Classical Mechanics (CM):

We ought to have some background to appreciate that QM really does just use CM and makes one slight modification that then changes the nature of the problem we need to solve but much of the physics remains the same.

Outline of This paper:

1. Defining a Hamiltonian in CM (which then QM uses as the basis of all mechanics)
2. The conservation of energy, and how that leads to Newton's Laws of motion.
3. Showing applications of different types of potentials in CM and solving the motion.
4. Relating the momentum to the velocity.
5. Deriving the Classical Virial Theorem which relates the Kinetic and Potential Energies.
6. Discussing the mechanics of a charged particle going around a nucleus.

Part 1:

The first point is that we need in both QM and CM a statement about the energy. This statement is used in both forms of mechanics with a Hamiltonian. The Hamiltonian is the total energy (the sum of the kinetic and potential energies) written in terms of the position (q or x) and momentum (p) of the particle. Hamilton formulated his ideas for classical mechanics. They are the bases for quantum mechanics as well. Basically the classical Hamiltonian is just the sum of the kinetic and potential energies. Generally the kinetic energy (T) depends only on the momentum of the particle, and the potential energy (V) depends only on the position of the particle. The Hamiltonian is $H = T + V$.

For most situations: $T = \frac{1}{2}mv^2 = \frac{1}{2}m\vec{v}\cdot\vec{v} = \frac{\vec{p}\cdot\vec{p}}{2m}$, and V is just the potential energy as a result of the environment and therefore depends only on the position of the particle $V = V(q)$, and $V \neq V(p,\dot{q})$. The dot" notation is quite convenient and is just shorthand for taking the time derivative. The velocity (v) is: $v \equiv v_q \equiv \frac{dq}{dt} \equiv \dot{q}$.

The potential energies we will consider will be:

$V = 0$ free space

$V = mgx$ gravity on the surface of the earth

$V = \frac{1}{2}kx^2$ Hooks Law (springs) Harmonic Motion

$V = -\frac{Ze^2}{r}$ Coulomb's Law

$V = -\frac{mM}{r}$ gravity in general

Part 2:

There are a couple of important ideas that help us understand the point of Hamilton's expression. The first is: The energy is conserved. Which means it does not change in time. Classically if an object moves into a region of lower potential it speeds up so that the total energy of the particle is constant. Therefore:

$$\frac{dH}{dt} = 0$$

This is very important. The consequence is that we can derive Newton's laws of motion from this simple requirement:

$$H = T + V \qquad T = \frac{\vec{p} \cdot \vec{p}}{2m} \qquad V = V(r)$$

$$0 = \frac{dH}{dt} = \frac{dT}{dt} + \frac{dV}{dt}$$

Using these definitions of T and V and the chain rule we have:

$$\frac{dT}{dt} = \frac{\vec{p}}{m} \cdot \frac{d\vec{p}}{dt} \qquad \frac{dV}{dt} = \frac{dr}{dt} \cdot \frac{dV(r)}{dr}$$

Then combining with the definition of momentum: $\frac{\vec{p}}{m} = \frac{d\vec{r}}{dt}$ yields:

$$0 = \frac{dT}{dt} + \frac{dV}{dt}$$

$$0 = \frac{\vec{p}}{m} \cdot \frac{d\vec{p}}{dt} + \frac{dr}{dt} \cdot \frac{dV(r)}{dr}$$

$$0 = \frac{\vec{p}}{m} \cdot \left\{ \frac{d\vec{p}}{dt} + \frac{dV(r)}{dr} \right\}$$

The part in the brace is set to zero: This is Newton's Law of motion. Each term is related to the force on the particle. The force comes from the potential (specifically, the gradient of the potential) $F = -\frac{dV(r)}{dr}$ and the force causes the particle momentum to change in time $F = \frac{d\vec{p}}{dt}$. A system is said to be "conserved" (energetically) when there are no external forces acting on it. Then only the internal forces are used to change particle momentum:

$$\frac{d\vec{p}}{dt} = -\frac{dV(r)}{dr}$$

$$F = \frac{d\vec{p}}{dt} = -\frac{dV(r)}{dr}$$

This expression, combined with the definition of the momentum give us two equations and two unknowns (x,p) that can be determined:

Newton's Equations
$\vec{p} = m\frac{d\vec{r}}{dt}$
$\frac{d\vec{p}}{dt} = -\frac{dV(r)}{d\vec{r}}$

Part 3: Applications of Newton's Laws
Let's see how these (Newton's) equations are used to solve for the position, and momentum (x,p) of a particle as a function of time.
Case 1: Free space: $V = 0$
Because there is no potential there is no force on a particle, so the momentum is conserved. $\frac{d\bar{p}}{dt} = 0$, which implies $p(t) = p(0) \equiv p_o$, a constant. We can now solve for the position given a constant momentum from the definition of the momentum:

$$m\frac{d\bar{r}}{dt} = \bar{p} = \bar{p}_o$$

As p is a constant this equation is solved by integrating (both sides of the equation) from time 0 to t:

$$m\int_{t'=0}^{t}\frac{dr}{dt'}dt' = \int_{t'=0}^{t} p_o dt' = p_o t$$

$$mr(t) - mr(0) = p_o t$$

$$\bar{r}(t) = \bar{r}_o + \tfrac{1}{m}\bar{p}_o t$$

So the particle moves at a constant velocity and the position changes linearly with time depending only on where it started and how much momentum it had initially (in the three directions).

Case 2: Gravity on the surface of the earth: $V = mgx$
This is a simplified gravity potential where x is the distance off the surface of the earth: $x = r - r_o$, and the potential energy is that relative to the earth's surface. The two fundamental equations of motion are:

$$p = m\frac{dx}{dt} \quad \text{and} \quad \frac{dp}{dt} = -\frac{dV(x)}{dx}$$

For this application the specific equations of motion are that the momentum is subjected to a constant force:

$$p = m\frac{dx}{dt} \quad \text{and} \quad \frac{dp}{dt} = -mg$$

The equations can be integrate up, to obtain the momentum (or velocity), using the second equation, and then integrated to get the position, using the first equation.

$$\frac{dp}{dt} = -mg \xrightarrow{\text{Integrate}} p(t) - p(0) = -mgt$$

$$m\frac{dx}{dt} = p = p_o - mgt \xrightarrow{\text{Integrate}} mx(t) - mx_o = p_o t - \tfrac{1}{2}mgt^2$$

$$x(t) = x_o + v_o t - \tfrac{1}{2}gt^2$$

Notice the momentum become more negative as time increases, the object is falling. The position has a quadratic dependence on time; the particle is approaching earth faster and faster. $x(t)$ gets smaller in time. There is a catch to this that is not shown directly, and that is that x cannot be negative. Once the particle hits the ground the equations stop applying, unless the particle is going down a mine shaft or a fault in the earth, then things change slightly. But assuming the potential only applies when the particle is in the air, everything is OK.

There are two arbitrary constants: where the particle starts and how fast it is going at the start $\left(x_o, p_o = mv_o\right)$. These are boundary conditions. Each first order differential equation is solved in turn here but one could write a second order D.E. directly for x (but not for p):

$$-mg = \frac{dp}{dt} = m\frac{d^2x}{dt^2}$$

$$\frac{d^2x}{dt^2} = -g$$

Notice that the single equation can be solved on its own, but requires two boundary conditions. Also notice the change in position of the particle is the same regardless of mass. Often this is the method of solution shown in introductory physics; I think it is more confusing that the method used above, but it is popular as it gets rid of an intermediate variable (the momentum). I disagree with this approach and urge you to focus on how the problem was solved above. After all we are interested in the momentum as much as the position.

The third case: $V = \frac{1}{2}kx^2$ The Harmonic Oscillator

For this example the two fundamental equations become:

$$\boxed{\begin{aligned} m\frac{dx}{dt} &= p \\ \frac{dp}{dt} &= -\frac{dV(x)}{dx} = -kx \end{aligned}}$$

These equations are quite symmetric in x and p. We can combine the two to get a second order D.E. in terms of either x or p.

$$m\frac{d^2x}{dt^2} = \frac{dp}{dt} \qquad\qquad m\frac{dx}{dt} = p$$

$$\frac{dp}{dt} = -kx \qquad\qquad \frac{d^2p}{dt^2} = -k\frac{dx}{dt}$$

$$\boxed{\frac{d^2x}{dt^2} = -\frac{k}{m}x} \qquad\qquad \boxed{\frac{d^2p}{dt^2} = -\frac{k}{m}p}$$

Notice this shows you can think about the equations of motion in either x or p, independently. So either way works fine. The two equations (in the boxes here) are identical in form, just x and p are interchangeable symbols. This means that the same solution applies equally well to either quantity. This might seem contradictory to the idea that the momentum is proportional to the time change of x, but we will see that that is not a problem In Q.M. one says that either a "momentum space" or "position space" representation of the problem is equally valid. Unlike the previous two problems this cannot be integrated to obtain either p or x as a function of time. One must solve either the two coupled first order D.E.s or either of the second order D.E. (in the boxes). The solution is well known, because the second derivative of a cosine or sine function gives that function back (times a constant). Therefore these are the two independent solutions.

$$p = p(t) = A\sin(\omega t + \phi_o)$$

$$\frac{d^2 p}{dt^2} = -\omega^2 p$$

Therefore the frequency, ω, of oscillation is tied to the spring force constant and the mass: $\omega = \sqrt{\frac{k}{m}}$. [FN: This is an eigenvalue problem where p is the eigenfunction.] A, ϕ_o are constants one can set: From the appropriate expression the position can be determined:

$$x = \frac{-1}{k}\frac{dp}{dt} = -\frac{\omega}{k}A\cos(\omega t + \phi_o)$$

$$x = \frac{-A}{\sqrt{km}}\cos(\omega t + \phi_o)$$

The Hamiltonian or energy is rather interesting. We said at the beginning that the equations of motion should show that the Hamiltonian is independent of time. So let's substitute our solutions to (x,p) and evaluate the Hamiltonian as a function of time:

$$H = \tfrac{1}{2m} p^2 + \tfrac{k}{2} x^2$$

$$H = \tfrac{1}{2}\frac{A^2}{m}\left\{\sin^2(\omega t + \phi_o) + \frac{km}{\left(\sqrt{km}\right)^2}\cos^2(\omega t + \phi_o)\right\}$$

$$H = \frac{A^2}{2m}$$

As one knew at the beginning the Hamiltonian must be a constant of the motion, so it is independent of time. The energy (which is the Hamiltonian <u>after</u> it has been evaluated as a function of time) is proportional to the square of the amplitude of the motion. Let's consider the physical interpretation of these equations. Set $\phi_o = 0$. In this case the boundary conditions (or the values of x and p at time zero) tell us that the spring has been pulled to its maximum extension to the left, negative x. The extension (x_o) is: $x_o = \frac{-A}{\sqrt{km}}$. As the particle is released (at time zero) it proceeds to the right (increasing x, *i.e.* a smaller negative number), and the momentum increases. When the particle crosses the y axis at $x = 0$, the time is $\omega t = \frac{\pi}{2}$, or 90degrees, it has gone 1/4 of its cycle. At this point the momentum is maximal (and contains all the energy), and $p = p_{\max} = A$, and we find that the maximum in x is: $x_{\max} = \frac{p_{\max}}{\sqrt{km}}$

Part 4: The momentum and the velocity.

We are all pretty well conditioned to accept that the momentum is just the mass times the velocity. This is not always the case. The definition of momentum needs to go beyond what we already know. The reason the momentum is associated with a coordinate is that it is derived from the energetics of the system in terms of that coordinate's velocity. Recall that momentum is actually call the "conjugate momentum", meaning a relation to a coordinate velocity. So here is how the momentum is officially defined. [FN: This is intended to explain why the usual definition of momentum and why we need to do more.] The full definition of momentum: The kinetic and potential energies are written in terms of coordinates, q, and velocities: $\dot{q} \equiv \frac{dq}{dt}$. Both T and V can be functions of both $\{q, \dot{q}\}$. The official definition of momentum conjugate to q is:

$$p = p_q = \frac{d\left(T - V\right)}{d\dot{q}}$$

This may seem like it came out of thin air, but we really do not want to develop the equation that leads to this. Even though this is the really complete definition, for our purposes, the potential is <u>not</u> a function of the velocity (which only happens when magnetic fields are present) and so the more typical definition is:

$$p_q = \frac{dT}{d\dot{q}}$$

The derivative, w.r.t. the velocity, by the way, means to take the derivative but consider the coordinate (q) to <u>not</u> be a function of $v_q \equiv \dot{q}$. The kinetic energy, T, is always the same, regardless of problem but sometimes it is written in different coordinate systems, and so it can look a little different. Let's start with Cartesian coordinates:

$$T = \tfrac{1}{2} m \left\{ v_x^2 + v_y^2 + v_z^2 \right\}$$

$$v_q = \dot{q} \text{ or grouped as a vector } \vec{v} = \dot{\vec{r}}$$

The derivative of T w.r.t. any one of the three coordinate-velocities then gives the momentum conjugate to that coordinate. The result is the usual result:

$$p_x = \frac{dT}{dv_x} = \tfrac{1}{2} m \frac{d\left\{ v_x^2 + v_y^2 + v_z^2 \right\}}{dv_x} = m v_x$$

The same holds for y or z. This is the definition of the momentum and its relation to the velocity. Now T and V can be rewritten in terms of the momentum and position rather than velocities and positions. $T = \frac{1}{2m} p^2$ Then the Hamiltonian can be formed: H=T+V, where T and V are written in terms of momentum and position.

That seems straightforward, and gives the momentum we all know. So why do we need to go though all of this? Well, if we have a problem of a stationary orbit (the electron around the nucleus or the earth around the sun) the potential is a central force potential and written in terms of the distance between the particles, so spherical coordinates are more appropriate because then V depends only on one of the three coordinates (the r).

So we need to know what the kinetic energy looks like when we write it in terms of spherical coordinates. From the definition of the Cartesian coordinates in terms of spherical coordinates we can write the velocity and then the kinetic energy. We use the above definition to find the momentum conjugate to the radius and the angles.
The relation between Cartesian and polar coordinates:

$$\vec{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = r\begin{pmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{pmatrix}$$

The velocity is

$$\vec{v} = \dot{\vec{r}} = \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} = \dot{r}\begin{pmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{pmatrix} + r\dot{\theta}\begin{pmatrix} \cos\theta\cos\phi \\ \cos\theta\sin\phi \\ -\sin\theta \end{pmatrix} + r\sin\theta\dot{\phi}\begin{pmatrix} -\sin\phi \\ \cos\phi \\ 0 \end{pmatrix}$$

The kinetic energy is found now from the inner product of the velocity vectors:

$$\vec{v}\cdot\vec{v} =$$

$$\dot{r}^2\begin{pmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{pmatrix}^\dagger \cdot \begin{pmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{pmatrix} + \left(r\dot{\theta}\right)^2\begin{pmatrix} \cos\theta\cos\phi \\ \cos\theta\sin\phi \\ -\sin\theta \end{pmatrix}^\dagger \cdot \begin{pmatrix} \cos\theta\cos\phi \\ \cos\theta\sin\phi \\ -\sin\theta \end{pmatrix} + \left(r\sin\theta\dot{\phi}\right)^2\begin{pmatrix} -\sin\phi \\ \cos\phi \\ 0 \end{pmatrix}^\dagger \cdot \begin{pmatrix} -\sin\phi \\ \cos\phi \\ 0 \end{pmatrix}$$

$$+CrossTerms$$

$$T = \tfrac{1}{2}m\vec{v}\cdot\vec{v} = \tfrac{1}{2}m\left(\dot{r}^2 + \left(r\dot{\theta}\right)^2 + \left(r\sin\theta\dot{\phi}\right)^2\right)$$

The cross terms all vanish and the inner products of the vectors with angle parts all give 1. So the answer is not too bad given the messy intermediate matrix multiplication. From here now we can obtain the momentum conjugate to $\left\{\dot{r},\dot{\theta},\dot{\phi}\right\}$. This is where the more general definition of momentum comes in handy.

$$T = \tfrac{1}{2}m\left(\dot{r}^2 + \left(r\dot{\theta}\right)^2 + \left(r\sin\theta\dot{\phi}\right)^2\right)$$

$$p_r = \frac{dT}{d\dot{r}} = m\dot{r} \qquad p_\theta = \frac{dT}{d\dot{\theta}} = mr^2\dot{\theta} \qquad p_\phi = \frac{dT}{d\dot{\phi}} = m\left(r\sin\theta\right)^2\dot{\phi}$$

$$T = \tfrac{1}{2}\left(p_r\cdot\dot{r} + p_\theta\dot{\theta} + p_\phi\dot{\phi}\right)$$

Notice that we have now defined the momenta conjugate to each of the velocities and the kinetic energy can be written in a form like that from Cartesian coordinates: $T = \tfrac{1}{2}\vec{p}\cdot\vec{v}$.
We can finish up by writing T fully in terms of the momenta, using the conventional definition of the moment of inertia, I:

$$I = mr^2$$

$$T = \tfrac{1}{2m}p_r^{\,2} + \tfrac{1}{2I}\left(p_\theta^{\,2} + \frac{p_\phi^{\,2}}{\sin^2\theta}\right)$$

The momenta are not nearly as tidy as they were in Cartesian coordinates. Also, the kinetic energy now depends on the coordinates not just the velocities. In particular $p_\theta = I\dot{\theta}$, which looks like the angular analogue of linear momentum, but I contains the r coordinate (not $\dot{r}$). So this makes T more complicated and causes us to rethink how we handle the mechanics of motion going around a central force.

Part 5: The virial theorem

The problem we would like to do is the gravitational attraction or charge attraction (Coulomb's law) of one particle orbiting around another. These two problems are identical as the potential is proportional to $|r|^{-1}$ in both cases. The potential is also negative in both cases which lead to bound orbits. However, is really too tedious and time consuming for us.

We can do the orbit problem a different way. And that is to first prove the viral theorem for classical mechanics. This is not too hard to do for the cases we are considering: The virial (G(t)) is x time p (or r time $p = p_r$). We follow this product in time:

$$G(t) = r \cdot p$$

$$\frac{dG(t)}{dt} = \frac{dr}{dt} \cdot p + r \cdot \frac{dp}{dt}$$

These two terms can each be evaluated (using Newton's Laws, above) in terms of T and V:

$$\frac{dr}{dt} \cdot p = \frac{m}{m}\frac{dr}{dt} \cdot p = \frac{1}{m} p \cdot p = 2T$$

$$r \cdot \frac{dp}{dt} = -r \cdot \frac{dV}{dr}$$

For cases where the potential energy is proportional to r to some power $V = ar^n$ we have:

$$r \cdot \frac{dp}{dt} = -r \cdot \frac{dV}{dr} = -ra \cdot \frac{dr^n}{dr} = -nV$$

Combining all of the parts, we can relate the virial to T and V:

$$\frac{dG(t)}{dt} = 2T - nV$$

Now average over time (from 0 to time T). The angular braces, $\langle \ \rangle$, mean the same as averaging. These braces will be used a lot in this course.

$$\left\langle \frac{d}{dt} G \right\rangle = \frac{1}{t} \int_{t=0}^{t} \frac{dG(t')}{dt'} dt' = \frac{G(t) - G(0)}{t} = 2\langle T \rangle - n \langle V \rangle$$

If we average over a time such that the system returns to where it was at time zero: Then $G(t) - G(0) = 0$, and we have the relation between the kinetic and the potential. So for any system that is periodic, over a period of the motion we have the relation:

$$2\langle T \rangle = n \langle V \rangle$$

For the Harmonic oscillator n=2 and we have that the mean kinetic and mean potential energies are the same, and for the bound electron, where n=-1, the relation is:

$$2\langle T\rangle = -\langle V\rangle .$$

Part 6: Motion in a central force

From the virial theorem we can obtain some insight into motion of a particle in an orbit of constant radius. Above we wrote the kinetic energy in terms of polar coordinates. We can simplify the kinetic energy expression by saying the particle is confined to a specific radius (so $\dot{r} = 0$), and is rotating in the x-y plane (so $\dot{\theta} = 0$ and $\sin\theta = 1$), in a stationary orbit. The kinetic energy expression simplifies in this case to:

$$T = \tfrac{1}{2} m\left(r\dot{\phi}\right)^2 = \tfrac{1}{2} I\dot{\phi}^2$$

And the angular velocity, $\omega = \dot{\phi}$, is constant (time independent). Then (both T and V are always constant):

$$2\langle T\rangle = -\langle V\rangle$$

$$2\tfrac{1}{2} I\omega^2 = \frac{Ze^2}{r}$$

$$r^3\omega^2 = \frac{Ze^2}{m}$$

The problem is solved by equating all the constants of the motion. This ties together the radius to the angular velocity: as the radius of a stationary orbit increases the angular velocity must slow down. All quantities are constants of motion but the constants are not independent of each other. This is true for planets as well. The further out planets must orbit the sun at slower rates, and they exactly follow this law.

However, showing that one is allowed to say that the angular velocity is a constant of the motion takes a bit of work. This really is a good place to stop.

Beyond here lie Dragons:

Part 7 The mechanics of motion in a central force

For the more general case we will need to use Hamilton's equations of motion (which are a variant on Lagrange's equations), which reduce to Newton's laws for the simpler cases we have developed.
We need, for later, the idea of the angular momentum and how that relates to the conjugate mometa we defined in part 4. So the definition of angular momentum, written as a vector, initially in Cartesian coordinates is:

$$\vec{L} = m\vec{r} \times \vec{v} = \vec{r} \times \vec{p}$$

This is very useful for a particle traveling in a central force, because:

$$\dot{\vec{L}} = m\dot{\vec{r}} \times \vec{v} + \vec{r} \times \dot{\vec{p}} = m\vec{v} \times \vec{v} - \vec{r} \times \vec{\nabla} V = 0$$

The angular momentum is a constant of the motion because the two terms vanish. The first term vanishes identically because the cross product of a vector with itself must be zero. The second term vanishes because the potential only depends on r. So the gradient of the potential is in the same direction as r (points outward, radially only) and so that cross product must vanish, as well. To show this explicitly:

$$r^2 = x^2 + y^2 + z^2$$

$$\frac{dr^{-1}}{dx} = \frac{d\left(r^2\right)^{-1/2}}{dx} = \tfrac{-1}{2}\left(r^2\right)^{-3/2}\frac{dr^2}{dx} = -\left(r^2\right)^{-3/2} x = \frac{-x}{r^3}$$

$$-\vec{\nabla} r^{-1} = \frac{\vec{r}}{r^3}$$

This makes the angular momentum a constant of the motion. Now we need to write L out in Cartesian and polar coordinates and show that:

$$\vec{L} \cdot \vec{L} = \left(p_\theta{}^2 + \frac{p_\phi{}^2}{\sin^2\theta} \right)$$

From this it follows that the Kinetic Energy is: $T = \frac{1}{2m} p_r{}^2 + \frac{1}{2I} L^2$

Finding L in polar coordinates:

$$\vec{L} = m\vec{r} \times \vec{v}$$

$$= mr\begin{pmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{pmatrix} \times \left\{ \dot{r}\begin{pmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{pmatrix} + r\dot{\theta}\begin{pmatrix} \cos\theta\cos\phi \\ \cos\theta\sin\phi \\ -\sin\theta \end{pmatrix} + r\sin\theta\dot{\phi}\begin{pmatrix} -\sin\phi \\ \cos\phi \\ 0 \end{pmatrix} \right\}$$

$$= mr^2\begin{pmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{pmatrix} \times \left\{ \dot{\theta}\begin{pmatrix} \cos\theta\cos\phi \\ \cos\theta\sin\phi \\ -\sin\theta \end{pmatrix} + \sin\theta\dot{\phi}\begin{pmatrix} -\sin\phi \\ \cos\phi \\ 0 \end{pmatrix} \right\}$$

$$= mr^2 \left\{ \dot{\theta}\begin{pmatrix} -\sin\phi \\ \cos\phi \\ 0 \end{pmatrix} - \sin\theta\dot{\phi}\begin{pmatrix} \cos\theta\cos\phi \\ \cos\theta\sin\phi \\ -\sin\theta \end{pmatrix} \right\}$$

And we can summarize this as: $\vec{L} = I\vec{\omega}$, which gives the full definition of the angular velocity in terms of the polar angles.

The z component of L is, rigorously: $L_z = I\sin^2\theta \cdot \dot{\phi} = p_\phi$.

Now get the square of the angular momentum:

$$\vec{L}\cdot\vec{L} = I^2 \left\{ \dot{\theta}^2 \begin{pmatrix} -\sin\phi \\ \cos\phi \\ 0 \end{pmatrix} \cdot \begin{pmatrix} -\sin\phi \\ \cos\phi \\ 0 \end{pmatrix} + \left(\sin\theta\dot{\phi}\right)^2 \begin{pmatrix} \cos\theta\cos\phi \\ \cos\theta\sin\phi \\ -\sin\theta \end{pmatrix} \cdot \begin{pmatrix} \cos\theta\cos\phi \\ \cos\theta\sin\phi \\ -\sin\theta \end{pmatrix} \right\} + CrossTerms$$

The cross terms cancel and the direct vector products are 1:

$$\vec{L}\cdot\vec{L} = I^2 \left\{ \dot{\theta}^2 + \left(\sin\theta\dot{\phi}\right)^2 \right\}$$

$$\vec{L}\cdot\vec{L} = \left\{ \left(I\dot{\theta}\right)^2 + \left(\frac{I\sin^2\theta\dot{\phi}}{\sin\theta} \right)^2 \right\} = p_\theta^2 + \frac{p_\phi^2}{\sin^2\theta}$$

Consider the particle to be rotating in the x-y plane ($\sin\theta = 1$). The Kinetic energy is:

$$T = \tfrac{1}{2m} p_r{}^2 + \tfrac{1}{2I} p_\phi{}^2$$

The angular momentum is the momentum conjugate to the angular variable, and the angular velocity is $\omega = \dot{\phi}$. Hamilton’s equations of motion are:

$$\dot{q}_k = \frac{\partial H}{\partial p_k} \quad \text{and} \quad \dot{p}_k = -\frac{\partial H}{\partial q_k}$$

The first equation is the definition of the conjugate variable and the second equation is Newton’s law, for a force. Now let's apply these equations to see how to solve for the motion of a particle constrained to be in the x-y plane under the influence of a central force.

The first equation is that for the phi-rotation: $\dot{p}_\phi = 0$ or $p_\phi = I\omega$ is a constant of the motion. Notice the angular velocity may change in time, we don't know what that does

yet, as the distance changes. Now the radial momentum equation of motion is more complex and more interesting (I suppose):
The first equation is just the relation between the radial velocity and the momentum. The second one gives a description of the forces involved:

$$\dot{r} = \frac{\partial H}{\partial p_r} \text{ and } \dot{p}_r = -\frac{\partial H}{\partial r}$$

$$\dot{r} = \frac{\partial T + V}{\partial p_r} = \frac{\partial T}{\partial p} = \tfrac{1}{2}\frac{\partial}{\partial p_r}\left(\frac{1}{m}\left(p_r\right)^2 + \frac{p_\phi^2}{I}\right) = \frac{p}{m}$$

$$\dot{p}_r = -\frac{\partial T + V}{\partial r} = \frac{p_\phi^2}{rI} - \frac{Ze^2}{r^2}$$

If the radial distance does not change we have:

$$0 = \dot{p}_r = \frac{p_\phi^2}{rI} - \frac{Ze^2}{r^2}$$

$$mr\omega = \frac{Ze^2}{r^2}$$

At this point we have a first order differential equation for the change in the radial component and everything else is a constant:

$$m\dot{r} = \frac{p_\phi^2}{rI} - \frac{Ze^2}{r^2}$$

$$mr^3\dot{r} = \frac{p_\phi^2}{m} - rZe^2$$