There are 5 pages; with 6 problems. Be sure you have all the pages before you start.
Be sure your name is on the exam before you start.
This is a timed 50 minute exam.
To receive full credit on all problems you must show your work or reasoning.

Point Values will be posted on here later.
Q1) (Q12.4) a) What did Einstein postulate to explain that the kinetic energy of the emitted electrons in the photoelectric effect depends on the frequency?

Einstein used Planck’s relation $E = h\omega = h\nu$ for the light field to explain that light had particle like behavior and imparted energy according to its frequency.

b) How does Einstein’s postulate differ from the predictions of classical physics?

Classical Physics contends the energy of a light field is proportional to its amplitude squared, which is the intensity of the light, and does not depend on the frequency of the light at all.

Q2) (P12.17) a) When light is emitted from the hydrogen atom an electron moves from one state to another state. What two quantum level numbers (integers) would characterize the largest possible energy separation of atomic states.

An excited electron is in the highest (bound) energy state, $n = \infty$, and goes to the ground state $n = 1$.

b) What is the largest or highest frequency of light possible in the discrete emission spectrum of atomic hydrogen.

The energy difference then is: $\Delta E = E_\infty - E_1 = 22 \cdot 10^{-19} J$, and so the photon must have energy to match:

$E_{\text{photon}} = h\omega = h\nu = 22 \cdot 10^{-19} J$

$\nu = \frac{22 \cdot 10^{-19}}{2\pi \hbar} = \frac{11}{\pi} \cdot 10^{34-19} = 3 \cdot 10^{15} \text{ sec}^{-1}$ (UV)
Q3) (Q13.2) Given a set of functions \{\psi_n\} where \( n = 1, 2, 3 \cdots \)

a) What does it mean to say they are orthogonal?

Geometrically they point in different directions, algebraically the inner product is zero, for continuous functions of \( x \) the inner integral vanishes:
\[
\int \psi_n^* \psi_m \, dx = 0 \text{ for } m \neq n
\]

b) What does it mean that they are normalized?

Geometrically each function has unit norm as a vector pointing in the direction described by that function. For continuous functions
\[
\int \psi_n^* \psi_n \, dx = 1
\]

c) What does it mean that they are complete?

There are enough functions to describe any other function over the domain of \( x \), provided the function you want to describe is a “good” function, i.e. is single valued, continuous, does not blow up, and well behaved in that the function is square integrable, i.e. can be normalized itself.

Q4) (P13.13) a) Determine whether the function \( f(\theta) = 3 \cos^2 \theta - 1 \) is an eigenfunction of the operator \( \hat{O} = \frac{1}{\sin \theta} \frac{d}{d\theta} \sin \theta \frac{d}{d\theta} \)
\[
\hat{O} f(\theta) = \frac{1}{\sin \theta} \frac{d}{d\theta} \left( \sin \theta \frac{d}{d\theta} \left( 3 \cos^2 \theta - 1 \right) \right)
\]
\[
\sin \theta \frac{d}{d\theta} \left( 3 \cos^2 \theta - 1 \right) = -6 \cos \theta \sin^2 \theta
\]
\[
\hat{O} f(\theta) = -\frac{6}{\sin \theta} \frac{d}{d\theta} \left( \cos \theta \sin^2 \theta \right) = -\frac{6}{\sin \theta} \left( -\sin^3 \theta + 2 \cos^2 \theta \sin \theta \right) = 6 \left( \sin^2 \theta - 2 \cos^2 \theta \right)
\]
\[
\hat{O} f(\theta) = 6 \left( 1 - \cos^2 \theta - 2 \cos^2 \theta \right) = 6 \left( 1 - 3 \cos^2 \theta \right)
\]

b) If it is an eigenfunction, what is the eigenvalue. If not an eigenfunction write: Not

Yes, it is an eigenfunction, with eigenvalue -6: \( \hat{O} f(\theta) = -6 \left( 3 \cos^2 \theta - 1 \right) = -6 f(\theta) \)
Q5) (P13.29) a) Show whether the following two vectors are orthogonal:

\[ \phi_1 = \begin{pmatrix} 0.7 \\ 2 \end{pmatrix} \quad \text{and} \quad \phi_2 = \begin{pmatrix} 1.4 \\ -0.49 \end{pmatrix} \]

Orthogonality

\[ 0 \equiv \phi_1^\dagger \phi_2 = \begin{pmatrix} 0.7 \\ 2 \end{pmatrix} \begin{pmatrix} 1.4 \\ -0.49 \end{pmatrix} = (0.7)(1.4) - (2)(-0.49) = 0.98 - 0.98 = 0 \]

So the two vectors are orthogonal, the inner product is zero.

b) Are they orthogonal: Answer Yes or No: Yes, they are orthogonal

c) Find \( \psi_2 \): a vector proportional to \( \phi_2 \) (of part a) that is normalized:

The question asks that you find a vector related to the one above, and that one is normalized:

\[ \psi_2 = N \cdot \phi_2 = N \begin{pmatrix} 1.4 \\ -0.49 \end{pmatrix} \]

\[ 1 = \psi_2^\dagger \psi_2 \]

Now you can solve for the norm, N:

\[ \psi_2 = N \cdot \phi_2 = N \begin{pmatrix} 1.4 \\ -0.49 \end{pmatrix} \]

\[ 1 = \psi_2^\dagger \psi_2 = \left( N^\dagger N \right) \begin{pmatrix} 1.4 \\ -0.49 \end{pmatrix} \begin{pmatrix} 1.4 \\ -0.49 \end{pmatrix} = N^\dagger N \left( 1.4^2 + 0.49^2 \right) \]

\[ \left( N^\dagger N \right) = \frac{1}{1.4^2 + 0.49^2} = \frac{1}{2.2} \]

\[ N = \frac{1}{1.48} \]

\[ \psi_2 = N \cdot \phi_2 = \sqrt{\frac{1}{2}} \begin{pmatrix} 1.4 \\ -0.49 \end{pmatrix} = \begin{pmatrix} 0.95 \\ -0.33 \end{pmatrix} \]

As a check on the norm: \( 0.95^2 + 0.33^2 = 1.01 \), close enough for our rough numbers.
Q6) (P15.12) Consider the wave function \( \psi(x) = \sqrt{\frac{2}{a}} \left[ \sin \left( \frac{2\pi x}{a} \right) + \sin \left( \frac{4\pi x}{a} \right) \right] \), as a superposition of two individual functions which are each eigenfunctions of the total energy operator with eigenvalues: \( E_2 \) and \( E_4 \).

a) Is \( \psi(x) \) an eigenfunction for the total energy operator for the particle in the box?

\[
\hat{H} \psi(x) = \sqrt{\frac{2}{a}} \left[ \hat{H} \sin \left( \frac{2\pi x}{a} \right) + \hat{H} \sin \left( \frac{4\pi x}{a} \right) \right] = \sqrt{\frac{2}{a}} \left[ E_2 \sin \left( \frac{2\pi x}{a} \right) + E_4 \sin \left( \frac{4\pi x}{a} \right) \right]
\]

Because \( E_4 = 4E_2 \) one cannot recover the original function. Therefore the superposition is not an eigenfunction of the energy operator.

b) Is the superposition wave function (part a) normalized (show work). (Hint: do not do any integrals involving trig functions yourself.) Use the integral given on the front page:

\[
\int \psi(x)^* \psi(x) \, dx = \frac{2}{a} \int \left\{ \sin \left( \frac{2\pi x}{a} \right) + \sin \left( \frac{4\pi x}{a} \right) \right\} \left\{ \sin \left( \frac{2\pi x}{a} \right) + \sin \left( \frac{4\pi x}{a} \right) \right\} \, dx
\]

\[
= \frac{2}{a} \int \left\{ \sin^2 \left( \frac{2\pi x}{a} \right) + 2 \sin \left( \frac{2\pi x}{a} \right) \sin \left( \frac{4\pi x}{a} \right) + \sin^2 \left( \frac{4\pi x}{a} \right) \right\} \, dx
\]

\[
= \frac{2}{a} \left[ \frac{a}{2} + 2 \cdot 0 + \frac{a}{2} \right] = 2
\]

So no, it is not normalized, but now that you have the norm you can renormalize the wave function or just remember that the norm is 2.

c) Using postulate 4, write the average energy associated with the wave function (part a) in terms of \( E_2 \), and whatever other constants you may need.

Combining parts A and B above into the average of the energy expression:

\[
\langle E \rangle = \frac{\int \psi(x)^* \hat{H} \psi(x) \, dx}{\int \psi(x)^* \psi(x) \, dx} = \sqrt{\frac{2}{a}} \frac{2}{\int \psi(x)^* \left\{ E_2 \sin \left( \frac{2\pi x}{a} \right) + E_4 \sin \left( \frac{4\pi x}{a} \right) \right\} \, dx}{2} = \frac{E_2 + E_4}{2} = E_2 \left( \frac{1 + 4}{2} \right) = E_2 \frac{5}{2}
\]