A look at the two momentum integrals we need to do. Let's leave the m and n in the integrals to keep track of how the terms come out, even though we will set m=1 and n=2, (or vise versa). The wave function is a superposition

$$\Psi(x,t) = ce^{-i\frac{E_n}{\hbar}t}\phi_n + de^{-i\frac{E_m}{\hbar}t}\phi_m$$

and it is normalized when: $c^*c + d^*d = 1$

The momentum term comes down to:

$$\langle p \rangle = \Psi(x,t) = \int_0^a \left(ce^{-i\frac{E_n}{\hbar}t}\phi_n\right)^* \left(de^{-i\frac{E_m}{\hbar}t}\hat{p}\phi_m\right)dx + \int_0^a \left(de^{-i\frac{E_m}{\hbar}t}\phi_m\right)^* \left(ce^{-i\frac{E_n}{\hbar}t}\hat{p}\phi_n\right)dx$$

We have used postulate 4, and taken advantage of the normalized wave function, and the fact that the direct terms in the cross product multiplication vanish and we are left only with the cross terms. We can simplify this a bit by removing constants from the integrals:

$$\langle p \rangle = \left(ce^{-i\frac{E_n}{\hbar}t}\right)^* \left(de^{-i\frac{E_m}{\hbar}t}\right)\int_0^a \phi_n\hat{p}\phi_m dx + \left(de^{-i\frac{E_m}{\hbar}t}\right)^* \left(ce^{-i\frac{E_n}{\hbar}t}\right)\int_0^a \phi_m\hat{p}\phi_n dx$$

and I left the stars of the eigenfunctions only for this case because they are real functions, but this does not always apply. Now I will show that the integrals are equal because we can use differentiation by parts:

$$\int_0^a \frac{d\phi_n\phi_m}{dx}dx = \int_0^a \phi_n\frac{d\phi_m}{dx}dx + \int_0^a \phi_m\frac{d\phi_n}{dx}dx$$

$$\int_0^a \frac{d\phi_n\phi_m}{dx}dx = \phi_n\phi_m\Big|_0^a = 0$$

$$\int_0^a \phi_n\frac{d\phi_m}{dx}dx = -\int_0^a \phi_m\frac{d\phi_n}{dx}dx$$

Now we need to evaluate each of these to show this relation is true. This is closely related to the property that with the I-hbar term present the integrals are equal in the sense that:

$$-i\hbar\int_0^a \phi_n^*\frac{d\phi_m}{dx}dx = \int_0^a \phi_m i\hbar\frac{d\phi_n^*}{dx}dx = \left(\int_0^a \phi_m^*\left(-i\hbar\frac{d\phi_n}{dx}\right)dx\right)^* = \left(\int_0^a \phi_m^*\left(\hat{p}\phi_n\right)dx\right)^*$$

$$\int_0^a \phi_n^*\left(p\phi_m\right)dx = \left(\int_0^a \phi_m^*\left(\hat{p}\phi_n\right)dx\right)^*$$

If we assume these two integrals are related and are pure real, and c and d are pure real, we have the terms nearly canceling but the phase from the complex exponentials gives a term that does not cancel:

$$\omega = \tfrac{1}{\hbar}\left(E_m - E_n\right)$$

$$\langle p \rangle = -i\hbar cd\int_0^a \phi_n\frac{d\phi_m}{dx}dx\left\{e^{-i\omega t} - e^{+i\omega t}\right\} = -2\hbar cd\sin(\omega t)\int_0^a \phi_n\frac{d\phi_m}{dx}dx$$

This is close enough to the desired answer. It is important to realize that the momentum is oscillating harmonically but with sine term, compared to the position that oscillates with the cosine term. So the two both oscillate harmonically but out of phase. This makes physical sense. When the particle is near the end of the box the momentum (or velocity) slows to zero and turns around. This shows the momentum exists, oscillates in time and is pure real. Now we need to show that the two integrals with different derivatives are indeed the same (up to the sign differenc). It is sufficient to show that the one integral does not depend on m or n order at the end of the day.

$$I = \int_0^a \phi_n \frac{d\phi_m}{dx}\, dx = \frac{2}{a}\int_0^a \sin\left(\frac{\pi n x}{a}\right) \frac{d \sin\left(\frac{\pi m x}{a}\right)}{dx}\, dx$$

$$y = \frac{\pi x}{a}$$

$$I = \frac{2}{a}\int_0^\pi \sin(ny) \frac{d \sin(my)}{dy}\, dy = \frac{2}{a} m \int_0^\pi \sin(ny)\cos(my)\, dy$$

Right now it looks like the m in front will spoil the integral, in that it will have an m that is not exchangeable with the n. But this does not happen. We need the trig identities:

$$\sin(n+m)y = \sin ny \cos my + \sin my \cos ny$$

$$\sin(n-m)y = \sin ny \cos my - \sin my \cos ny$$

So we add these two identities to get the integrand above:

$$\sin(n+m)y + \sin(n-m)y = 2 \sin ny \cos my$$

Substituting into the integral:

$$I = \frac{1}{a} m \int_0^\pi \sin(n+m)y + \sin(n-m)dy$$

$$I = -\frac{1}{a} m \left\{ \frac{\cos(n+m)y}{n+m} + \frac{\cos(n-m)y}{n-m} \right\}_0^\pi$$

We notice that both combinations of m and n $n \pm m$ are odd integers when one is even and the other is odd, and that is the case of interest here. (We could consider the other cases later). In this case the cos of an odd integer time pi is –1 and the cos of zero is one. Therefore, each of the terms in the numerators evaluates to –2.

$$I = -\frac{1}{a} m \left\{ \frac{-2}{n+m} + \frac{-2}{n-m} \right\} = \frac{2}{a} m \left\{ \frac{n-m+n+m}{(n+m)(n-m)} \right\} = \frac{2}{a} m \left\{ \frac{2n}{(n^2 - m^2)} \right\}$$

$$I = \frac{4}{a} \left\{ \frac{mn}{(n^2 - m^2)} \right\}$$

The result is that the integral is symmetric in m and n, so at the end you can't tell whether the derivative was taken on the m or n eigenfunction. And on swapping m and n the sign changes as we showed above that it should. So this looks right.