Biost/Stat 533
Spring 2008

Homework Assignment #2
(due in class Monday, April 7, 2008)

Reading: Seber & Lee (S&L): Chapter 2

Homework:

1. (*S&L 1a.1*) Prove that if $\mathbf{a}$ is a vector of constants with the same dimension of the random vector $\mathbf{X}$, then

$$E[(\mathbf{X}-\mathbf{a})(\mathbf{X}-\mathbf{a})^T] = Var[\mathbf{X}] + (E[\mathbf{X}]-\mathbf{a})(E[\mathbf{X}]-\mathbf{a})^T.$$

If $Var[\mathbf{X}] = (\sigma_{ij})$, show

$$E[||\mathbf{X}-\mathbf{a}||^2] = \sum_i \sigma_{ii} + ||E[\mathbf{X}]-\mathbf{a}||^2.$$

2. (*S&L 1a.3*) Let $\mathbf{X} = (X_1, X_2, \ldots, X_n)'$ be a vector of random variables, and let $Y_1 = X_1, Y_i = X_i - X_{i-1} (i = 2, 3, \ldots, n)$. If the Y_i are mutually independent random variables each with unit variance, find $Var[\mathbf{X}]$.

3. (*S&L 1a.4*) If $X_1, X_2, \ldots, X_n$ are random variables satisfying $X_{i+1} = \rho X_i$, where ρ is a constant, and $var(X_1) = \sigma^2$, find $Var[\mathbf{X}]$.

4. (*S&L 1b.2*) If $X_1, X_2, \ldots, X_n$ are independent random variables with comon mean μ and variances $\sigma_1^2, \sigma_2^2, \ldots \sigma_n^2$, find $var[\bar{X}]$. Prove that $\sum_i (X_i - \bar{X})^2/[n(n-1)]$ is an unbiased estimate of $var[\bar{X}]$.

5. (*S&L 1b.5*) Let $X_1, X_2, \ldots, X_n$ be independently and identically distributed (IID) as $N(\theta, \sigma^2)$. Define

$$S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2,$$

and

$$Q = \frac{1}{2(n-1)}\sum_{i=1}^{n-1}(X_{i+1} - X_i)^2.$$

(a) Prove that $var[S^2] = 2\sigma^4/(n-1)$.

(b) Show Q is an unbiased estimator of σ^2.

(c) Find the variance of Q and show that, as $n \to \infty$, the efficiency of Q relative to S^2 is $\frac{2}{3}$.

6. (*S&L 1Misc.2*) Let $\mathbf{X} = (X_1, X_2, X_3)'$ with

$$Var[\mathbf{X}] = \begin{pmatrix} 5 & 2 & 3 \\ 2 & 3 & 0 \\ 3 & 0 & 3 \end{pmatrix}.$$

(a) Find the variance of $X_1 - 2X_2 + X_3$.

(b) Find the variance matrix of $\mathbf{Y} = (Y_1, Y_2)'$, where $Y_1 = X_1 + X_2$ and $Y_2 = X_1 + X_2 + X_3$.