lambda\mathbf{x}$ and $\mathbf{x}'\mathbf{x} = 1$ then $\mathbf{x}$ is an eigenvector for $\mathbf{A}$ and its corresponding eigenvalue is λ.

For a symmetric matrix $\mathbf{A}$ there exists an orthogonal matrix T such that

(i) $\mathbf{T}'\mathbf{AT} = \Lambda$

(ii) $\text{rank}(\mathbf{A}) = \#$ non-zero eigenvalues

(iii) $\text{tr}(\mathbf{A}) = \Sigma\lambda_i$ and $|\mathbf{A}| = \prod \lambda_i$

A symmetric matrix $\mathbf{A}$ is positive definite if $\mathbf{x}'\mathbf{Ax} > 0$ for all non-zero $\mathbf{x}$

A symmetric matrix $\mathbf{A}$ is positive semi-definite if $\mathbf{x}'\mathbf{Ax} \geq 0$ for all non-zero $\mathbf{x}$

Positive definite matrices have positive eigenvalues and positive semi-definite matrices have nonnegative eigenvalues.

If $\mathbf{A}$ is positive definite then there exists a non-singular $\mathbf{B}$ such that $\mathbf{A} = \mathbf{BB}'$.

Idempotent means $\mathbf{P}^2 = \mathbf{P}$

A matrix that is symmetric and idempotent is called a projection matrix.

Projection matrix with rank r:

(i) has r eigenvalues $= 1$ and remaining eigenvalues $= 0$.

(ii) trace = rank

(iii) Positive semi-definite

Let $\mathbf{V}$ be a vector space and let Ω be a subspace. If $\mathbf{Y} \in \mathbf{V}$ then $\mathbf{Y} = w_1 + w_2$ uniquely where $w_1 \in \Omega$ and $w_2 \in \Omega^{\perp}$

2 Random vectors Z

$$\begin{aligned}
E(\mathbf{AZB}+\mathbf{C}) &= \mathbf{A}E(\mathbf{Z})\mathbf{B}+\mathbf{C} \\
\mathrm{cov}(\mathbf{Z}) &= [\mathrm{cov}(\mathbf{Z}_i, \mathbf{Z}_j)] \\
&= E[(\mathbf{Z}-E(\mathbf{Z}))(\mathbf{Z}-E(\mathbf{Z}))'] \\
&= E(\mathbf{ZZ}') - E(\mathbf{Z})E(\mathbf{Z}) \\
\mathrm{cov}(\mathbf{Z}) \text{ is positive semi-definite} & \\
\mathrm{cov}(\mathbf{X}, \mathbf{Y}) &= \mathrm{cov}(\mathbf{X}_i, \mathbf{Y}_i) \\
&= E[(\mathbf{X}-E\mathbf{X})(\mathbf{Y}-E\mathbf{Y})]' \\
\mathrm{cov}(\mathbf{AX}, \mathbf{BY}) &= \mathbf{A}\mathrm{cov}(\mathbf{X}, \mathbf{Y})\mathbf{B}'
\end{aligned}$$

Quadratic forms: Let $E(\mathbf{X}) = \boldsymbol{\mu}$ and $\mathrm{cov}(\mathbf{X}) = \boldsymbol{\Sigma}$.

$E((\mathbf{X}-\boldsymbol{\mu})'\mathbf{A}(\mathbf{X}-\boldsymbol{\mu})) = \mathrm{tr}(\mathbf{A}\boldsymbol{\Sigma})$ and $E(\mathbf{X}'\mathbf{A}\mathbf{X}) = \mathrm{tr}(\mathbf{A}\boldsymbol{\Sigma}) + \boldsymbol{\mu}'\mathbf{A}\boldsymbol{\mu}$.

We have three definitions for the multivariate normal distribution $\mathbf{Y}\ N_n(\boldsymbol{\mu}, \boldsymbol{\Sigma})$:

(1) The density function of $\mathbf{Y}$ is

$$f_{\mathbf{Y}}(\mathbf{y}) = (2\pi)^{-n/2}|\boldsymbol{\Sigma}|^{-1/2} exp\{-\frac{1}{2}(\mathbf{y}-\boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\mathbf{y}-\boldsymbol{\mu})\}.$$

(This definition only works when $\boldsymbol{\Sigma}$ is positive definite.)

(2) The moment generating function of $\mathbf{Y}$ is

$$M_{\mathbf{Y}}(\mathbf{t}) \equiv E[e^{\mathbf{t}'\mathbf{Y}}] = \exp\{\boldsymbol{\mu}'\mathbf{t} + \frac{1}{2}\mathbf{t}'\boldsymbol{\Sigma}\mathbf{t}\}$$

(3) $Y = \mathbf{AZ} + \boldsymbol{\mu}$ where $\mathbf{Z} = (\mathbf{Z}_1 \ldots, \mathbf{Z}_k)$ are independent N(0, 1) and $\mathbf{AA}' = \boldsymbol{\Sigma}$.

3 Some facts related to least squares estimation

A least squares estimate of $\boldsymbol{\beta}$ must satisfy the normal equations $\mathbf{X}'\mathbf{Y} = \mathbf{X}'\mathbf{X}\hat{\boldsymbol{\beta}}$.

$\mathbf{a}'\boldsymbol{\beta}$ is estimable if and only if $\mathbf{a} \in \mathcal{R}(\mathbf{X}')$

$\mathrm{var}(\mathbf{a}'\hat{\boldsymbol{\beta}}) = \boldsymbol{\sigma}^2\mathbf{a}'(\mathbf{X}'\mathbf{X})^{-}\mathbf{a}$

If $\mathrm{rank}(\mathbf{X}) = r < p$ then we can impose identifiability constraints on the parameters. In vector notation a constraint is $\mathbf{h} = (\mathbf{h}_0, \ldots, \mathbf{h}_{p-1})$ such that $\mathbf{h}'\boldsymbol{\beta} = 0$. We need $s = p - r$ constraints and they should be linearly independent of each other and the rows of $\mathbf{X}$.

4 Generalized least squares

Generalized least squares pertains to the more general assumption $\mathrm{cov}(\boldsymbol{\varepsilon}) = \boldsymbol{\sigma}^2\mathbf{V}$ for some known positive definite $\mathbf{V}$.

$\boldsymbol{\beta}^* = (\mathbf{X}'\mathbf{V}^{-1}\mathbf{X})^{-1}\mathbf{X}'\mathbf{V}^{-1}\mathbf{Y}$ is the generalized least squares estimate.

$\boldsymbol{\beta}^*$ is unbiased and $\mathrm{cov}(\boldsymbol{\beta}^*) = \boldsymbol{\sigma}^2(\mathbf{X}'\mathbf{V}^{-1}\mathbf{X})^{-1}$

The GLS and OLS estimates are the same if and only if $\mathcal{R}(\mathbf{V}^{-1}\mathbf{X}) = \mathcal{R}(\mathbf{X})$.