Bayesian modeling of inseparable space-time variation in disease risk

Leonhard Knorr-Held

Laina Mercer

Department of Statistics
UW

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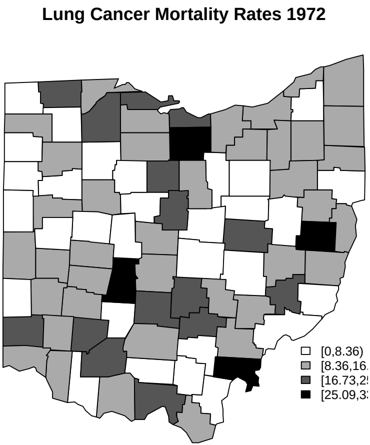
Motivation

Ohio Lung Cancer Example



The Model

Stage 1

Address the situation where the disease variation cannot be separated into temporal and spatial main effects and spatio-temporal interactions become and important feature.

$$\eta_{it} = \mu + \alpha_t + \gamma_t + \theta_i + \phi_i + \delta_{it}$$

- μ - overall risk level
- α_t - temporally structured effect of time t
- γ_t - independent effect of time t
- θ_i - spatially structured effect of county i
- ϕ_i - independent effect of county i
- δ_{it} - space $\times$ time interaction

The Model

Stage 2 - Independent Priors

Prior for γ :

$$
\begin{aligned}
p(\gamma|\lambda_\gamma) &\propto \exp\left(-\frac{\lambda_\gamma}{2}\sum_{t=1}^{T}\gamma_t^2\right) \\
&= \exp\left(-\frac{\lambda_\gamma}{2}\gamma^T K_\gamma \gamma\right)
\end{aligned}
$$

where $K_\gamma = I_{TxT}$.

Prior for ϕ:

$$
\begin{aligned}
p(\phi|\lambda_\phi) &\propto \exp\left(-\frac{\lambda_\phi}{2}\sum_{i=1}^{n}\phi_t^2\right) \\
&= \exp\left(-\frac{\lambda_\phi}{2}\phi^T K_\phi \phi\right)
\end{aligned}
$$

where $K_\phi = I_{nxn}$.

The Model

Stage 2 - First Order Random Walk

$$\alpha_{t+1} - \alpha_t \sim \mathcal{N}(0, \lambda_\alpha^{-1}), \quad t = 1, \ldots, T-1$$

The Model

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$$\begin{aligned} p(\alpha|\lambda_\alpha) &\propto \lambda_\alpha^{(T-1)/2} \exp\left(-\frac{\lambda_\alpha}{2} \sum_{t=1}^{T-1} (\alpha_{t+1} - \alpha_t)^2\right) \\ &= \lambda_\alpha^{(T-1)/2} \exp\left(-\frac{\lambda_\alpha}{2} \alpha^T K_\alpha \alpha\right) \end{aligned}$$

The Model

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Where,

$$K_\alpha = \begin{pmatrix} 1 & -1 & & & \\ -1 & 2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -1 & 1 \end{pmatrix}$$

and has rank $T-1$.

The Model

Stage 2 - First Order Random Walk

With

$$p(\alpha|\lambda_\alpha) \propto \lambda_\alpha^{(T-1)/2} \exp\left(-\frac{\lambda_\alpha}{2}\alpha^T K_\alpha \alpha\right)$$

we have that

$$\alpha_t|\alpha_{-t}, \lambda_\alpha \sim \mathcal{N}\left(\frac{1}{2}(\alpha_{t-1} + \alpha_{t+1}), 1/(2\lambda_\alpha)\right)$$

The Model

Stage 2 - ICAR Prior

$$\theta_i - \theta_j \sim \mathcal{N}(0, \lambda_\theta^{-1}), \quad \text{for neighbors } i \text{ and } j$$

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Stage 2 - ICAR Prior

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Where our precision matrix has elements:

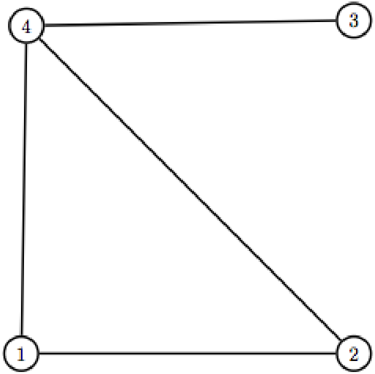
$$K_{\theta,ij} = \begin{cases} n_i & i = j \\ -1 & i \sim j \\ 0 & \text{otherwise} \end{cases}$$

and has rank $n - 1$.

The Model

Stage 2 - ICAR Prior

$$\text{An example } K_\theta = \begin{pmatrix} 2 & -1 & 0 & -1 \\ -1 & 2 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ -1 & -1 & -1 & 3 \end{pmatrix}$$



With

$$p(\theta|\lambda_\theta) \propto \lambda_\theta^{(n-1)/2} \exp\left(-\frac{\lambda_\theta}{2}\alpha^T K_\theta \theta\right)$$

we have that

$$\theta_i|\theta_{-i}, \lambda_\theta \sim \mathcal{N}\left(\frac{1}{n_i}\sum_{j:j\sim i}\theta_j, \frac{1}{n_i \lambda_\theta}\right)$$

Type I Independent - Independent multivariate gaussian prior

$$
\begin{aligned}
p(\delta|\lambda_\delta) &\propto \exp\left(-\frac{\lambda_\delta}{2}\sum_{i=1}^{n}\sum_{t=1}^{T}(\delta_{it})^2\right) \\
&= \exp\left(-\frac{\lambda_\delta}{2}\delta^T K_\delta \delta\right)
\end{aligned}
$$

Where $K_\delta = K_\gamma \otimes K_\phi = I_{nT\times nT}$.

Type II Temporal trends differ by area - first order random walk prior

$$
\begin{aligned}
p(\delta|\lambda_\delta) &\propto \exp\left(-\frac{\lambda_\delta}{2}\sum_{i=1}^{n}\sum_{t=1}^{T-1}(\delta_{it}-\delta_{i,t-1})^2\right) \\
&= \exp\left(-\frac{\lambda_\delta}{2}\delta^T K_\delta \delta\right)
\end{aligned}
$$

Where $K_\delta = K_\alpha \otimes K_\phi$ (rank $n(T-1)$).

The Model

Stage 2 - ICAR prior for δ

Type III Spatial trends differ over time - Intrinsic autoregression prior

$$\begin{aligned} p(\delta|\lambda_\delta) &\propto \exp\left(-\frac{\lambda_\delta}{2}\sum_{i\sim j}\sum_{t=1}^{T}(\delta_{it}-\delta_{jt})^2\right) \\ &= \exp\left(-\frac{\lambda_\delta}{2}\delta^T K_\delta \delta\right) \end{aligned}$$

Where $K_\delta = K_\gamma \otimes K_\theta$ (rank $(n-1)T$).

Type IV Spatio-temporal interaction - conditional depends on first and second order neighbors

$$\begin{aligned} p(\delta|\lambda_\delta) &\propto \exp\left(-\frac{\lambda_\delta}{2}\sum_{i\sim j}\sum_{t=2}^{T}(\delta_{it}-\delta_{jt}-\delta_{i,t-1}+\delta_{j,t-1})^2\right) \\ &= \exp\left(-\frac{\lambda_\delta}{2}\delta^T K_\delta \delta\right) \end{aligned}$$

Where $K_\delta = K_\alpha \otimes K_\theta$ (rank $(n-1)(T-1)$).

Overall level can be absorbed by both θ and α in main effects model and interaction model I.

- recenter θ and α after each iteration to have mean zero
- omit μ and recenter θ or α

Additional constraints for interaction models II, III, and IV:

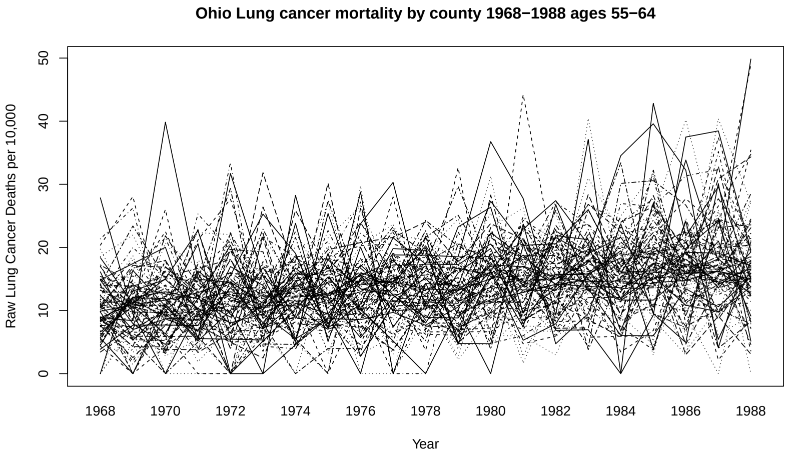
II recenter δ_{it} row-wise (across time)

III recenter δ_{it} column-wise (over space)

IV recenter δ_{it} row-wise and column-wise

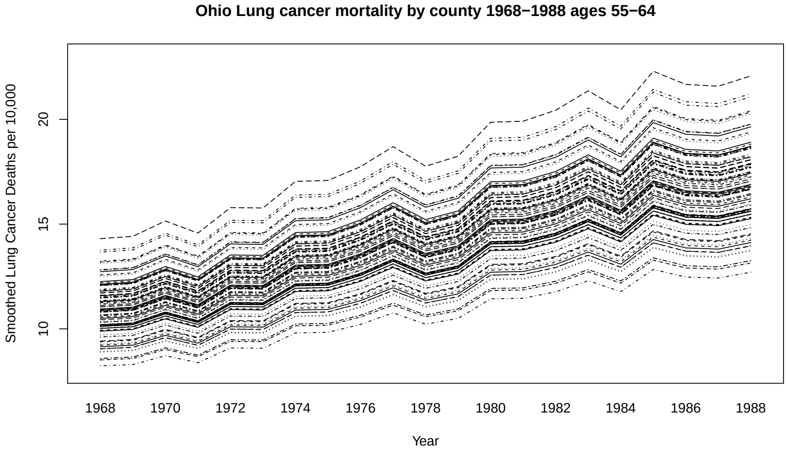
Example

Ohio Lung Cancer without smoothing



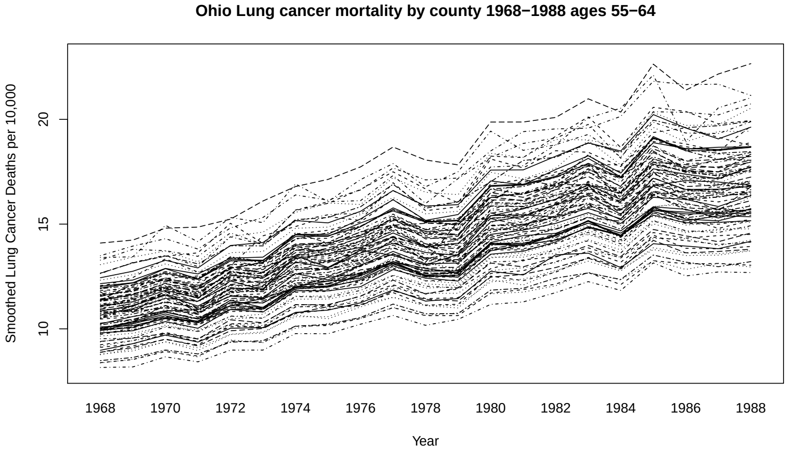
Example

Ohio Lung Cancer with main effects smoothing



Example

Ohio Lung Cancer with main effects + space-time interaction smoothing



What's next?

Currently coding all 5 models on a very small (5 nodes and 5 time points) graph with

- Winbugs
- INLA
- MCMC

To do

- 'tuning in an automated fashion' for univariate Metropolis updating
- block updating

The End

Questions?

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