Homework Assignment #4
(due in class Monday, April 28, 2008)

Reading: Seber & Lee (S&L): Continue reading in chapter 3

Homework:

1. (a) Let $Y_1, \ldots, Y_n$ be a random sample from a distrubition with mean θ and finite variance σ^2. Find the BLUE of θ (and justify that it is, in fact, the Best Linear Unbiased Estimate).

(b) Explain the statement in lectures notes 8 that $RSS/(n-p)$ is a generalization of the sample variance.

We will complete the calculations for the simple one–way ANOVA model. By considering both $r = p$ and $r < p$ cases, one can understand more clearly the properties of $\hat{\beta}$ and $\hat{\theta}$.

2. To begin, consider the full rank version of the model:

$$\mathbf{Y}^{2J\times 1} = \begin{bmatrix} Y_{11} \\ \vdots \\ Y_{1J} \\ Y_{21} \\ \vdots \\ Y_{2J} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \vdots & \vdots \\ 1 & 0 \\ 0 & 1 \\ \vdots & \vdots \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix} + \begin{bmatrix} \epsilon_{11} \\ \vdots \\ \epsilon_{1J} \\ \epsilon_{21} \\ \vdots \\ \epsilon_{2J} \end{bmatrix} = \mathbf{X}^{2J\times 2}\boldsymbol{\beta}^{2\times 1} + \boldsymbol{\varepsilon}^{2J\times 1}$$

(a) Interpret the model parameters.

(b) Compute $\hat{\boldsymbol{\beta}}$. Is it unique (yes/no)? Explain.

(c) Compute $\mathbf{X}\hat{\boldsymbol{\beta}} \equiv \hat{\boldsymbol{\theta}}$. Is it unique (yes/no)? Explain.

(d) Compute the hat matrix (always unique) $\mathbf{P} = \mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'$.

3. Now consider a non-full rank version of the model:

$$\mathbf{Y}^{2J\times 1} = \begin{bmatrix} 1 & 1 & 0 \\ \vdots & \vdots & \vdots \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ \vdots & \vdots & \vdots \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} \mu \\ \alpha_1 \\ \alpha_2 \end{bmatrix} + \begin{bmatrix} \epsilon_{11} \\ \vdots \\ \epsilon_{1J} \\ \epsilon_{21} \\ \vdots \\ \epsilon_{2J} \end{bmatrix} = \mathbf{X}^{2J\times 3}\boldsymbol{\beta}^{3\times 1} + \boldsymbol{\varepsilon}^{2J\times 1}$$

(a) Interpret the model parameters.

(b) Recall that $\hat{\boldsymbol{\beta}} = (\mathbf{X}'\mathbf{X})^{-}\mathbf{X}'\mathbf{Y}$. There are an infinite number of generalized inverses for $\mathbf{X}'\mathbf{X}$. Here are two of them:

$$(\mathbf{X}'\mathbf{X})_1^{-} = J^{-1}\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad (\mathbf{X}'\mathbf{X})_2^{-} = J^{-1}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Use $(\mathbf{X}'\mathbf{X})_1^{-}$ and $(\mathbf{X}'\mathbf{X})_2^{-}$ to compute $\hat{\boldsymbol{\beta}}$. Is $\hat{\boldsymbol{\beta}}$ unique (yes/no)? Explain.

(c) Compute $\mathbf{X}\hat{\boldsymbol{\beta}} \equiv \hat{\boldsymbol{\theta}}$ for both cases in (a). Is $\hat{\boldsymbol{\theta}}$ unique (yes/no)? Compare your results with problem 2(c).

(d) Compute the hat matrix $\mathbf{P} = \mathbf{X}(\mathbf{X}'\mathbf{X})^{-}\mathbf{X}'$ and compare your results with problem 2(d).

4. Let $Y_i = \beta_0 + \beta_1 x_i + \epsilon_i$ $(i = 1, \ldots, n)$, where $E[\epsilon] = \mathbf{0}$ and var$[\epsilon]$=$\sigma^2\mathbf{I}$. Prove that the least squares estimates of β_0 and β_1 are uncorrelated if and only if $\bar{x} = 0$.

5.

(a) (*S& L 3b.4*) Let

$$Y_i = \beta_0 + \beta_1(x_{i1} - \bar{x}_1) + \beta_2(x_{i2} - \bar{x}_2) + \epsilon_i \quad i = 1, 2, \ldots, n,$$

where $\bar{x}_j = \sum_{i=1}^{n} x_{ij}/n$, $E[\epsilon] = \mathbf{0}$, and $Var[\epsilon] = \sigma^2\mathbf{I}$. If $\hat{\beta}_1$ is the least squares estimator of β_1, show that

$$\text{var}[\hat{\beta}_1] = \frac{\sigma^2}{\sum_{i=1}^{n}(x_{i1} - \bar{x}_1)^2(1 - r_{12}^2)},$$

where r_{12} is the correlation coefficient of the pairs (x_{i1}, x_{i2}).

(b) Comment on the impact of using highly correlated predictors x_1 and x_2 in a linear model.

6. (*S& L 3c.1*) Suppose that $\mathbf{Y} \sim N(\mathbf{X}\boldsymbol{\beta}, \sigma^2\mathbf{I})$, where $\mathbf{X}$ is $n \times p$ of rank p.

(a) Find var$[S^2]$.

(b) Evaluate $E[(\mathbf{Y}'\mathbf{A}\mathbf{Y} - \sigma^2)^2]$ for $A = \frac{1}{n-p+2}[\mathbf{I} - \mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}']$.

(c) Prove that S^2 does not have minimum mean squared error among estimates of σ^2.