Solving convolution problems

PART I: Using the convolution integral

The convolution integral is the best mathematical representation of the physical process that occurs when an input acts on a linear system to produce an output. If $x(t)$ is the input, $y(t)$ is the output, and $h(t)$ is the unit impulse response of the system, then continuous-time convolution is shown by the following integral.

$$y(t) = x(t) * h(t) \equiv \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)\,d\tau$$

In it, τ is a dummy variable of integration, which disappears after the integral is evaluated.

Example 1: unit step input, unit step response

Let $x(t) = u(t)$ and $h(t) = u(t)$.

$$y(t) = \int_{-\infty}^{+\infty} u(\tau)u(t-\tau)\,d\tau$$

The challenging thing about solving these convolution problems is setting the limits on t and τ. I usually start by setting limits on τ in terms of t, then using that information to set limits on t.

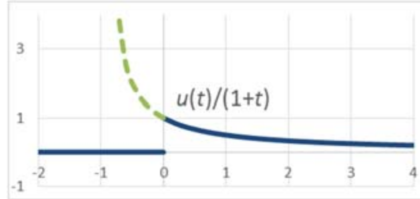
- The unit step function $u(\tau)$ makes the integrand zero for $\tau < 0$, so the lower bound is 0.
- The unit step function $u(t-\tau)$ makes the integral zero for $\tau > t$, so the upper bound is t.
- Once we have used the step functions to determine the limits, we can replace each step function with 1.

$$y(t) = \int_{0}^{t} 1\,d\tau$$

This integral produces $y(t) = t$. However, when we used t to set a limit on τ, we also created a limit on t. In this case, $y(t)$ is zero when $t < 0$, because we have already set $0 < \tau < t$ and there is no τ that satisfies $0 < \tau < 0$. Therefore, the answer to the convolution problem is

$$y(t) = \begin{cases} t, & t > 0 \\ 0, & t < 0 \end{cases} \quad \text{or } y(t) = t\,u(t)$$

A system with $h(t) = u(t)$ is known as an integrator, and can be made from an amplifier with a capacitor in it, or pretty much any system that accumulates an input and does not leak.



Example 2: Unit step input, $1/x$ response

Let $x(t) = u(t)$ and $h(t) = u(t)/(t+1)$. Convolution is commutative, so we can swap the t and $t-\tau$ and write the integral in either of these two ways.

$$y(t) = \int_{-\infty}^{+\infty} u(t-\tau)\frac{u(\tau)}{\tau+1}d\tau = \int_{-\infty}^{+\infty} u(\tau)\frac{u(t-\tau)}{t-\tau+1}d\tau$$

The version on the left looks easier, so let's try it.

- The unit step function $u(\tau)$ makes the integrand zero for $\tau < 0$, so the lower limit is 0.
- The unit step function $u(t-\tau)$ makes the integrand zero for $\tau > t$, so the upper limit is t.
- Once we have used the step functions to determine the limits, we can replace each step function with 1.

$$y(t) = \int_0^t \frac{1}{\tau+1}d\tau$$

This integral produces $y(t) = \ln(t+1)$. However, the fact that t is the upper limit on the range $0 < \tau < t$ means that $y(t)$ is zero when $t < 0$. Therefore, the solution is

$$y(t) = \ln(t+1)u(t).$$

Example 3: pulse input, unit step response.

Let $x(t) = u(t) - u(t-2)$, $h(t) = u(t)$.

$$y(t) = \int_{-\infty}^{+\infty}[u(\tau)-u(\tau-2)]u(t-\tau)d\tau$$

The integrand is zero when $\tau < 0$ and $\tau > 2$, so that *at most* the integrand is non-zero when $0 < \tau < 2$. It is also zero when $\tau > t$. This sets up three intervals for t. First, when $t < 0$ there is no way that $0 < \tau < 2$ and $\tau < t$. Therefore, for $t < 0$, $y(t) = 0$. Next, when $0 < t < 2$, the integrand is 1 when $0 < \tau < t$, making 0 and t the limits of integration. Finally, when $t > 2$, the value of t does not matter any more and the limits of integration are 0 and 2. Thus:

$$t<0 \qquad y(t)=0$$

$$0<t<2 \qquad y(t)=\int_0^t 1\,d\tau = t$$

$$2<t \qquad y(t)=\int_0^2 1\,d\tau = 2$$

Problem 1:
Use the convolution integral to find the convolution result $y(t) = u(t) * \exp(-t)u(t)$, where $x*h$ represents the convolution of x and h.

PART II: Using the convolution sum

The convolution summation is the way we represent the convolution operation for sampled signals. If x(n) is the input, y(n) is the output, and h(n) is the unit impulse response of the system, then discrete- time convolution is shown by the following summation.

$$y(n) = \sum_{k=-\infty}^{+\infty} x(k)h(n-k)$$

In it, k is a dummy variable, which disappears when the summation is evaluated.

Discrete signals or functions are often sequences of numbers that are pretty easy to write in a table, but are not easy to write as a function. A good example is a sequence { **1** –1 }, i.e. $h(0) = 1$, $h(1) = -1$, and h(n) = 0 everywhere else. The bold number indicates where n=0. This could be written as the sum of three step functions, or as two step functions times a cosine, or something else complicated, or it could just be written { 1, –1 }. Therefore, it makes the most sense to convolve these signals in a table or graphically rather than as functions in a sum. A summation with step functions is shown in example 1; a summation with a table is shown in example 2, and a graphical example is shown in an older version of the Lecture 9 slides (see link in the course calendar).

Example 1: unit step input, unit step response
Let x(n) = u(n) and h(n) = u(n).

$$y(n) = \sum_{k=-\infty}^{+\infty} u(k)\, u(n-k)$$

As with the continuous transform, start by setting the limits on n and k. For $k < 0$ and for $k > n$, $u(k) = 0$. Therefore, the summation has a lower limit of 0 and an upper limit of n. For $0 < k < n$, the summand is one.

$$y(n) = \sum_{k=0}^{n} 1$$

Of course, when $n < 0$ then the limits do not permit any summing at all, so $y(n) = 0$. When $n \geq 0$, the sum is n. Therefore, the answer is $y(n) = n \cdot u(n)$, which is a ramp beginning at 0,0.

Example 2: input is {1, 1} pulse, response is {+1, −1}

Remember we are summing over k. The input is zero for $k < 0$, so the lower limit is 0. The system response is zero for $k < n$, so the upper limit is n, and the output is zero for $n < 0$. Therefore we start summing for $n = 0$.

n	***k*=0**	***k*=1**	**sum**
0	x(0)h(0–0) = 1*1 = 1	x(1)h(0–1) =1*0 = 0	1
1	x(0)h(1–0) = 1*(−1) = −1	x(1)h(1-1) =1*1=1	0
2	x(0)h(2–0) = 1*0 = 0	x(1)h(2-1) = 1*(−1) = −1	−1
3	x(0)h(3–0) = 1*0 = 0	x(1)h(3-1) = 1*0 = 0	0

The result is that $y(n)$ = { **1** 0 −1 }, starting at n=0.

Problem 2.

Find the output $y(n) = x(n)*h(n)$, where $x(n)$ = { **1**, 1 } and $h(n)$ = { **3**, 2, 1 }.

Both $x(n)$ and $h(n)$ are zero for $n < 0$, and the bold number shows where $n = 0$.

You may use either a table or the graphical method for finding $y(n)$.