Collision of gold nuclei at RHIC accelerator

Animation by Jeffery Mitchell (Brookhaven National Laboratory). Simulation by the UrQMD Collaboration

Physics 311
Special Relativity

Lecture 10:

ENERGY-MOMENTUM

Today's lecture plan

- Collisions
- Conservation of energy and momentum
- Energy-momentum: a 4-vector
- Spacetime map, now with energy-momentum
- Magnitude of energy-momentum
- Relativistic momentum
- Relativistic energy ($E = mc^2$)

Collisions are very complex...

• Consider a collision. Say, two cars collide. The process is very complex... Bent metal, glass chips flying, ...

... or very simple!

- Two objects rush towards each other, smash, pieces fly... If we only look at projectiles and the products, it’s quite simple!

Particle collisions at high energy

• Details of collision dynamics may be very complex, and are in fact beyond the the scope of Special Relativity (e.g. accelerations may be enormous, Quantum Mechanics is also involved...)

• But if all we look at is what goes in and what comes out, and ignore the details of collision itself, Special Relativity has a lot to say where classical mechanics fails miserably.

• A dramatic example of that is a collision of high-energy particles. These particles fly at very near speed of light and, it turns out, carry an enormous amount of energy, far greater than the classical kinetic energy $mv^2/2$.

• When cars collide, pieces that fly out are pieces of the cars – wheels, fenders, glass...

• When high-energy particles collide, pieces that fly are sometimes new particles, which are not necessarily "parts" of the colliding particles.

Birth of new particles

- Can two Cooper Minis collide to make a Hummer???

Birth of new particles

- Can two Cooper Minis collide to make a truck???

- This doesn't seem to make sense, yet this happens all the time at Fermilab and other high-energy facilities. Two puny protons give birth to massive top quarks and other heavy things, all because of the huge am  ergy they carry.

Energy-momentum

• We know all too well that energy is conserved, as well as the total momentum, *in any physical process*. Collisions included.

• Special Relativity makes a *special* point of the fact that *both* quantities are conserved.

• Special Relativity also likes to put things of seemingly different nature – like space and time – under one roof. (Still remember the interval?..)

• Following that route, energy and momentum in Special Relativity are combined into a 4-vector called simply "Energy-momentum". (Taylor and Wheeler insist on "Momenergy", but I will not yield this time.)

• Recall: 4-vectors are not our everyday 3-dimensional vectors, they have this funny metric in Special Relativity:

$$\text{magnitude}^2 = (\text{time part})^2 - (\text{space part})^2.$$

Components of energy-momentum

• Recall also that we write a 4-vector **A** as {$\mathbf{A}^0$, $\mathbf{A}^1$, $\mathbf{A}^2$, $\mathbf{A}^3$}, where superscript "0" corresponds to the time-component, and superscripts "1", "2" and "3" to the space-component.

• The time part of the energy-momentum 4-vector is a scalar. We set it to be the particle's energy – which is just a number, a scalar.

• The space part of the energy-momentum 4-vector is a regular 3-vector, and is represented by the particle's momentum.

• What about the units? Energy is measured in Joules, or ergs, or calories, or electron-volts, while momentum is measured in N·s or eV/c, or g cm s^{-1}...

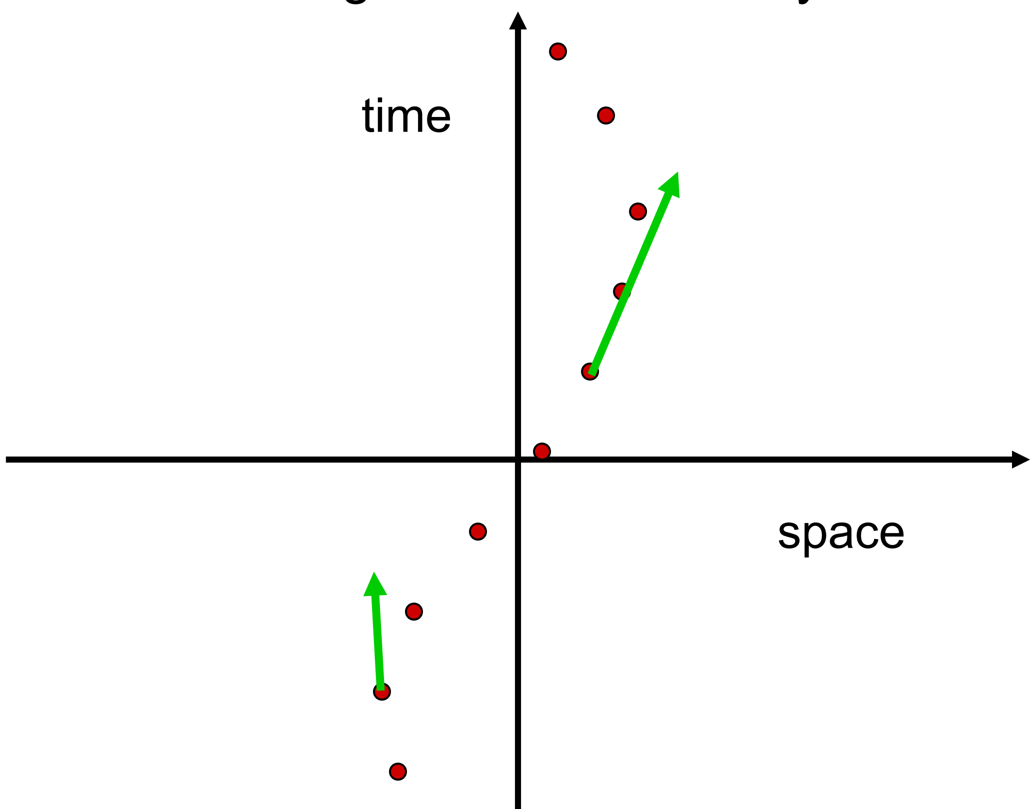
• Just like with the interval, where we've used speed of light *c* as the conversion factor between space and time, we'll find that the same conversion factor applies for energy and momentum.

Energy-momentum on spacetime map

• We'll determine the exact expression for the energy-momentum 4-vector components momentarily. For now, look at the spacetime map.

• Energy-momentum 4-vector magnitude is proportional to particle's mass. Just assume it for now, we'll see why and how.

• Energy-momentum has a direction. It's a vector on the spacetime map. Direction of the vector? Along the worldline! Why? We'll see that, too.



Expression for energy-momentum

• We claim that the magnitude of energy-momentum is the particle's mass. Recall that a 4-vector's magnitude must be invariant in all inertial frames. The time-part (energy) and the space-part (momentum) of the energy-momentum will change, but the "rest mass" will remain the same.

• The direction of energy-momentum is that of the worldline. This is why:

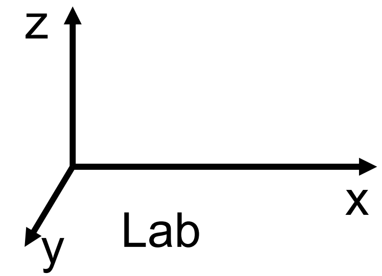
• We'll write the energy-momentum as:

P = (mass) x (d **spacetime displacement**)/(d proper time)

• It is the derivative of spacetime displacement with respect to proper time, and thus it is tangential with respect to the worldline. We did a similar derivation in the case of the 4-velocity.

• ... wait! It ***is exactly*** like we did in the case of the 4-velocity!!! That's right, energy-momentum is nothing but mass times 4-velocity. But it's more than just sum (well, product) of it's parts.

4-velocity components:



- "Time"-component:

$$U^0 = cdt/cdt' = (\boldsymbol{u}\gamma d\boldsymbol{r'} + \gamma dt')/dt'$$
$$U^0 = \boldsymbol{u}\gamma(d\boldsymbol{r'}/dt') + \gamma(dt'/dt') = \gamma$$

- "Space"-components:

$$U^1 = dx/cdt' = (\gamma dx' + \boldsymbol{u_x}\gamma dt')/dt'$$
$$U^1 = \gamma(dx'/dt') + \boldsymbol{u_x}\gamma(dt'/dt') = \boldsymbol{u_x}\gamma$$

- The other two:

$$U^2 = \gamma(dy'/dt') + \boldsymbol{u_y}\gamma(dt'/dt') = \boldsymbol{u_y}\gamma$$

$$U^3 = \gamma(dz'/dt') + \boldsymbol{u_z}\gamma(dt'/dt') = \boldsymbol{u_z}\gamma$$

- The whole 4-velocity vector is then:

$$U = \{\gamma, \gamma\boldsymbol{u_x}, \gamma\boldsymbol{u_y}, \gamma\boldsymbol{u_z}\}$$

Components of energy-momentum revealed

• Components of energy-momentum of a particle are now easily calculated:

$$\mathbf{P^0} = \text{energy} = \quad m\,(dt/d\tau) = \gamma\, m$$

$$\mathbf{P^1} = \text{x-momentum} = \quad m\,(dx/d\tau) = \gamma\, mu_x$$

$$\mathbf{P^2} = \text{y-momentum} = \quad m\,(dy/d\tau) = \gamma\, mu_y$$

$$\mathbf{P^3} = \text{z-momentum} = \quad m\,(dz/d\tau) = \gamma\, mu_z$$

• Here, τ is the proper time – the interval measured in the particle's rest frame, and m is the *rest mass* of the body.

• We can write energy-momentum as $\mathbf{P} = \{\gamma m, \gamma m\boldsymbol{u_x}, \gamma m\boldsymbol{u_y}, \gamma m\boldsymbol{u_z}\}$.

• Notice that I've conveniently omitted speed of light – c – in all these equations. It's there, but it's equal to 1, and is unitless.

Energy-momentum magnitude

• Recall: the claim was that the 4-velocity absolute value is invariant, just like the interval is. What is this value for energy-momentum?

$$
\begin{aligned}
|\mathbf{P}| &= ((E)^2 - [(p_{\boldsymbol{x}})^2 + (p_{\boldsymbol{x}})^2 + (p_{\boldsymbol{x}})^2])^{1/2} \\
&= ((\gamma m)^2 - [(\gamma m \boldsymbol{u_x})^2 + (\gamma m \boldsymbol{u_x})^2 + (\gamma m \boldsymbol{u_x})^2])^{1/2} \\
&= \gamma m\,(1 - \boldsymbol{u}^2\,)^{1/2} \\
&= \gamma m(1/\gamma) \\
|\mathbf{P}| &= m
\end{aligned}
$$

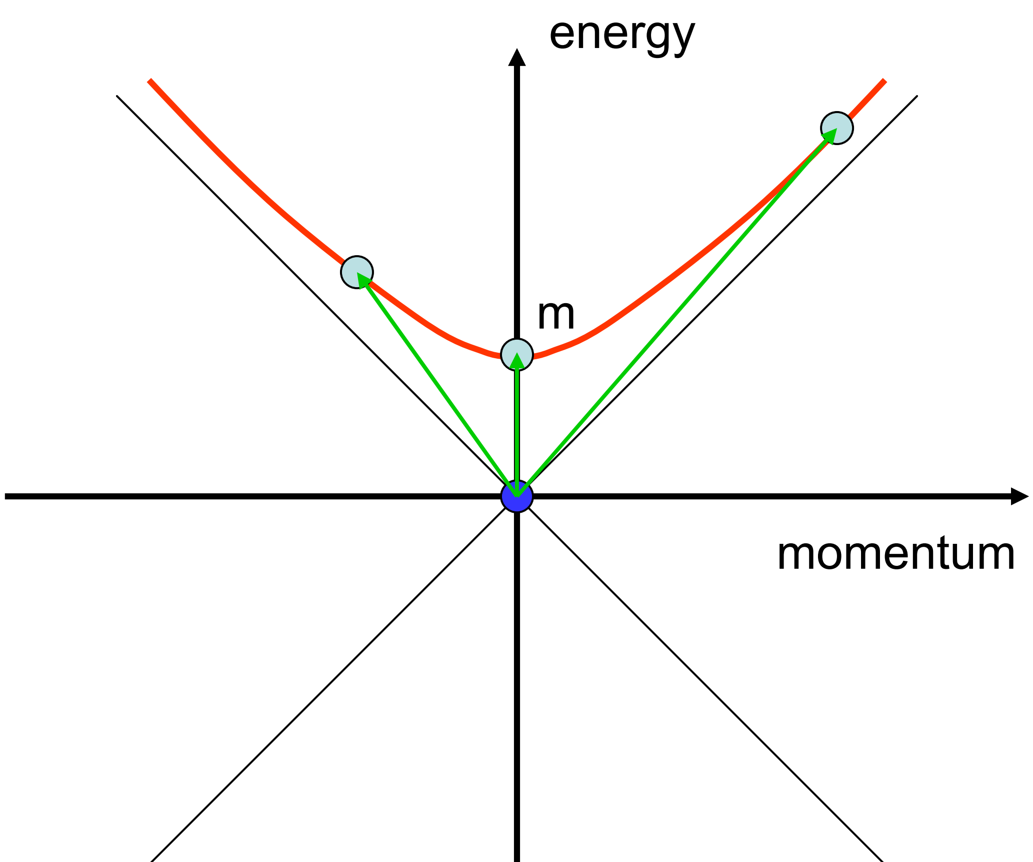
• We can write this as:

$$\boxed{m^2 = E^2 - \mathbf{p}^2}$$

• This is one of the most important equations in physics. It connects energy, momentum and mass.

Energy-momentum plot

• Energy-momentum 4-vector can be plotted on a graph similar to the spacetime map. **P** will appear as a vector whose end traces a familiar-looking hyperbola as we hop from one frame to another.



Momentum: the "space part"

• In Newtonian mechanics, momentum was defined as

$$\mathbf{p} = m\,(d\mathbf{r}/dt)$$

• What is the difference between this and the relativistic expression for the "space part" of energy-momentum? The definition of time!

• Back in the days of Newton time was universal and the same for every observer. The sundial gave no hint that time may flow at different rates for different objects...

• It is not so in relativity! That's why when defining the energy-momentum we use the only "universal time" of the particle – its proper time, the invariant interval. We write:

$$\mathbf{P} = m\,(d\mathbf{r}/d\tau) = \gamma m\,(d\mathbf{r}/dt)$$

• The relativistic momentum is thus γ times larger than in Newtonian mechanics.

Low speed vs. high speed

• At low speed $\gamma = (1 - v^2)^{-1/2}$ is very close to unity, so difference between relativistic and Newtonian momentum is negligible.

• At high speed... Well, *how high* speed? Newtonian mechanics doesn't have a speed limit. The speed can be infinite, thus the momentum can be infinite, too. At a finite speed, say, the speed of light, Newtonian momentum is finite.

• In Relativity, no particle can move faster than light. Yet, particle's momentum can be arbitrarily large! How?

• As $v \to c = 1$, $\boxed{\gamma = (1 - v^2)^{-1/2} \to (1 - 1^2)^{-1/2} = \infty}$

• You can always *increase particle's momentum* but when you approach the speed of light, this increase of momentum comes from the stretch factor γ rather than increasing velocity.

• In high energy accelerators particles' momentum can be increased 100-fold without noticeably changing particles' speed.

Energy: the "time part"

• In Newtonian mechanics, particle's kinetic energy was:

$$K = \frac{1}{2} mv^2$$

• This value goes to zero when the particle is not moving. It was quite natural to say that the object at rest has no energy.

• Relativistic energy is

$$E = m\,(dt/d\tau) = \gamma m$$

• At $v = 0$, $\gamma = 1$ and $E = E_{rest} = m$. Object at rest has energy! Can this be reconciled with Newtonian mechanics?

• The answer is that relativistic energy is the *total energy* of the particle, which includes the *rest energy* E_{rest} and the *kinetic energy*:

$$E = m + K_{rel}$$

• DO NOT CONFUSE Newtonian K and relativistic K_{rel}! You will not get relativistic total energy by simply adding the rest mass of an object to its Newtonian kinetic energy!

Kinetic energy: Newtonian vs. relativistic

- Relativistic kinetic energy:

$$K_{rel} = E - m = \gamma m - m = m(\gamma - 1)$$

- Again, at low speed we have:

$$\begin{aligned} K_{rel} = m(\gamma - 1) \quad &= m((1 - v^2)^{-1/2} - 1) \\ &\approx m((1 - (-1/2)v^2) - 1) \quad \text{(true for } v << 1\text{)} \\ &= m(1 + 1/2\, v^2 - 1) \\ &= \boxed{1/2mv^2 = K_{Newtonian}} \end{aligned}$$

- At high speed (close to speed of light), $\gamma >> 1$, so $K_{rel} \approx \gamma m >> K_{Newtonian}$. Just like with the relativistic momentum, the relativistic kinetic energy has no limit, even though the speed is limited by *c*.

The celebrated $E = mc^2$

• In all our derivations so far we've been omitting speed of light c – a unitless unity. That's why we had expressions like "$E = m$" and so on.

• Einstein's famous formula for objects rest energy is, of course, **$E = mc^2$**, but for us $c^2 = 1$.

• But has this formula always been so beautiful and clear? It's a curious fact that in Einstein's original paper there was no "$E = mc^2$". What he had was a clumsy-looking

$$K_0 - K_1 = (L/V^2)(v^2/2)$$

• What is this thing, anyway?!

• Einstein looked at kinetic energies, K_0 and K_1 of two light sources , as seen by a stationary and a moving observer – in Newtonian limit! Each light source was emitting a plane wave of energy L/2. If the kinetic energies of the sources were different in the moving frame, this could only mean that energy radiated away by the source reduces the mass of the object! If you now recall $K = m(v^2/2)$, you'll see that L/V^2 is "mass"...

Back to collisions

• Relativistic description of energy and momentum, especially the famous $E = mc^2$, allows us to see how new, and heavy, particles can be born in high-energy collisions.

• Relativity demands conservation of energy-momentum, but does not set any limits as to where the energy or momentum come from.

• This means that an enormous kinetic energy of incoming high-energy particles can be converted into an (equally enormous) rest mass of a new particle, of be split between a bunch of smaller particles.

• Conservation of energy-momentum also leads to understanding of nuclear processes – fusion and fission.

• In an even more dramatic manifestation of energy-momentum conservation, a photon, massless particle of light, given enough energy can create matter. These are the topics of our following lectures.