Homework Key #4

1 See the key of the midterm.

2. See the key of the sample midterm.

3 Consider the following model:

$$\begin{aligned} Y_1 &= \tau_1 + \tau_2 + \tau_3 + \epsilon_1 \\ Y_2 &= \tau_1 + \tau_3 + \epsilon_2 \\ Y_3 &= \tau_2 + \epsilon_3 \end{aligned}$$

(a) Write out the model in matrix form. What is the rank of the design matrix?
(b) Which (if any) of the individual parameters are estimable?
(c) Is $\tau_1 - 2\tau_2 + \tau_3$ estimable? Explain how you know.
(d) Find the BLUE of $\tau_1 - 2\tau_2 + \tau_3$. In addition, find another estimate of $\tau_1 - 2\tau_2 + \tau_3$ that is not the BLUE.

Solution: (a) We can write

$$\begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \tau_1 \\ \tau_2 \\ \tau_3 \end{bmatrix} + \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \end{bmatrix}.$$

Letting $\mathbf{Y} = \begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \end{bmatrix}, \mathbf{X} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \beta = \begin{bmatrix} \tau_1 \\ \tau_2 \\ \tau_3 \end{bmatrix}, \epsilon = \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \end{bmatrix}$, we can also write

$$\mathbf{Y} = \mathbf{X}\beta + \epsilon.$$

The Gaussian elimination gives the rank of the design matrix $\mathbf{X}$ of 2.
(b) Let $\mathbf{e}_i \in \mathbb{R}^3$ be the vector the ith element of which is 1 and the rest of which is 0. Then $\mathbf{e}_i^T\beta = \tau_i$. Define the augmented matrices given by $[\mathbf{X}^T\ \mathbf{e}_i]$. The Gaussian elimination shows that $\mathbf{e}_1 \notin \mathcal{R}(\mathbf{X}^T)$, $\mathbf{e}_3 \notin \mathcal{R}(\mathbf{X}^T)$ but $\mathbf{e}_2 \in \mathcal{R}(\mathbf{X}^T)$. Applying the Lemma 10.2.1, we conclude that τ_2 is estimable but that other two parameters are not.
(c) Let $\mathbf{c} = (1, -2, 1)^T$. Then $\mathbf{c}^T\beta = \tau_1 - 2\tau_2 + \tau_3$. As in the previous problem, define the augmented matrix $[\mathbf{X}^T\ \mathbf{c}]$ and perform the Gaussian elimination to conclude that $\tau_1 - 2\tau_2 + \tau_3$ is estimable by the Lemma 10.2.1.
(d) Let

$$\mathbf{X}_1 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ and } \mathbf{X}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}.$$

Note that $\mathbf{X} = [\mathbf{X}_1 \, \mathbf{X}_2]$. Note also that $\mathbf{X}$ has two independent columns and those correspond to columns of $\mathbf{X}_1$. Thus, a generalized inverse of $\mathbf{X}^T\mathbf{X}$ is given by

$$\mathbf{X}^T\mathbf{X}^- = \begin{bmatrix} (\mathbf{X}_1^T\mathbf{X}_1)^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$$

with

$$(\mathbf{X}_1^T\mathbf{X}_1)^{-1} = \frac{1}{3}\begin{bmatrix} 2 & \text{-1} \\ \text{-1} & 2 \end{bmatrix}.$$

This gives a least squares estimate $\hat{\beta}$ of β by

$$\hat{\beta} \equiv (\mathbf{X}^T\mathbf{X})^-\mathbf{X}^T\mathbf{Y} = \frac{1}{3}\begin{bmatrix} Y_1 + 2Y_2 - Y_3 \\ Y_1 - Y_2 + 2Y_3 \\ 0 \end{bmatrix}.$$

Because $\mathbf{c}^T\beta$ is estimable by part(c), the estimate $\mathbf{c}^T\hat{\beta}$ given by

$$\mathbf{c}^T\hat{\beta} = \frac{-Y_1 + 4Y_2 - 5Y_3}{3}$$

is the BLUE of $\tau_1 - 2\tau_2 + \tau_3$ by the Gauss-Markov Theorem. A possible linear unbiased estimate of $\tau_1 - 2\tau_2 + \tau_3$ is $Y_2 - 2Y_3$ because

$$E[Y_2 - 2Y_3] = E[Y_2] - 2E[Y_3] = (\tau_1 + \tau_3) - 2\tau_2 = \tau_1 - 2\tau_2 + \tau_3$$

and

$$Y_2 - 2Y_3 = \mathbf{d}^T\mathbf{Y}$$

where $\mathbf{d} = (0, 1, -2)^T$. This is not the BLUE of $\tau_1 - 2\tau_2 + \tau_3$ because the BLUE is unique by the Gauss-Markov Theorem.