How to describe classical events (every day life) using wave functions:

Often in thinking about quantum mechanics, one imagines that the wave function description of physical events is restricted to quantum systems. This makes the picture of the world seem more abstract (and irrelevant) than is necessary. One can use the basic machinery usually associated with quantum events to describe every-day classical phenomenon. In the previous discussion which showed that the structure of QM (which is really linear algebra) arose from requirements of measurement, and this is not the exclusive purview of the Quantum world.

So let's apply the wave function picture to the idea of marriage. One is either married or not married. (Yes, there are subtleties along the way; but the WF picture does handle those; so bear with me.) So we have two states of matrimony (as the phrase goes). Those two states will be described by m for married and s for single (or not-married). The notation then for the two state is $|m\rangle$ and $|s\rangle$, respectively. In QM/Linear Algebra we would say that to represent the system we have a two dimensional space spanned by the two bases vectors (i.e. state functions) m and s. There are associated with each state, the outcome of either married or single (these are the fundamental observables of the system). A person is either married or not. If you ask somebody, are you married you either get (yes or no); (married or single); or (m or s). After you ask you know if an individual is m or s, but before you ask don't know. We will give examples of how these ideas can be applied to individuals or to populations taken collectively. (The application of linear algebra to QM is called Dirac notation, after P.A.M. Dirac who popularized its use.)

Because the two states are mutually exclusive (you are either married or single) we imply that the (two basis) functions are orthogonal. Like the geometric analogy they point in two different directions. So the dot product equals zero. So we note:

$$\langle m|s\rangle = 0$$
$$\langle m|m\rangle = \langle s|s\rangle = 1 .$$

This is called the orthogonality (or orthonormal) condition on the basis functions and the comparison is called the inner product. And the basis functions (representing the m and s states) are unit vectors that point in the married and single direction, respectively, so they are normalized (have length of 1).
To aid in understanding the inner-product think of these two states as vectors that are orthogonal or point in different directions:

$$|m\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ and } |s\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

The inner product is then the vector multiplication. For example:

$$\langle s|m\rangle = \langle s|\cdot|m\rangle = \begin{pmatrix} 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 0\cdot 1 + 1\cdot 0 = 0$$

Assuming that about ½ the people in the U.S. are married the (population) wave function should be a 50:50 mix of the two possible states. Therefore, if we just add up the parts for the two states (called a "linear superposition (a sum) of two states") the coefficient representing the contribution for each state should be the same for both state: $\left|\psi\right\rangle = \frac{1}{\sqrt{2}}\left|m\right\rangle + \frac{1}{\sqrt{2}}\left|s\right\rangle$. (The reason for the value assigned to the coefficient will be made clear later.) This is an example of a more general form of the wave function:

$$\left|\psi\right\rangle = c_m \left|m\right\rangle + c_s \left|s\right\rangle .$$

The reason we can add the parts up is because we are just making a vector that has a part that is pointing to the married direction and a part that is pointing to the single direction. Just like x and y on a graph. This is just vector addition.

The wave function must be normalized so that it accounts for the entire population, so it can be normalized to either N (the number of people it represents; statistically) or just 1, which means that we have unit outcome always; which is to say a person is either married or not married. The math representing normalization goes this way:

$$1 = \left\langle \psi \middle| \psi \right\rangle = \left(c_m^* \left\langle m\right| + c_s^* \left\langle s\right|\right)\left(c_m \left|m\right\rangle + c_s \left|s\right\rangle\right)$$
$$1 = c_m^* c_m \left\langle m \middle| m \right\rangle + c_m^* c_s \left\langle m \middle| s \right\rangle + c_s^* c_m \left\langle s \middle| m \right\rangle + c_s^* c_s \left\langle s \middle| s \right\rangle$$

Because the basis vectors are ortho-normal the full wave normalization simplifies:

$$1 = c_m^* c_m 1 + c_m^* c_s 0 + c_s^* c_m 0 + c_s^* c_s 1$$
$$1 = c_m^* c_m + c_s^* c_s$$

So we can describe a person's matrimonial state by the coefficients c_m and c_s. The normalization means that these quantities (the coefficients) are tied together, which is not too surprising: Each coefficient must be between 0 and 1, so if $c_m = 0$ then $c_s = 1$ and the person that this wave function describes is single. Can a person be in a superposition of states? If you are dating you can say the chances of being married are higher so yes you can have a superposition that represents the various stages of dating leading to marriage.

The wave function can equally well be applied to the population as a whole, as mentioned above. The example above said half the country is married. This could equally well mean that an individual is "half-way" married – i.e. engaged – so making the transition from single to married (or getting divorced and making the transition from married to single). However, at any time when asked a person will be able to say "married" or "single" (even if contemplating the other state).

The distinction between the individual measurement and the collective measurements happens in this example. If you ask an individual are you married you will receive either a yes or a no. In the back of the person's mind the thought may be well I'm considering changing state but the answer to the question is either yes or no. So the wave function allows for the possibility of change, yet the state of each individual is definite. So let's say we have an operator M which is the way of asking whether the person is married (or not). M operates on the wave function (ie. Asks the question – of the wave function) and produces the answer. Let's say 1 is yes and 0 is no (a simple truth table structure). So the individual is married or single then the wave function is either $\left|\psi\right\rangle = \left|m\right\rangle$ or $\left|\psi\right\rangle = \left|s\right\rangle$.

Now operate on the wave function and you get the wave function back but multiplied by the answer to the operation. For example if the person is married: $M\left|m\right\rangle = m\left|m\right\rangle$, and $m=1$.Or you could get 'yes' as in $M\left|m\right\rangle = 'yes'\left|m\right\rangle$. The individual is married, and you found out because the result of asking (using the M operator) got back the answer yes or 1 and the original state was undisturbed or you got back the original wave function, in this case, $\left|m\right\rangle$. If the individual is single then you get the definite answer of no: $M\left|s\right\rangle = s\left|s\right\rangle$, and here $s=0$. So again you get the answer to your question, and the answer is no (meaning single). (m and s are the 'values' of the answer, and are characteristic values associated with the operator M. German for characteristic is eigen, so they are called eigen values.)

Now if we ask the total population wave function are you married or single we don't get the wave function back. Let's see how this works: $M\left|\psi\right\rangle = \frac{1}{\sqrt{2}}M\left|m\right\rangle + \frac{1}{\sqrt{2}}M\left|s\right\rangle = \frac{1}{\sqrt{2}}m\left|m\right\rangle + \frac{1}{\sqrt{2}}s\left|s\right\rangle = \frac{1}{\sqrt{2}}\left|m\right\rangle$. We took advantage of the fact that our operator can ask the individual basis functions the question individually. (Mathematically this is called the linear property of operators; all operators we deal with are linear, and required to be so because they just ask these types of questions of wave functions.) But we did not get a definite answer because we did not get the original wave function back. In fact the new wave function is not even normalized. We see that we just got the coefficient for the married state back. So what does this imply? It means that as a total population we are not either married or single, but some of us are one way and some are the other. In fact we have messed up the wave function by asking the entire population "Are you married?" The act of asking caused the wave function to give us the ones that are married but the wave function is not normalized to one anymore it is normalized to the fraction of those who are married (a half). [Show that it is indeed normalized to ½.] So asking the question has selected for those in the affirmative, and now we are dealing with those who are married, the wave function is now the married state. So now we must renormalize it so that the new wave function represents all married people. In QM this is called the collapse of the wave function. Before you ask an individual about the marital status (all other things being equal) the answer could go either way (so the wave function is a 50:50 chance of being either way.) But then after you ask and get an answer you know the marital status of that individual and you must use that knowledge to update the person's wave function, as either one state or the other, not a mixture anymore. The updated wave function is said to have collapsed into a specific state as a result of asking (or measuring). I hope you see that the "collapse" really is the consequence of making a measurement and has nothing to do with any mysterious way quantum particles behave.

We can use the wave function to determine how many people are married, on average even though the wave function is not just simply married or not married. The wave function is not in a specific state with respect to the question "are you married" (Mathematically, we say the wave function is not an eigenfunction of the operator M.)

But the averages can be determined. Let's see how we get the averages. The rule determining the average from any wave is: $\langle \psi | M | \psi \rangle = \langle M \rangle$. The left hand side tells you how to evaluate for the average (of M) and the right hand side is just a symbol for the fact that you will determine the average fraction of M, the fraction of the average of people married. {We will discuss where this formula comes from later.}
Taking advantage of what we did above (with the normalization of the wave function):

$$M|\psi\rangle = \frac{1}{\sqrt{2}}|m\rangle$$

$$\langle M \rangle = \langle \psi | M | \psi \rangle = \langle \psi | \left(M |\psi\rangle \right) = \frac{1}{\sqrt{2}}\langle \psi | m \rangle$$

So we have taken advantage of knowing what M did to the wave function and now we ask for the inner product of the wave function and the married state. We evaluated the basis functions above so we can evaluate this inner product from those basis function inner products: $\langle \psi | m \rangle = |\psi\rangle = \frac{1}{\sqrt{2}}\langle m | m \rangle + \frac{1}{\sqrt{2}}\langle s | m \rangle = \frac{1}{\sqrt{2}}1 + \frac{1}{\sqrt{2}}0 = \frac{1}{\sqrt{2}}$. Using this for the projection of the married state onto the total wave function in the above average expression we get:

$$\langle M \rangle = \frac{1}{\sqrt{2}}\langle \psi | m \rangle = \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}} = \frac{1}{2}$$

This tells us what we originally knew. Half of the population is married. Hopefully you can see where this number came from, it is the coefficient in front of the married state, but it is the coefficient squared, c_m^2 or later as $c_m^* c_m$, that determines the fractional amount of people married (or the probability of being married).

Any wave function describing the marital status of people then can be written very generally in terms of the married or single states. (Mathematically: The basis functions let us span a two dimensional space. Anything like a chalk board or a piece of paper where one can put an x-y axis system is a two dimensional space.)

Above we saw that we were able to use the M operator to determine the fraction of people married from the wave function (using the math of inner products). Another operator is the unity operator. The unity operator is important. It has two meaning. The first is it asks the question "Are you married or single?" The answer to that must universally be yes (no excluded middle allowed here.) So we had better recovery everybody. A second, related, interpretation is it enables us to know if we asked everybody whether they were married or single. Either way, like normalization the unity operator had better give us unity (or one) at the end of the day. Mathematically then $\langle 1 \rangle = \langle \psi | 1 | \psi \rangle$. This follows from the operation:

$$1|\psi\rangle = 1|\psi\rangle$$

$$\langle 1 \rangle = \langle \psi | 1 | \psi \rangle = \langle \psi | \psi \rangle = 1$$

Maybe it looks like little progress but this ties together the idea that the wave function must be normalized with the idea that the measurement of 1 must include everybody and so you must get a one to clearly state that you queried all people. The general form of the wave function in terms of basis function is:

$$|\psi\rangle = c_m |m\rangle + c_s |s\rangle$$

This general form combined with the requirement that the wave function be normalized puts a requirement on the coefficients that:

$$1 = c_m^* c_m + c_s^* c_s$$

So the married and single coefficients are not really ever linearly independent. This constraint ties them together, so that there really is only one independent coefficient. This makes some sense in that if one is married $c_m = 1$, and that person is not single; therefore $c_s = 0$. This general normalization condition is much like the trigonometric relation that: $1 = \cos^2\theta + \sin^2\theta$. So with no loss of generality (except that the coefficients give up the possibility of being complex for now): $c_m = \cos\theta$. The idea that the basis vectors span a two dimensional space is now made more concrete. The x axis (abscissa) can be the married axis and the y axis (ordinate) can be the single axis and the wave function is a vector of unit length that points (like the hour hand of the clock) to either married or single or anywhere in between. The angle theta is the angle that determines the direction the vector is pointing. If $\theta = 0$, then $\cos\theta = \cos 0 = 1$ and this is the married state; if $\theta = 90° = \frac{\pi}{2}$ then we have the single state.

The coefficients and hence θ progress with time. At birth the individual is single. In an idealized case then one remains single from birth to young adulthood and then marries. So as a function of time theta is zero and the switches to $\pi/2$ and stays there ("'til death do us part"). So the value of θ is a useful indicator of the marital status of the individual. There are some notable exceptions which can in fact be tracked with a θ as a function of time. For example Elizabeth Taylor has been married about eight times over a sixty year span. My numbers are not very accurate here; and I do not mean to pick on E.T. personally, but just wish to illustrate how the wave function can oscillate or rotate in a two dimensional space. So the θ of her life is periodic. . Excluding the early years, she marries every decade or so. Start her out as single at time zero: $\theta_{ET} = \frac{\pi}{2} + \omega t$, and $\omega \approx \frac{\pi}{10} yr^{-1}$, is the angular rate in radians per year. We can assume that when $\theta = 0$ is the same state as $\theta = \pi$, so the space is folded back on itself. (Curiously enough electrons do this too, it is a property of a two dimensional space.) A wave function of this sort does not often give a clear-cut answer of married or single. And we can discuss the fine details of whether or not this does reflect the status of people who marry many times (serially).

We have covered how a wave function may be written (or expanded) in terms of basis functions (as a superposition of states). The states require an operator to distinguish them (Here the operator is M, the married state. An equally valid operator is the S operator, the single state one, and conservation requires: 1=M+S.} What is interesting is that the wave function in terms of these states can be used to compute any average quantity desired. We could go on to compute the fraction of people married, the root-mean-squared fraction of people married, the width of the distribution of people married etc. The coefficients can contain the time dependence of the wave function and this shows how the wave function rotates in the space pointing from married to single in time. The general language for this is we are in an N dimensional Hilbert space. The example we have chosen is N=2. It is particularly relevant to QM because electron spin is in a two dimensional space; and electrons do pair up in orbitals as a consequence of spin.

Questions: What do the operators M, S, and 1 look like if you assume the vector form of the two states m and s shown above (first page). Show that they give the answers stated above when operating on the various wave functions.

If the “answers” are ’red’, ‘green and ‘blue’ (3 distinct answers) what does the interrogation matrix look like and what do the basis vectors look like. Show how the same matrix can produce each answer individually when the correct basis vector is used as the wave function.

In the case where there are two answers, ‘yes’ and ‘no’ what does the interrogation matrix look like? Suppose an answer is ‘maybe’ and that answer is neither distinct state, but a 50:50 mix of the two, how is that different from either the yes or no states?

Explain how the eigenvalue problem “asks a question” and gives an answer, and the answer is contained in the wave function.