Biost/Stat 533
Spring 2008

Homework Assignment #3
(due in class Monday, April 21, 2008)

Reading: Seber & Lee 3.1

Homework:

1. Let $\mathbf{Y} \sim N(\boldsymbol{\mu}, \sigma^2\mathbf{I})$. Define matrices $\mathbf{A}_1 = \frac{1}{3}\mathbf{J}J'$, $\mathbf{A}_2 = \frac{1}{2}\begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, and $\mathbf{A}_3 = \frac{1}{6}\begin{bmatrix} 1 & 1 & -2 \\ 1 & 1 & -2 \\ -2 & -2 & 4 \end{bmatrix}$. Define $Q_i = \mathbf{Y}'\mathbf{A_i}\mathbf{Y}$.
(a) Find the distribution of Q_1, Q_2, Q_3.
(b) Prove or disprove: the Q_i are pairwise independent.

2. (*S&L 3a.1*) Recall our definition of $\hat{\boldsymbol{\beta}}$: $\hat{\mathbf{Y}}$ is the projection of $\mathbf{Y}$ onto the column space of $\mathbf{X}$, and $\hat{\boldsymbol{\beta}}$ is a vector such that $\hat{\mathbf{Y}} = \mathbf{X}\hat{\boldsymbol{\beta}}$.

Show that if $\mathbf{X}$ has full rank,

$$(\mathbf{Y} - \mathbf{X}\boldsymbol{\beta})'(\mathbf{Y} - \mathbf{X}\boldsymbol{\beta}) = (\mathbf{Y} - \mathbf{X}\hat{\boldsymbol{\beta}})'(\mathbf{Y} - \mathbf{X}\hat{\boldsymbol{\beta}}) + (\hat{\boldsymbol{\beta}} - \boldsymbol{\beta})'\mathbf{X}'\mathbf{X}(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta}),$$

and hence, deduce that the left side is minimized when $\hat{\boldsymbol{\beta}} = \boldsymbol{\beta}$.

3. Suppose $\hat{\boldsymbol{\beta}}_1 \neq \hat{\boldsymbol{\beta}}_2$ are two different least-squares estimates of $\boldsymbol{\beta}$. Show there are infinitely many least-squares estimates of $\boldsymbol{\beta}$.

4. Let $\mathbf{P}$ be the projection operator onto $\mathcal{R}(\mathbf{X})$. For least-squares estimation, recall that $\hat{\boldsymbol{\varepsilon}} = (\mathbf{I} - \mathbf{P})\mathbf{Y}$. Derive
(a) $E(\hat{\boldsymbol{\varepsilon}})$
(b) $\text{cov}(\hat{\boldsymbol{\varepsilon}})$
(c) $\text{cov}(\hat{\boldsymbol{\varepsilon}}, \mathbf{PY})$
(d) $E[RSS]$