Separable covariance arrays via the Tucker product

by Peter Hoff

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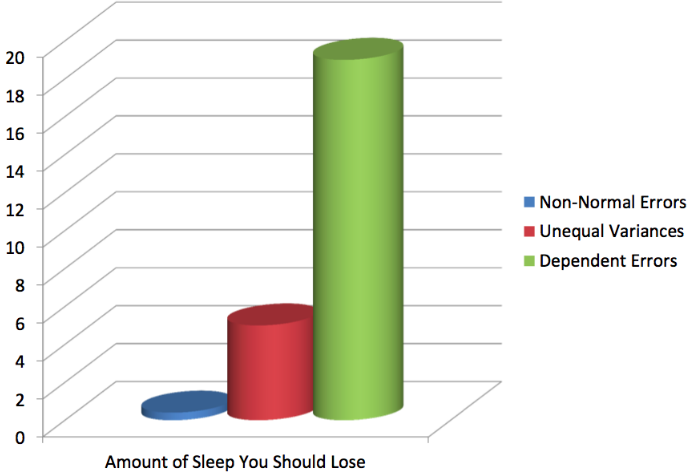
Correlated Errors are Bad!



Lecture slide from Biostat 533

Review of Multivariate Analysis

Multivariate normal model, $\mathbf{y} \in \mathbb{R}^m$:

$$\mathbf{z} = \{z_j : j = 1, \ldots, m\} \overset{\text{iid}}{\sim} \text{normal}(0, 1)$$
$$\mathbf{y} = \boldsymbol{\mu} + \mathbf{A}\mathbf{z} \overset{\text{iid}}{\sim} \text{multivariate normal}(\boldsymbol{\mu}, \boldsymbol{\Sigma} = \mathbf{A}\mathbf{A}^T)$$

Matrix-variate normal model, $\mathbf{Y} \in \mathbb{R}^{m_1 \times m_2}$:

$$\mathbf{Z} = \{z_{i,j}\}_{i=1}^{m_1}{}_{j=1}^{m_2} \overset{\text{iid}}{\sim} \text{normal}(0, 1)$$
$$\mathbf{Y} = \mathbf{M} + \mathbf{A}\mathbf{Z}\mathbf{B}^T \overset{\text{iid}}{\sim} \text{matrix normal}(\mathbf{M}, \boldsymbol{\Sigma}_1 = \mathbf{A}\mathbf{A}^T, \boldsymbol{\Sigma}_2 = \mathbf{B}\mathbf{B}^T)$$
$$\sim \text{matrix normal}(\mathbf{M}, \boldsymbol{\Sigma}_1 \circ \boldsymbol{\Sigma}_2)$$

Note that matrix-variate normal assumes separable covariance structure

What is separable covariance structure?

$$\mathbf{Y} \sim \text{matrix normal}(0, \boldsymbol{\Sigma}_1, \boldsymbol{\Sigma}_2)$$

- Covariance is product of row covariance and column covariance
$$\text{Cov}(Y_{ij}, Y_{kl}) = \Sigma_{1\,ik} \times \Sigma_{2\,jl}$$
- Reduced number of parameters to be estimated
- From $\frac{(np)\times(np+1)}{2}$ to $\frac{p(p+1)}{2} + \frac{n(n+1)}{2}$

Made up motivation - linear regression model

$$\mathbf{Y} = \mathbf{M} + \mathbf{E}$$

- $\mathbf{M}$ is the mean structure (for instance, $\mathbf{X}\boldsymbol{\beta}$, or ANOVA model)
- $\mathbf{E}$ is the error term

Made up motivation - Example 1

Suppose $\mathbf{y}_i \in \mathbb{R}^{m_1}$ is the outcome variable obtained by repeatedly taking measurements from subject i across time $j = \{1, \ldots, m_1\}$.

Appropriate Model:

$$\mathbf{y}_i = \mathbf{x}_i^T \beta + \boldsymbol{\epsilon}_i$$

$\boldsymbol{\epsilon}_i \sim$ multivariate normal$(\mathbf{0}, \mathbf{\Sigma})$

Made up motivation - Example 2

Suppose $Y_i \in \mathbb{R}^{m_1 \times m_2}$ and $y_{i,j,k}$ is the ith outcome variable for location j at time k.
Naive Model: Assume that the locations are not correlated

$$\mathbf{y}_{ij} = \mathbf{x}_{ij}^T \beta + \boldsymbol{\epsilon}_{ij}$$

$\boldsymbol{\epsilon}_{ij} \sim$ multivariate normal$(\mathbf{0}, \mathbf{\Sigma})$

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Really? Ignoring dependent errors after taking Biostat571?

More complicated models

Look at Laina Mercer's slides, or alternatively,

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$$\mathbf{Y}_i = \mathbf{\Theta}\mathbf{X}_i + \mathbf{E}_i$$

$\mathbf{E}_i \overset{iid}{\sim}$ matrix normal$(\mathbf{0}, \mathbf{\Sigma}_1, \mathbf{\Sigma}_2)$

Closely related to (Knorr-Held and Besag, 1998), it does not allow for space $\times$ time interactions.

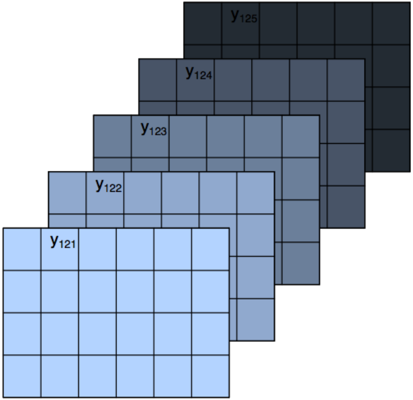
Citation: On matrix-variate regression analysis by Cinzia Viroli (2012)

What is an array?

Gene expression data set
$\mathbf{Y} = \{y_{i,j,k}\}$.

- i indexes the ith subject
- j indexes the jth gene
- k indexes the kth repeated measurement

Then, $y_{i,j,k}$ is the gene expression level for the jth gene of the ith subject, measured at time k.



Citation: Are a set of microarrays independent of each other by Brad Efron (2009)

Motivation - Example 3

Yearly change in log trade value (in 2000 dollars): $\mathbf{Y} = \{y_{i,j,k,l}\}$

- $i \in \{1, \ldots, 30\}$ indexes the exporting nation
- $j \in \{1, \ldots, 30\}$ indexes the importing nation
- $k \in \{1, \ldots, 6\}$ indexes the commodity type
- $t \in \{1, \ldots, 10\}$ indexes the year

Interested in modeling the mean $M_{ijk} = \mu_{i,j,k}$ across t measurements

What can we do?

Motivation - Example 3 cont

Interested in the model

$$y_{i,j,k,l} = \mu_{i,j,k} + \epsilon_{i,j,k,l}$$

- iid error model: $\epsilon_{i,j,k,l} \sim \text{normal}(0, \sigma^2)$
- multivariate error model: $\boldsymbol{\epsilon}_{i,j,k} \sim \text{multivariate normal}(\mathbf{0}, \boldsymbol{\Sigma})$
- matrix-variate error model: $\boldsymbol{\epsilon}_{i,j} \sim \text{matrix normal}(\mathbf{0}, \boldsymbol{\Sigma}_1, \boldsymbol{\Sigma}_2)$

Motivation - Example 3 cont

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- matrix-variate error model: $\boldsymbol{\epsilon}_{i,j} \sim \text{matrix normal}(\mathbf{0}, \boldsymbol{\Sigma}_1, \boldsymbol{\Sigma}_2)$

But all four dimensions are correlated!

$$\mathbf{E} \sim ???$$

Goal of the paper

Propose the Array Normal distribution for array data

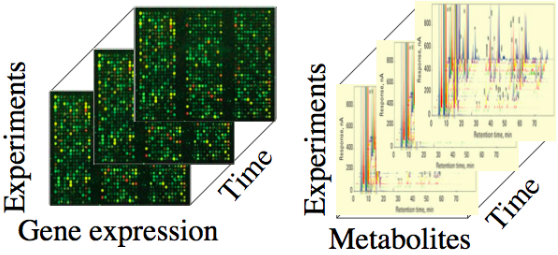
- model mean structure
- model covariance structure

Suppose $\mathbf{Y} \in \mathbb{R}^{m_1 \times \ldots \times m_k}$

$$\mathbf{Y} \sim \text{ array normal}(\mathbf{M}, \mathbf{\Sigma}_1, \ldots, \mathbf{\Sigma}_k)$$

Take home message until the next talk

- Array data is everywhere



- Most people assume certain dimensions are independent
- Maybe it is a good idea to model the dependencies after all!

Questions?