Homework #2

3-7 (10 points)

3-15 (20 points)

L-4 (10 points)

L-5 (30 points)

Physics 311
Special Relativity

Lecture 6:
Addition of velocities. 4-velocity.

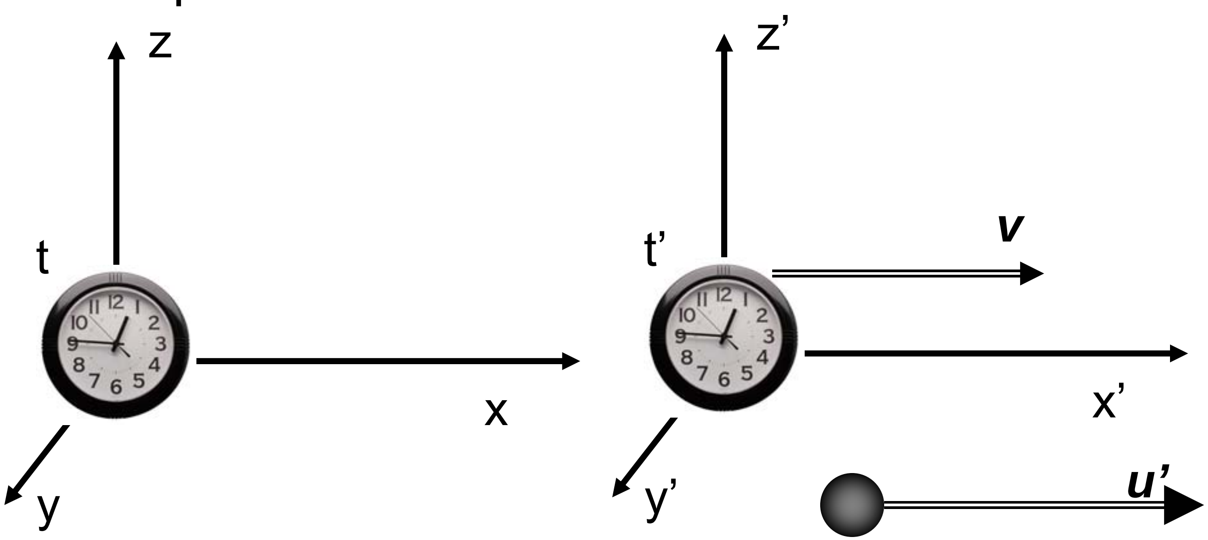
OUTLINE

- Addition of velocities – general case
- The extremes: speed of light and v<<c
- Transverse velocity
- 4-velocity – space and time, unite!
- 4-vectors are not scary, they are nice.

Addition of velocities

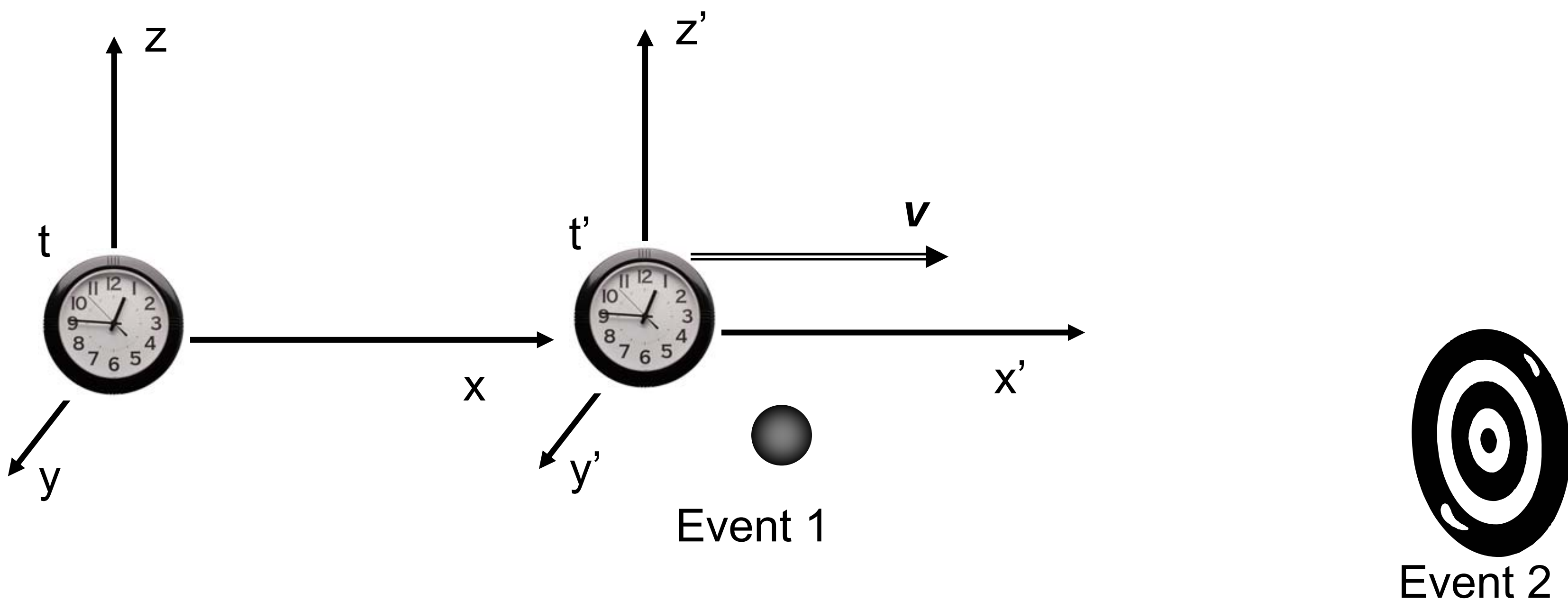
- Lab frame at rest, Rocket frame moving at speed ***v*** along the x-axis.
- The Lab coordinates are x, y, z and t;

  the Rocket coordinates are x', y', z' and t'
- Initial conditions: t = t' = 0; the origins coincide
- A bullet is fired at speed ***u'*** along the x axis in the Rocket frame.
- What is the speed of the bullet in the Lab frame?



Define and analyze events

- Event 1: the bullet is fired; coordinates: (x_1, t_1) and (x_1', t_1') (y,z are not important)
- Event 2: the bullet its the target; coordinates: (x_2, t_2) and (x_2', t_2')
- Bullet speed in Rocket frame: $(x_2' - x_1')/(t_2' - t_1') \equiv \Delta x'/\Delta t' = \boldsymbol{u'}$
- Bullet speed in Lab frame: $(x_2 - x_1)/(t_2 - t_1) \equiv \Delta x/\Delta t =$ ***?***



Transform time and distance, then divide

- $$t_1 = \boldsymbol{v}\gamma x_1' + \gamma t_1'$$
  $$t_2 = \boldsymbol{v}\gamma x_2' + \gamma t_2'$$
- $$x_1 = \gamma x_1' + \boldsymbol{v}\gamma t_1'$$
  $$x_2 = \gamma x_2' + \boldsymbol{v}\gamma t_2'$$
- Then: $$\Delta x = (x_2 - x_1) = \gamma(x_2' - x_1') + \boldsymbol{v}\gamma(t_2' - t_1') = \gamma\Delta x' + \boldsymbol{v}\gamma\Delta t'$$
  $$\Delta t = (t_2 - t_1) = \boldsymbol{v}\gamma(x_2' - x_1') + \gamma(t_2' - t_1') = \boldsymbol{v}\gamma\Delta x' + \gamma\Delta t'$$
- Bullet velocity in the Lab frame:

$$\boldsymbol{u} = \Delta x/\Delta t = (\cancel{\gamma}\Delta x' + \boldsymbol{v}\cancel{\gamma}\Delta t')/(\boldsymbol{v}\cancel{\gamma}\Delta x' + \cancel{\gamma}\Delta t')$$

$$= (\Delta x' + \boldsymbol{v}\Delta t')/(\boldsymbol{v}\Delta x' + \Delta t')$$ (time-stretch cancels!)

$$= (\Delta x'/\Delta t' + \boldsymbol{v})/(\boldsymbol{v}\Delta x'/\Delta t' + 1)$$ (divide by $\Delta t'$)

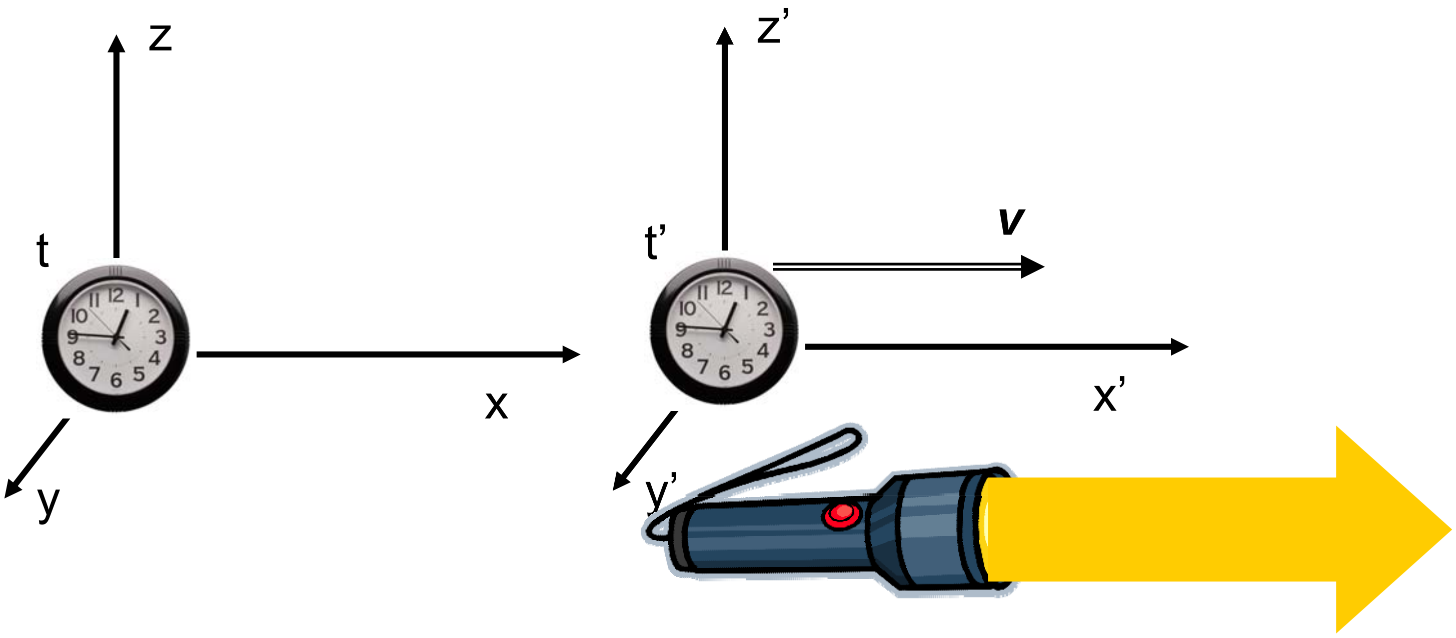
$$\boldsymbol{u} = (\boldsymbol{u'} + \boldsymbol{v})/(1 + \boldsymbol{u'v})$$

Speed of Light

- Special case: $\boldsymbol{u'} = c \equiv 1$. Then:

$$\boldsymbol{u} = (c + \boldsymbol{v})/(1 + c\boldsymbol{v}) = (1 + \boldsymbol{v})/(1 + \boldsymbol{v}) = 1 \equiv c$$

Speed of light is the same in all inertial frames!



Another extreme: $\boldsymbol{v} \ll 1$

- Then:

$\boldsymbol{u} = (\boldsymbol{u'} + \boldsymbol{v})/(1 + \boldsymbol{u'v}) \approx (\boldsymbol{u'} + \boldsymbol{v})/1 = \boldsymbol{u'} + \boldsymbol{v}$ (since $\boldsymbol{u'v} \ll 1$ even for $\boldsymbol{u'} = c = 1$)

The Galilean velocity addition!

- This is good news – a new theory should agree with the old theory where the old theory works, or where the effects of the new theory are not noticeable.

- What about the transformations for time and distance? (Notice: $\boldsymbol{v} \ll 1$ means that $\gamma \approx 1$)

$$t = \boldsymbol{v}\gamma x' + \gamma t'$$
$$x = \gamma x' + \boldsymbol{v}\gamma t'$$

Lorentz transformations

$$t = \boldsymbol{v}x' + t$$
$$x = x' + \boldsymbol{v}t'$$

"classical" transformations

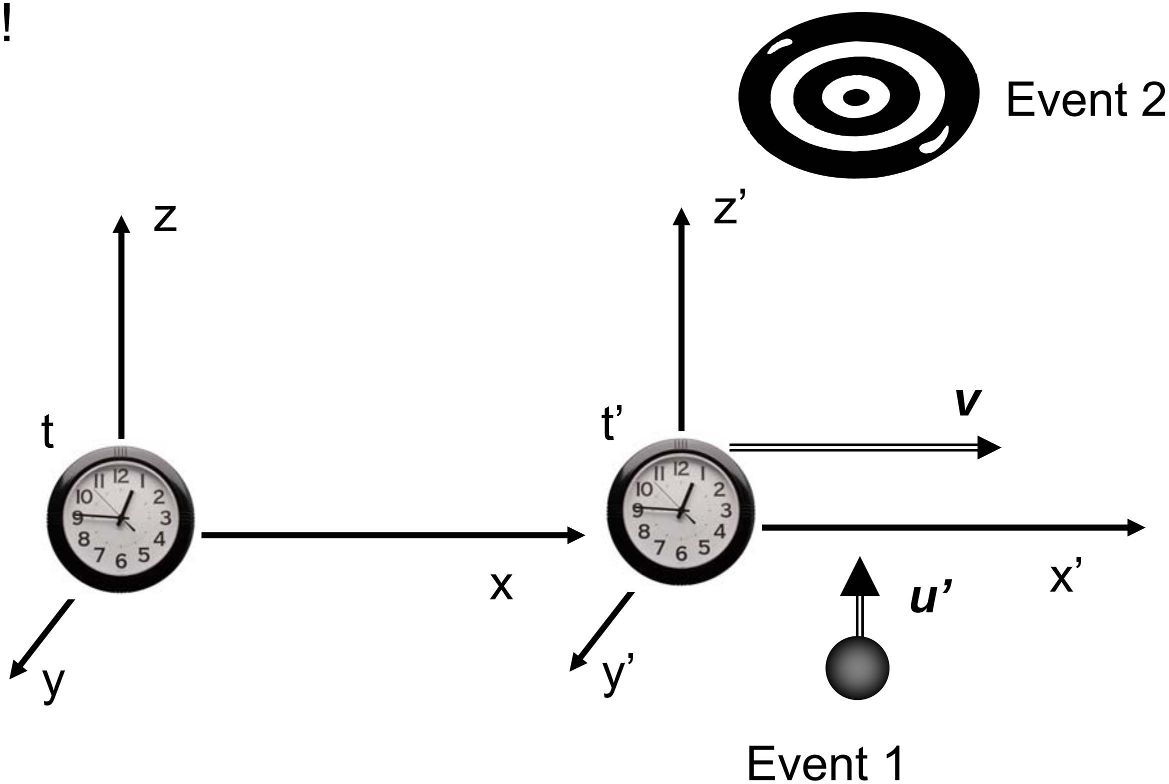
Not quite Galilean! Galileo assumed $c = \infty$, the term $\boldsymbol{v}x'$ is due to different synchronization of clocks

What about the orthogonal velocity?

• We have seen that the displacements orthogonal to the direction relative motion of reference frames do not change: y = y' and z = z'.

• Does this imply that the orthogonal velocity does not change either?

• NO! And why is that? Because time changes when we go between frames!

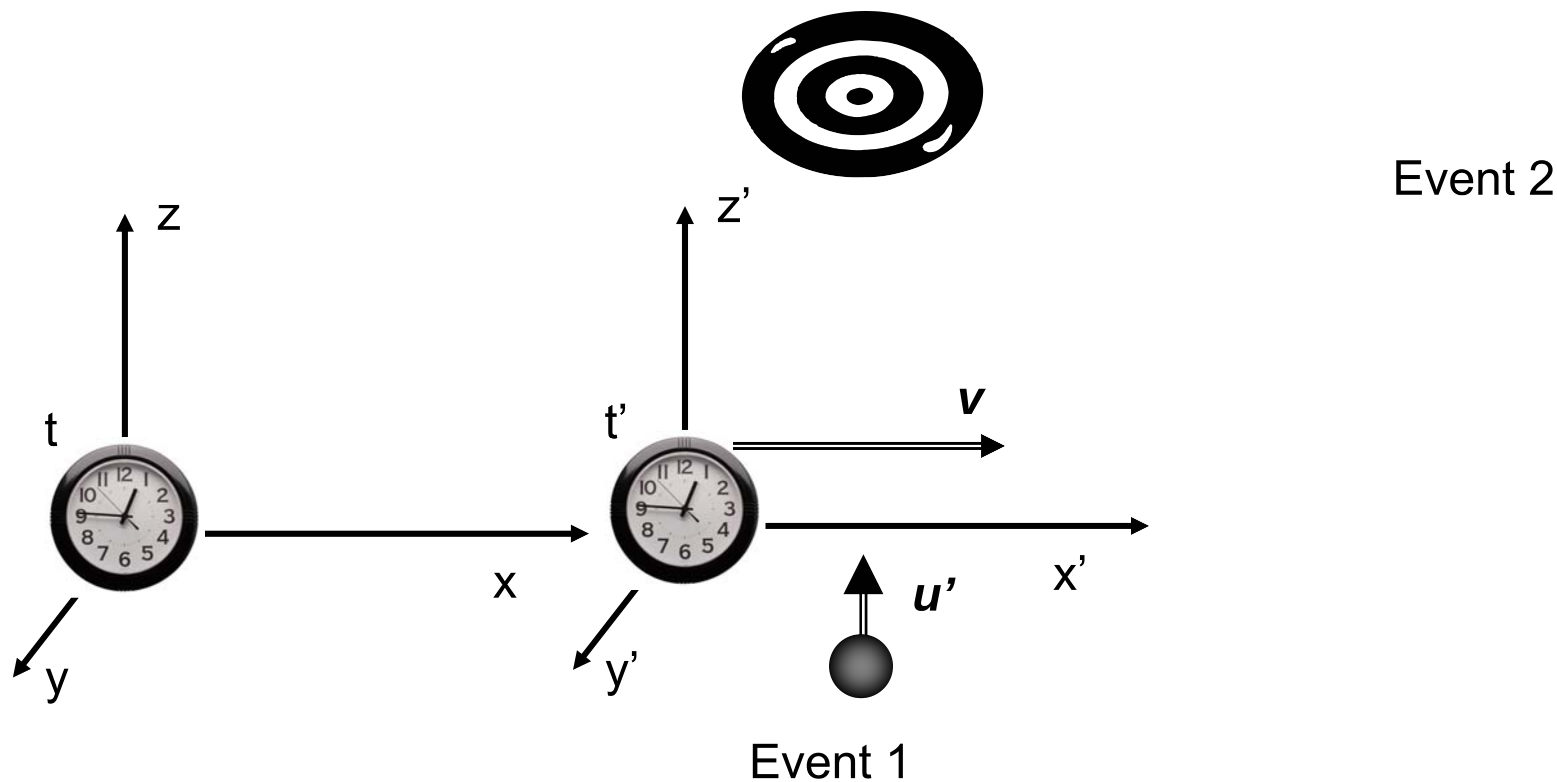


In the Lab frame...

• Velocity acquires a component along x according to the velocity addition formula: $\boldsymbol{u_x} = (0 + \boldsymbol{v})/(1 + 0\boldsymbol{v}) = \boldsymbol{v}$

• Assume that both ***v*** and ***u'*** are very close to speed of light. If the orthogonal velocity remained the same, then we would have:

$$\boldsymbol{u} = (\boldsymbol{v}^2 + \boldsymbol{u}'^2)^{1/2} > c\ !!!$$



So, orthogonal velocity must change - let's derive it!

- $t_1 = \boldsymbol{v}\gamma x_1' + \gamma t_1'$
  $t_2 = \boldsymbol{v}\gamma x_1' + \gamma t_2'$
- $z_1 = z_1'$
  $z_2 = z_2'$
- Then: $\Delta z = (z_2 - z_1) = (z_2' - z_1') = \Delta z'$
  $\Delta t = (t_2 - t_1) = \boldsymbol{v}\gamma(x_1' - x_1') + \gamma(t_2' - t_1') = \gamma\Delta t'$
- Orthogonal velocity in the Lab frame:

$$\boldsymbol{u} = \Delta z/\Delta t = \Delta z'/\gamma\Delta t' = \boldsymbol{u'}/\gamma = \boldsymbol{u'}(1 - \boldsymbol{v}^2)^{1/2}$$

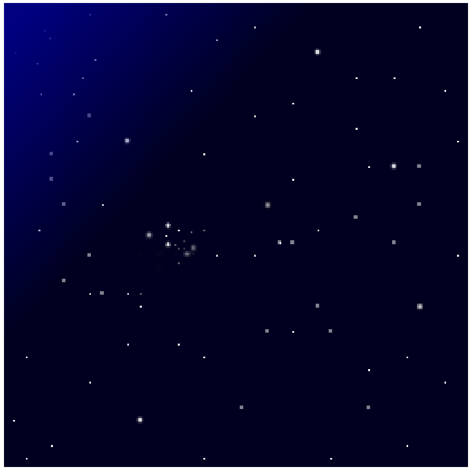
$$\boldsymbol{u} = \boldsymbol{u'}(1 - \boldsymbol{v}^2)^{1/2}$$

Lorentz transformations *separate* time and space!

• There are transformations for space, and then there are transformations for time:

$$t = \boldsymbol{v}\gamma x' + \gamma t'$$
$$x = \gamma x' + \boldsymbol{v}\gamma t'$$

• Space and time transform differently! What can we do to reunite them?



The march of 4-vectors

- In the spirit of treating space and time as one entity – the **spacetime** – we will introduce the "4-vectors".
- 4-vectors are 4-dimensional vectors whose three coordinates correspond to space, and the fourth (or first, as is usually the case) is related to time.
- You have already met one 4-vector: the displacement X = {ct,x,y,z}
- Other examples are: 4-velocity, energy-momentum, force-power.
- 4-vectors have special metric – for example, the length of the displacement 4-vector is $|X| = s = ((ct)^2 - (x^2 + y^2 + z^2))^{1/2}$ – the minus sign!
- 4-vectors transform between inertial frame as the interval – i.e. their absolute value is invariant.

4-velocity

- 4-velocity of a moving particle in an inertial frame is the first derivative of the displacement 4-vector measured in that frame with respect to particle's ***proper time*** τ:

$$U = ds/d\tau$$

- Let's assume that the particle is moving at velocity ***u*** with respect to the Lab frame. Displacement and time in the particle frame are x', t'; displacement and time in the Lab frame are x, t.

- The displacement 4-vector in the Lab frame: $s = \{ct, x, y, z\}$;
the proper time is the interval in the particle's frame: $\tau = s' = \{ct', 0, 0, 0\}$.

- Infinitesimals: $ds = \{cdt, dx, dy, dz\}$
$d\tau = \{cdt', 0, 0, 0\}$

4-velocity components

• "Time"-component:

$$U^0 = cdt/cdt' = (\boldsymbol{u}\gamma d\boldsymbol{r'} + \gamma dt')/dt'$$

We are considering a general case of particle moving along an arbitrary direction, so all velocity components are in general non-zero. Lorentz-transformation for time then depends on total velocity ***u*** and radius-vector ***r'***.

$$U^0 = \boldsymbol{u}\gamma(d\boldsymbol{r'}/dt') + \gamma(dt'/dt') = \gamma$$

(remember: $d\boldsymbol{r'}/dt' = 0$ is particle's velocity in its ***rest*** frame)

• The "time"-component is thus simply time-stretch factor γ. "How could it be?" - you ask, - " shouldn't it have dimensionality of velocity?"

• It should and it does! Our strange units simply hide it. Remember: our velocity is *unitless*. In fact, we can, and we should, write the "time"-component of 4-velocity as **$c\gamma$** (remembering that c = 1).

4-velocity components

- "Space"-components:

$$U^1 = dx/cdt' = (\gamma dx' + \boldsymbol{u}_x\gamma dt')/dt'$$

Here, $\boldsymbol{u}_x$ is the x-component of particle's velocity in the Lab frame. Lorentz transformations for x, y and z will depend on $\boldsymbol{u}_x$, $\boldsymbol{u}_y$ and $\boldsymbol{u}_z$, respectively.

$$U^1 = \gamma(dx'/dt') + \boldsymbol{u}_x\gamma(dt'/dt') = \boldsymbol{u}_x\gamma$$

- The other two:

$$U^2 = \gamma(dy'/dt') + \boldsymbol{u}_y\gamma(dt'/dt') = \boldsymbol{u}_y\gamma$$

$$U^3 = \gamma(dz'/dt') + \boldsymbol{u}_z\gamma(dt'/dt') = \boldsymbol{u}_z\gamma$$

- The whole 4-velocity vector is then:

$$U = \{c\gamma, \gamma\boldsymbol{u}_x, \gamma\boldsymbol{u}_y, \gamma\boldsymbol{u}_z\}$$

4-velocity magnitude

- Recall: the claim was that the 4-velocity absolute value is invariant, just like the interval is. What is this value?

- The 4-velocity vector: $U = \{c\gamma, \gamma \boldsymbol{u}_x, \gamma \boldsymbol{u}_y, \gamma \boldsymbol{u}_z\}$. It's absolute value:

$$|U| = ((c\gamma)^2 - [(\gamma \boldsymbol{u}_x)^2 + (\gamma \boldsymbol{u}_y)^2 + (\gamma \boldsymbol{u}_z)^2])^{1/2}$$
$$= \gamma\, (c^2 - \boldsymbol{u}^2)^{1/2}$$
$$= \gamma c\, (1 - (\boldsymbol{u}/c)^2)^{1/2} \quad = 1/\gamma!!!$$

$$|U| = c$$

- Well, true indeed, the speed of light is the same in all inertial frames, what can be better?!

- But what is the meaning of this? Sure, this seems strange – whatever the particle's 3-velocity $\boldsymbol{u}$ might be, the 4-velocity magnitude is *always* the speed of light!

4-velocity magnitude: the meaning

• Let's go into the particle rest frame. There, particle's 3-velocity components are all zero, the time-stretch factor γ is 1 (no time stretching in the rest frame!), and the 4-velocity there is:

$$U_{rest} = \{c, 0, 0, 0\}$$

• As you can see, the "time"-component of the 4-velocity is exactly the speed of light. Even though the particle is at rest, it is still traveling along the 4th dimension – time! That travel happens at the speed of light, so to speak.

• The 4-velocity has a nice way of reminding us that everything around us happens in spacetime, and even an object at rest in space is moving through time.

• If we accept that time-travel "velocity" is c, then the time-stretch factor γ has a very nice meaning: the time-travel "velocity" is γ times faster in moving frames ($U = \{c\gamma, \gamma \boldsymbol{u_x}, \gamma \boldsymbol{u_y}, \gamma \boldsymbol{u_z}\}$). The time is stretched, and we need to go faster to keep up!

4-vectors in general

• 4-vectors defined as any set of 4 quantities which transform under Lorentz transformations as does the interval. Such transformation is usually defined in the form of a matrix:

$$M = \begin{pmatrix} \gamma & -\gamma \boldsymbol{v} & 0 & 0 \\ -\gamma \boldsymbol{v} & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

• The transformation for the 4-velocity is then simply:

$U = MU'$, or for its components:

$$U^0 = \gamma(U^0)' - \gamma \boldsymbol{v}(U^1)'$$
$$U^1 = -\gamma \boldsymbol{v}(U^0)' + \gamma(U^1)'$$
$$U^2 = (U^2)'$$
$$U^3 = (U^3)'$$

• Notice that the "orthogonal" components of the 4-velocity ***do not change***!

4-vectors are useful!

- 4-vectors are very useful. Do not be intimidated by their apparent complexity. We’ll be seeing a lot more of them when we study the relativistic dynamics – force, momentum and energy.