Physics 311
Special Relativity

Lecture 5:

Invariance of the interval.
Lorentz transformations.

OUTLINE

- Invariance of the interval – a proof
- Derivation of Lorentz transformations
- Inverse Lorentz transformations

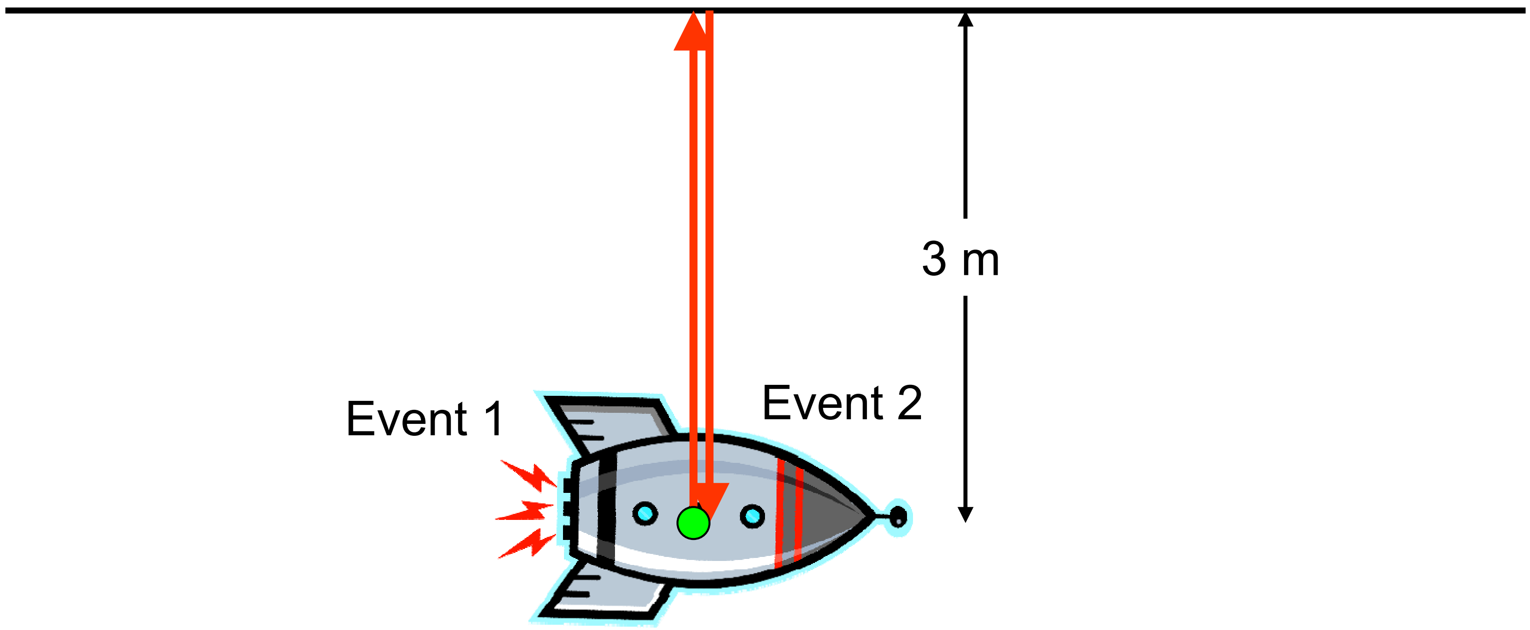
Space and time separation in moving frames

- Two events are recorded in two frames: the Lab frame and the Rocket frame (moving at v = 4/5c). The events are:
- Light flash is emitted from the Rocket normal to the direction of relative motion (Event 1)
- Light is reflected back and later detected at the Rocket (Event 2)
- The two events have zero space separation in the Rocket frame: the light was reflected right back to the rocket
- Distance to the reflecting mirror is 3 m, perpendicular to the direction of the Rocket motion. (Remember: the transverse dimension is the same in all frames, follows from the isotropy of space)
- Enough words, let's look at the events in the two frames!

1. The Rocket frame

- The events are separated by 0 m of space and 6 m of time

  $s^2 = 6^2 - 0^2 = 36$, thus $s = 6$

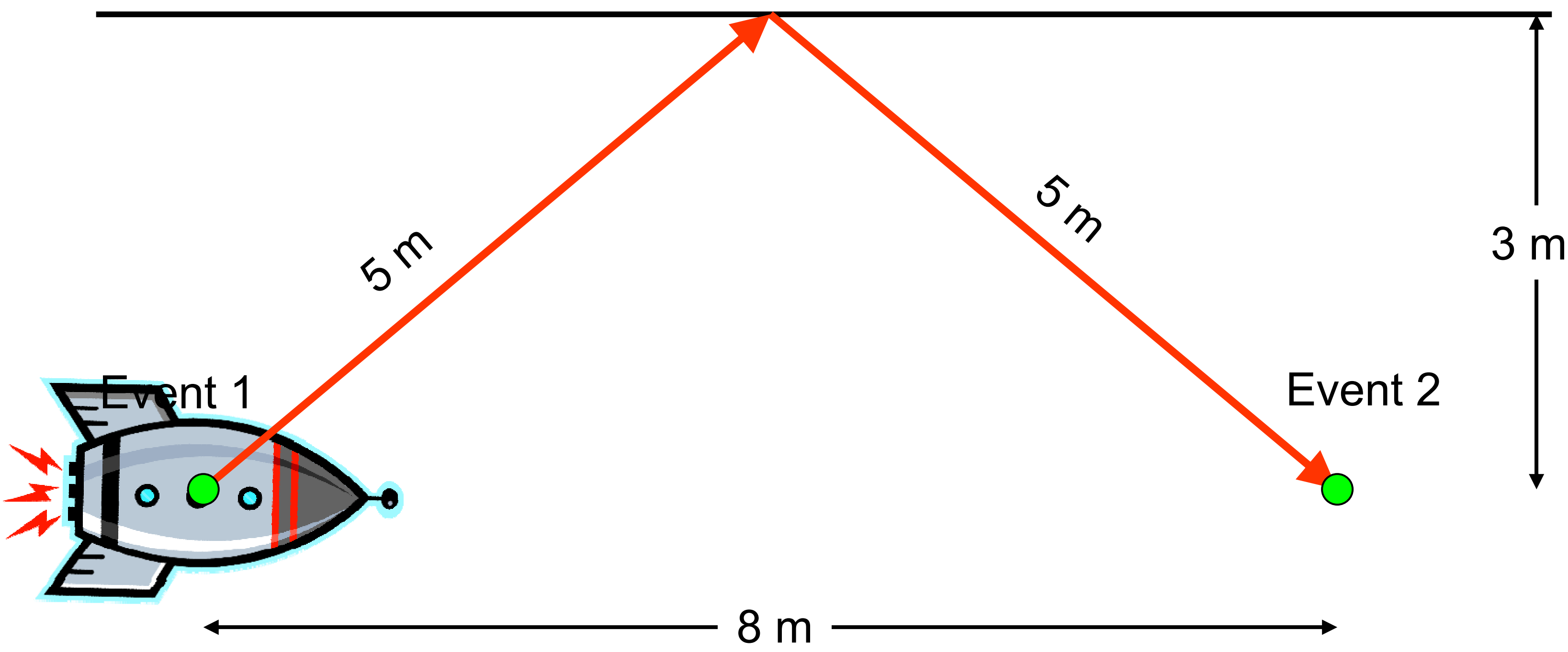


2. The Lab frame

• The events are separated by 8 m of space and 10 m of time (we've secretly used Pythagorean theorem to calculate that)

$s^2 = 10^2 – 8^2 = 36$, thus $s = 6$

(Notice how light travels in a different direction in the Lab frame)



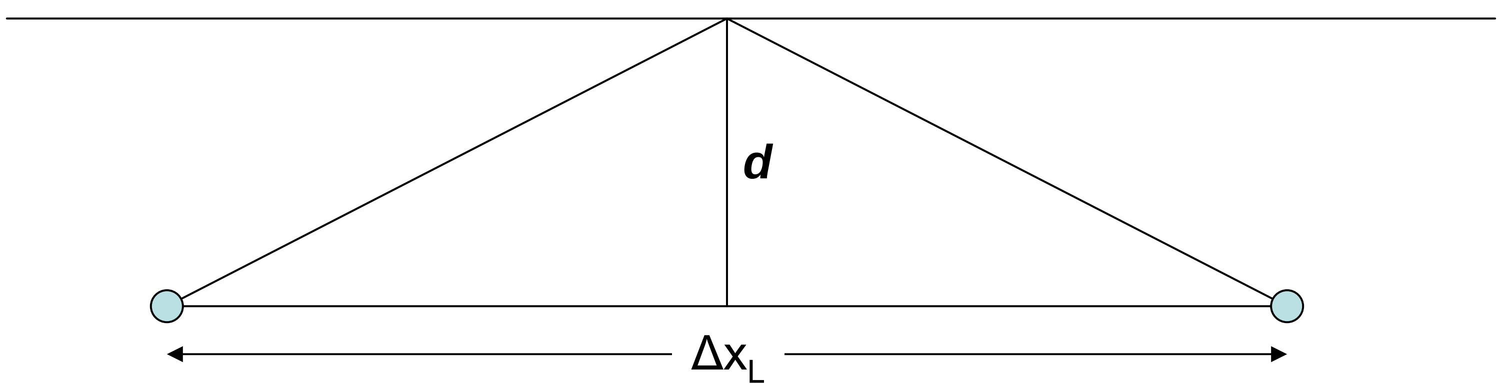
We’ve just proved a very important thing.....

6 = 6

General case

- Light flash is emitted (Event 1)
- Light is reflected back from a mirror ***d*** m away and detected (Event 2)
- The events are recorded in two frames: the Lab frame and the Rocket frame flying at velocity ***v***
- Rocket frame: $\Delta x_R = 0$, $\Delta t_R = 2\boldsymbol{d}$, $s = 2\boldsymbol{d}$
- Lab frame: $\Delta x_L = \boldsymbol{v}\Delta t_L$, $\Delta t_L = 2((\Delta x_L/2)^2 + (\Delta t_R/2)^2)^{1/2}$ (Pythagorean theorem)

$$s_L^2 = \Delta t_L^2 - \Delta x_L^2 = \Delta x_L^2 + \Delta t_R^2 - \Delta x_L^2 = \Delta t_R = 2\boldsymbol{d} \;\therefore$$



Well, it's a mildly general case, you say. But any two events can be represented this way!

Lorentz transformations

• Lorentz transformations give us means to calculate relationships between events in spacetime, including velocities measured in moving frames, electric and magnetic fields, time dilation and length contraction.

• More powerful than the intervals alone, Lorentz transformations are a ***useful mathematical tool***. Yet, they are ***not fundamental***, but simply follow from the relativity principle and the invariance of the speed of light.

• As any transformation of coordinates for inertial frames, Lorentz transformations must be linear:

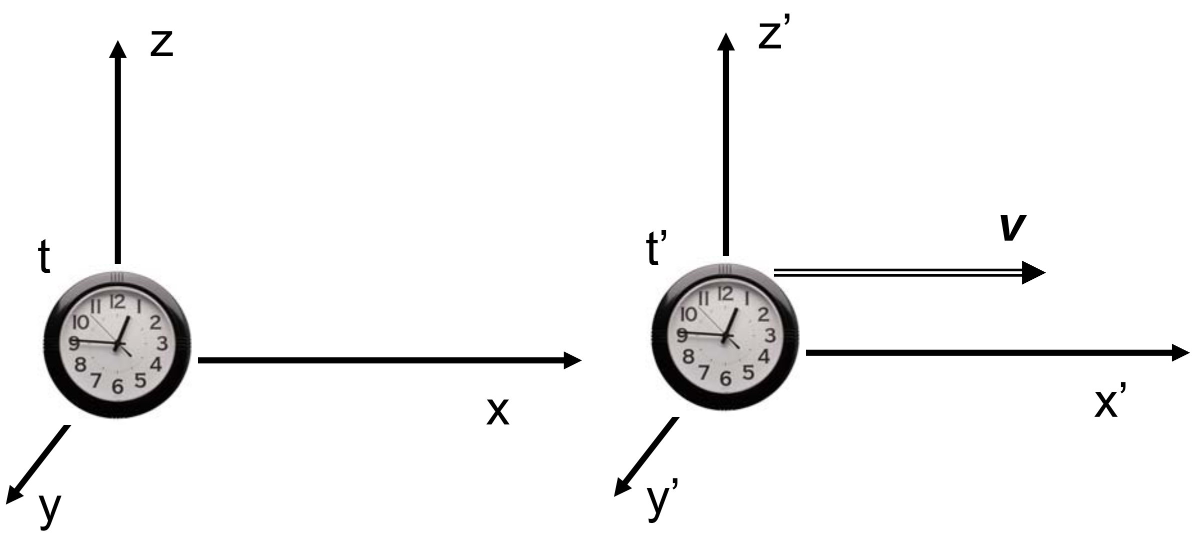
$$x = Ax' + Bt'$$
$$t = Cx' + Dt'$$

• As ***relativistic*** transformations, they must ***conserve the interval***.

Derivation – the setup

- The setup: Lab frame at rest, Rocket frame is moving at speed ***v*** along the x-axis.
- The Lab coordinates are x, y, z and t;

  the Rocket coordinates are x', y', z' and t'
- Initial conditions: t = t' = 0; the origins coincide



Derivation – the plan

• Use the invariance of the interval to derive coordinate and time transformations for events in the origin of the Rocket frame (x' = 0)

• Realize that any point in the Rocket frame can be the origin (i.e. the transformations must be linear, space is uniform...)

• Derive the transformations for arbitrary x' and t'

• Derive the inverse Lorentz transformation – from (x, t) to (x',t')

Step 1: event at x' = 0

- Light flash is emitted at x' = 0 at time t'
- Note: the orthogonal coordinates y and z do not change:

$$y = y'$$
$$z = z'$$

- Spark location in the laboratory frame is (the origins coincided at t = 0):

$$x = \boldsymbol{v}\, t$$

- Now use the invariance of the interval:

$$(t')^2 - (x')^2 = (t')^2 - 0 = t^2 - x^2 = t^2 - (\boldsymbol{v}\, t)^2 = t^2(1 - \boldsymbol{v}^2)$$

- (Remember: ***v*** is unitless, it is the ratio of v/c)

Time and distance transformations for x' = 0

- Now we can write down the transformation for time:

$$t' = t(1 - \boldsymbol{v}^2)^{1/2} \text{ , or}$$
$$t = t'/[(1 - \boldsymbol{v}^2)^{1/2}]$$

- Common notation: $1/[(1 - \boldsymbol{v}^2)^{1/2}] \equiv \gamma$ (a.k.a. *time stretch factor* or simply *Lorentz* γ*-factor*)

- Then:

$$t = \gamma t'$$

- Substituting this into $x = \boldsymbol{v}\, t$, we get transformation for x:

$$x = \boldsymbol{v}\, \gamma t'$$

- Is that all we need? Not really, there's a bit more work to do.

Any point is a reference point

- Recall: the transformations must be linear in x (and x'), which follows from the fact that any point in space in any frame cam be a reference point. (Which, in turn, follows form the fact that space is uniform.)
- The transformations must be linear in t (and t'), because any point in *time* can be chosen as the origin. (This follows from the fact that time is uniform.)
- So, we seek the following form for the transformations:

$$x = Ax' + Bt'$$
$$t = Cx' + Dt'$$

Lorentz transformations for arbitrary x and t

- We already know the coefficients B and D from the previous derivation:

$$B = \boldsymbol{v}\gamma \text{ and } D = \gamma$$

- To determine A and C let's consider an event at some (arbitrary) x' and t'. The interval in the two frames is:

$$s^2 = t^2 - x^2 = t'^2 - x'^2$$

- Substitute the expressions for x and t from the full Lorentz transformations:

$$(Cx' + \gamma t')^2 - (Ax' + \boldsymbol{v}\gamma t')^2 = t'^2 - x'^2$$

$$C^2x'^2 + \gamma^2t'^2 + 2C\gamma x't' - A^2x'^2 + \boldsymbol{v}^2\gamma^2t'^2 + 2A\boldsymbol{v}\gamma x't' = t'^2 - x'^2$$

... a mess! (But that's just a normal situation half-way through any derivation)

Lorentz transformations for arbitrary x and t

- Group together the coefficients of t'^2, x'^2 and $x't'$:

$$\gamma^2(1 - \boldsymbol{v}^2)t'^2 - (A^2 - C^2)x'^2 + 2\gamma(C - A\boldsymbol{v})x't' = t'^2 - x'^2$$

- Let's study these coefficients. The equality must be satisfied for each and every value of x' and t'. That means the coefficients on the right-hand side must be equal to the matching coefficients on the left-hand side:

$\gamma^2(1 - \boldsymbol{v}^2) = \gamma^2 \times 1/\gamma^2 = 1$ – very good!

$(A^2 - C^2)$ must equal 1, and

$2\gamma(C - A\boldsymbol{v})$ must equal 0.

- All we have to do is to solve a system of two equations with two unknowns.

Lorentz transformations for arbitrary x and t

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$$\gamma^2(1 - \boldsymbol{v}^2)t'^2 - (A^2 - C^2)x'^2 + 2\gamma(C - A\boldsymbol{v})x't' = t'^2 - x'^2$$

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$$\gamma^2(1 - \boldsymbol{v}^2) = \gamma^2 \times 1/\gamma^2 = 1 \text{ – very good!}$$

$$\begin{cases} (A^2 - C^2) = 1 \\ 2\gamma(C - A\boldsymbol{v}) = 0 \end{cases}$$

- All we have to do is to solve a system of two equations with two unknowns.

Lorentz transformations for arbitrary x and t

- The solution is simple:

$$A = \gamma$$
$$C = \boldsymbol{v}\gamma$$

- The complete Lorentz transformations are:

$$t = \boldsymbol{v}\gamma x' + \gamma t'$$
$$x = \gamma x' + \boldsymbol{v}\gamma t'$$
$$y = y'$$
$$z = z'$$

Inverse Lorentz transformations

- To find the inverse Lorentz transformations (i.e. from the Lab frame to the Rocket frame), need to solve the first two equations for x' and t':

$$\begin{cases} t = \boldsymbol{v}\gamma x' + \gamma t' \\ x = \gamma x' + \boldsymbol{v}\gamma t' \end{cases}$$

- Do this as an exercise! The solution is:

$$t' = -\boldsymbol{v}\gamma x + \gamma t \qquad t = \boldsymbol{v}\gamma x' + \gamma t'$$

$$x' = \gamma x - \boldsymbol{v}\gamma t \qquad x = \gamma x' + \boldsymbol{v}\gamma t'$$

$$y' = y \qquad y = y'$$

$$z' = z \qquad z = z'$$