Sequential Experimental Designs for Generalized Linear Models

Hovav A. Dror and David M. Steinberg, JASA (2008)

Bob A. Salim

May 14, 2013

Background (Motivation)

- Designing an experiment or a study
- Restrictions on sample size (due to cost, time, etc)
- How do we design an efficient experiment with valid estimate of the parameter?

Efficient Design

- Efficient experimental designs for GLMs depend on the unknown coefficients.
- Sequential design: next design point chosen based on current data.

What is Efficient?

- Optimality criterion: D-optimal
- D-optimality criterion: maximize the determinant of the information matrix $\mathbf{I}(\boldsymbol{\beta}$;d)
  $\boldsymbol{\beta}$ = parameters in the model; d = design
- Bayesian D-optimality criterion (Chaloner and Larntz, 1989):

$$\phi(d) = \int log(\mathbf{I}(\boldsymbol{\beta}; d)) d\pi(\boldsymbol{\beta}) \tag{1}$$

where $\pi(\boldsymbol{\beta})$ = prior distribution on $\boldsymbol{\beta}$.

Overview of Proposed Method: Approximate Design Criteria

- Discretized posterior

$$\phi_1(d) = \sum_{u=1}^{n} r_u log(\mathbf{I}(\boldsymbol{\beta}_u; d)) \tag{2}$$

where $r_u = L(\boldsymbol{\beta}_u) / \sum_{u=1}^{n} L(\boldsymbol{\beta}_u)$

- A faster approximation

$$\phi_2(d) = log(\mathbf{I}(\tilde{\beta}; d)) \tag{3}$$

where $\tilde{\beta} = \beta_{(g)}$ such that $\sum_{u=1}^{g} r_{(u)} \geq 0.5$ and $\sum_{u=g}^{n} r_{(u)} \geq 0.5$.

- Another approximation for determining augmentation horizon

$$\phi_3(d) = (1/p)\phi_2(d) - log(n) \tag{4}$$

Overview of Proposed Method: Algorithm

- In the beginning of the experiment, define the augmentation horizon (m).
- Find a locally D-optimal m-run augmentation to the current design, maximizing ϕ_2 at the current parameter median.
- Generate a candidate set for the augmentation consisting of the m points found in the previous step and their coordinatewise median.
- If the design points run thus far provide a nonsingular information matrix, choose the next design point as the candidate that gives the best ϕ_1 when added to the current design.
- If the design thus far does not provide a nonsingular information matrix, then choose the next design point from among the candidates by comparing the values of ϕ_1 for designs that consist of the points run thus far, the m-run augmentation, and the candidate.

Overview of Proposed Method: Augmentation Horizon

- Proposed to avoid problems, such as singular information matrix
- It is the number of observations (m) needed for highly efficient D-optimal design at prior median
- Determined at the start of the experiment

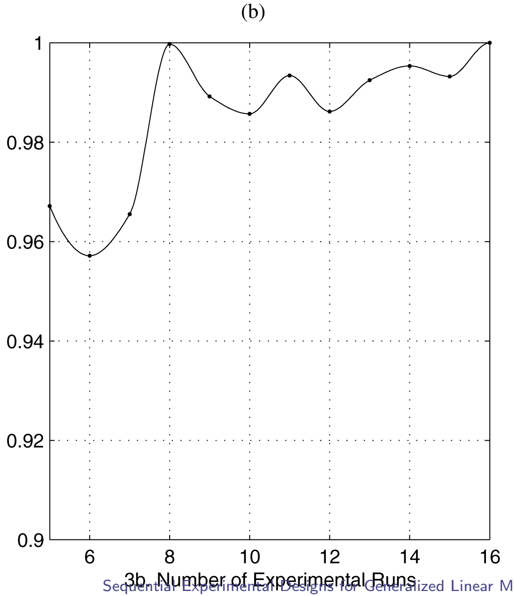
Overview of Proposed Method: Augmentation Horizon

- Find locally D-optimal designs at prior median for $n = p, ..., P$
- Define the efficiency of $d_n = \exp[\phi_3(d_n) - \phi_3(d*)]$
  where $d*$ is the design among $d_p, ..., d_P$ that maximizes $\phi_3(d)$
- Augmentation horizon m is the smallest value of n for which the efficiency is at least 99%

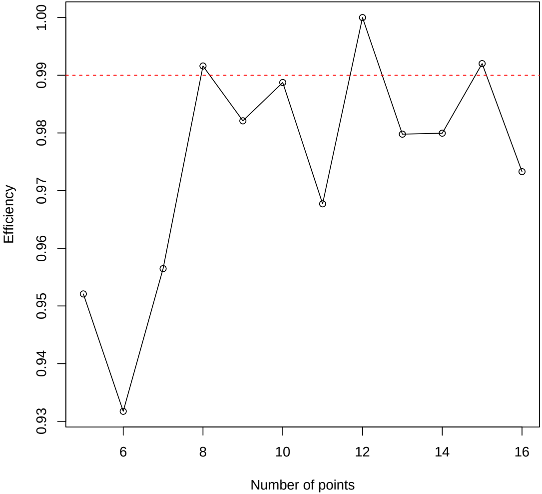
Finding Locally D-optimal Designs

- Given in their earlier paper (Dror and Steinberg, 2006)
- For linear regression, the information matrix is given by $\mathbf{X}^T\mathbf{X}$
- For GLMs, the information matrix is given by $\mathbf{X}^T\mathbf{W}\mathbf{X}$, where $W = V^{-1}(\boldsymbol{\mu})(d\boldsymbol{\mu}/d\boldsymbol{\eta})^2$. $\boldsymbol{\mu}$ is the vector of expected response, and $\boldsymbol{\eta}$ is the linear predictor $\mathbf{X}\beta$.
- For example, for logistic regression, $\mathbf{W}$ is a diagonal matrix with diagonal elements $w_{ii} = exp(\mathbf{X}_i\beta)/(1 + exp(\mathbf{X}_i\beta))^2$.
- Applies a row-exchange algorithm (Federov, 1972)

Back to the Augmentation Horizon



Back to the Augmentation Horizon



Explosives Testing Example

- Sensitivity Experiment in June 2006 at an industrial plant.
- Compare the performance of this method to Bruceton up-and-down method (default)
- Requirement: probability of detonation at 12V (or below) is $< 5\%$ and the probability of detonation at 25V (or above) is $> 95\%$.
- The authors used the following priors: $\mu \sim$ lognormal(log(17),0.5^2) and $\mu \sim$ lognormal(log(0.7),0.5^2)

Explosives Testing Example

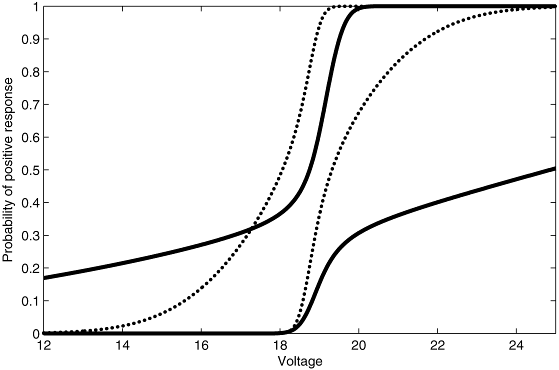


Figure 6. Comparison of the plant format (——) and our algorithm (·······) after 20 observations from each. The lines are pointwise 95% confidence intervals for the probability of response.

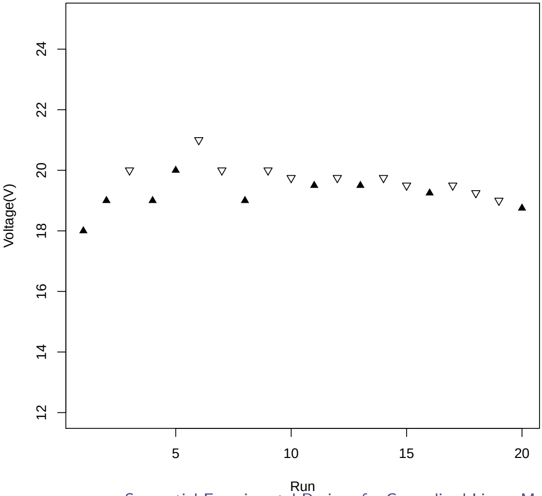
Explosives Testing Example

Difficulties in reproducing this result:

- Real test, actual response observed in the plant
- Can do simulation study, but the "true" parameters, and even the "true" response surface unknown

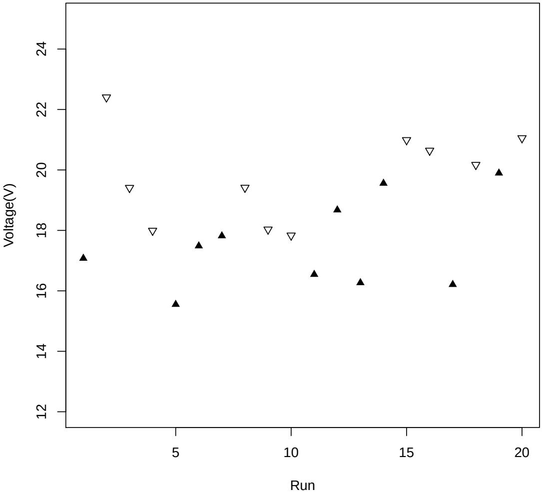
Explosives Testing Example: Bruceton

Simulation study. True parameter: $\mu = 19$ and $\sigma = 0.7$



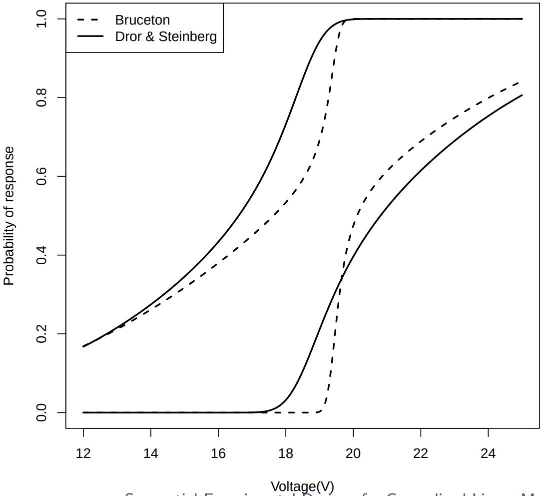
Explosives Testing Example: Dror & Steinberg

Simulation study. True parameter: $\mu = 19$ and $\sigma = 0.7$



Explosives Testing Example: Comparison of results

Simulation study. True parameter: $\mu = 19$ and $\sigma = 0.7$



Next Steps

- Figure out what's different in their codes and in mine.
- Simulation studies to compare results with other methods (Neyer, logit-MLE, Bayes-logit-MLE, etc)

References

Dror, H.A. and Steinberg, D.M. (2008). Sequential Experimental Designs for Generalized Linear Models. Journal of American Statistical Association 103:481, 288-298

Abdelbasit, K.M. and Plackett, R.L. (1983). Experimental Designs for Binary Data. Journal of American Statistical Association 101, 747-759

Chaloner, K. and Larntz, K. (1989). Optimal Bayesian Design Applied to Logistic Regression Experiments. Journal of Statistical Planning and Inference, 21, 191-208

Dixon, J.W. and Mood, A.M. (1948) A Method for Obtaining and Analyzing Sensitivity Data. *Journal of American Statistical Association* 43, 109-126

Robbins, H. and Monro, S. (1951). A Stochastic Approximation Method. Annals of Mathematical Statistics 29, 400-407