## 13 Summary

**The Schwarzschild metric**

In general, the metric provides a complete description of spacetime: the curvature of spacetime and the results of measurements carried out with rods and clocks. The metric for flat spacetime is the one that dominated our study of special relativity. However, special relativity cannot describe spacetime globally in the vicinity of a massive object. General relativity can do so, earning the name Theory of Gravitation.

The Schwarzschild metric describes spacetime exterior to the surface of any nonrotating, uncharged, spherically symmetric massive object. It describes spacetime everywhere around a nonrotating, uncharged black hole.

Several conventions make the Schwarzschild metric easy to understand and use:

**1. Satellite motion in a plane.** A light flash or test particle that moves through Schwarzschild geometry stays in a single spatial plane that passes through the center of the black hole. Describing motion on this plane requires only two space dimensions plus the time.

**2. Polar coordinates.** Motion with respect to a center is simply described using polar coordinates $r$ and $\phi$. For example, the metric for flat spacetime with two spatial dimensions goes from the Cartesian form

$$d\tau^2 = dt^2 - dx^2 - dy^2 \qquad \text{[34. flat spacetime]}$$

to the polar form

$$d\tau^2 = dt^2 - dr^2 - r^2 d\phi^2 \qquad \text{[9. flat spacetime]}$$

**3. Mass in units of meters.** We measure the mass $M$ of a planet, star, or black hole in units of meters. Equation [5] makes the conversion from mass $M_{kg}$ in kilograms to mass $M$ in meters, using $G$, the gravitational constant of Newtonian mechanics and $c$, the speed of light:

$$M = \frac{G}{c^2} M_{kg} = \left(7.424 \times 10^{-28} \frac{\text{meter}}{\text{kilogram}}\right) M_{kg} \qquad [5]$$

In length units, the mass of Sun is 1.477 kilometers and the mass of Earth is 0.444 centimeters.

**4. Radius as reduced circumference.** The presence of the black hole renders impossible the direct measurement of the radial coordinate $r$ of an object or satellite. Instead, define the radius as $r$ = (circumference)$/2\pi$, where the circumference is measured around the great circle of a stationary spherical shell concentric to the black hole or center of attraction. As a reminder of this process, we often call $r$ the *reduced circumference*.

**5. Time $t$ measured on far-away clocks.** To avoid the effects of curvature on clocks, calculate the time, called *bookkeeper time* or *far-away time*, that would be measured on clocks located in flat spacetime far from the attracting body. Give far-away time the symbol $t$. Light flashes are used for comparison of clock rates and also for communication between a far-away clock and a clock in curved regions of spacetime.

**Predictions from the Schwarzschild metric**

With these simplifying conventions the Schwarzschild metric in its timelike form can be written

$$d\tau^2 = \left(1 - \frac{2M}{r}\right)dt^2 - \frac{dr^2}{\left(1 - \frac{2M}{r}\right)} - r^2 d\phi^2 \qquad [10]$$

This metric "measures" the separation of a pair of events that have a timelike relation and that occur near one another in spacetime. Various choices of these two events lead to predictions verified by experiment:

**Prediction 1. Gravitational red shift.** Let the two events be sequential ticks of a clock at rest on a spherical shell near a black hole. *At rest* means that the space separation between events is zero: $dr = d\phi = 0$. The proper time $d\tau$ (defined as the time between the events in a frame in which they occur at the same place) is just the time $dt_{\text{shell}}$ read on the shell clock. Then the Schwarzschild metric tells us the relation between shell-time lapse and the lapse of far-away time:

$$dt_{\text{shell}} = \left(1 - \frac{2M}{r}\right)^{1/2} dt \qquad [19]$$

Instead of describing ticks on a clock, this equation can measure the period of a steady light wave emitted outward from a spherical shell at radius $r$. The equation predicts that the period $dt$ measured by a remote observer is *greater* than the period $dt_{\text{shell}}$ measured by the observer at the emitting clock. For visible light, longer period means redder light, so the general name for this effect is the *gravitational red shift*.

**Prediction 2. Curvature of space.** Let the two events occur at the ends of a measuring rod radially oriented with ends at two concentric spherical shells. And let these two events occur at the same far-away time. To analyze these two spacelike events, use the spacelike form of the Schwarzschild metric:

$$d\sigma^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \frac{dr^2}{\left(1 - \frac{2M}{r}\right)} + r^2 d\phi^2 \qquad [11]$$

For this example, $dt = d\phi = 0$, and the proper distance $d\sigma$ between them (defined as the separation between two events in a frame in which they occur at the same time) is just the radial separation measured by a shell observer:

$$dr_{\text{shell}} = \frac{dr}{\left(1 - \frac{2M}{r}\right)^{1/2}} \qquad [12]$$

The shell observer measures the distance $dr_{\text{shell}}$ between shells to be greater than the difference $dr$ between the reduced circumferences of the two shells.

**Reference frames**

General relativity allows use of any coordinate system whatsoever. We choose three coordinate systems convenient for our purposes: local free-float frames, local frames on spherical shells, and the global frame that employs Schwarzschild coordinates $r$, $\phi$, $t$. Observers can take measurements directly in free-float frames and on spherical shells, but these measurements are local. In contrast, Schwarzschild coordinates describe events that can span all of spacetime near a massive body, but no one observer can make these measurements directly. Instead we speak of the *Schwarzschild bookkeeper* who records and analyzes events measured by others.

A shell observer and a passing free-float observer compare their local measurements using special relativity, including the Lorentz transformation. The shell observer and the Schwarzschild bookkeeper compare their measurements using equations [12] and [19]. The tangential distance $r d\phi$ is the same in both systems.

One can construct in imagination a *Schwarzschild lattice* of spherical shells, each stamped with the reduced circumference $r$, angle $\phi$, and covered with clocks reading far-away time $t$. The Schwarzschild lattice can in principle start near the horizon and extend outward indefinitely (from an isolated body). The Schwarzschild coordinates of any event outside the horizon can then be read directly using this lattice. We give the name *Schwarzschild observer* to this collection of shells and clocks.