Biost/Stat 533
Spring 2008

Homework Assignment #6
(due in class Monday, May 12, 2008)

Reading: Seber & Lee (S&L): Chapter 4 (you should have finished Chapter 3 through 3.10)

Homework:

1. (Seber & Lee 3k.4) Let $Y_1, \ldots, Y_n$ be random variables with common mean θ and dispersion matrix $\sigma^2\mathbf{V}$, where $v_{ii} = 1$ and off-diagonal entries $v_{ij} = \rho$. Find the generalized least squares estimate of θ and show that it is the same as the ordinary least squares estimate. *Hint:* $\mathbf{V}^{-1}$ takes the same form as $\mathbf{V}$.

2. (similar to Seber & Lee 3k.5) Let $\mathbf{Y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\varepsilon}$ where $\boldsymbol{\varepsilon} \sim N_n(\mathbf{0}, \sigma^2\mathbf{V})$, $\mathbf{X}$ is $n \times p$ of rank p, and $\mathbf{V}$ is a known positive-definite $n \times n$ matrix. Let $\boldsymbol{\beta}^*$ be the GLS estimate of $\boldsymbol{\beta}$. Prove that

(a) $Q = (\mathbf{Y} - \mathbf{X}\boldsymbol{\beta}^*)'\mathbf{V}^{-1}(\mathbf{Y} - \mathbf{X}\boldsymbol{\beta}^*)/\sigma^2 \sim \chi^2_{n-p}$

(b) Find an unbiased estimate of σ^2.

(c) Define $\mathbf{Y}^* = \mathbf{X}\boldsymbol{\beta}^*$. If $\mathbf{Y}^* = \mathbf{P}^*\mathbf{Y}$, what is $\mathbf{P}^*$? Show that $\mathbf{P}^*$ is idempotent but not, in general, symmetric. (Extra credit if you work out an example where $\mathbf{P}^*$ is not symmetric.)

3. (Seber & Lee 3MISC.10) If $\mathbf{X}$ is not of full rank, show that any solution $\boldsymbol{\beta}$ of

$$\mathbf{X}'\mathbf{V}^{-1}\mathbf{X}\boldsymbol{\beta} = \mathbf{X}'\mathbf{V}^{-1}\mathbf{Y}$$

minimizes $(\mathbf{Y} - \mathbf{X}\boldsymbol{\beta})'\mathbf{V}^{-1}(\mathbf{Y} - \mathbf{X}\boldsymbol{\beta})$.