Name Key

Bioen 326 2013 FINAL EXAM

Rules: Closed Book Exam: Please put away all notes and electronic devices

Reminders: We give partial credit, so include equations and steps.

Equations Provided on the Exam:

- $F_R = \int F_d(x)dx$ or $F_R = \int F_d(\vec{r})d\vec{r}$
- $x_R = \frac{\int xF_d(x)dx}{F_R}$, or $x_R = \frac{\int xF_d(\vec{r})d\vec{r}}{F_R}$
- $\sigma_{av} = \frac{\sigma_x+\sigma_y}{2}$
- $R = \sqrt{\left(\frac{\sigma_x-\sigma_y}{2}\right)^2 + \tau^2}$
- $\sigma_{x\theta} = \sigma_{av} + \frac{\sigma_x-\sigma_y}{2}\cos(2\theta) + \tau\sin(2\theta)$
- $\tau_\theta = -\frac{\sigma_x-\sigma_y}{2}\sin(2\theta) + \tau\cos(2\theta)$
- $\sigma_{1,2} = \sigma_{av} \pm R$
- $\tau_{MAX} = R$
- $\epsilon_x = \frac{1}{E}\sigma_x - \frac{\nu}{E}\sigma_y - \frac{\nu}{E}\sigma_z$
- $\gamma_{xy} = \frac{1}{G}\tau_{xy} = \frac{2(1+\nu)}{E}\tau_{xy}$
- $\sigma_x = \frac{E}{(1+\nu)(1-2\nu)}((1-\nu)\epsilon_x + \nu\epsilon_y + \nu\epsilon_z)$
- $\tau_{xy} = G\gamma_{xy} = \frac{E}{2(1+\nu)}\gamma_{xy}$
- $e = \frac{\Delta V}{V_0} = \epsilon_x + \epsilon_y + \epsilon_z = \frac{1-2\nu}{E}(\sigma_x + \sigma_y + \sigma_z)$
- $\delta = \int_0^L \frac{N(x)}{E(x)A(x)}dx$
- $k_a = \frac{EA}{L}; F = k_a\delta$
- $k_t = \frac{GI_p}{L}; T = k_t\phi$
- $\int_A y^2 dA = I$
- $I = \frac{\pi}{4}r^4$
- $I = \frac{H^3W}{12}$
- $I = \frac{H^3W}{36}$, (neutral plane is at H/3 from flat edge).
- $\int_A r^2 dA = I_p$
- $I_p = \frac{\pi}{2}r^4$
- $\kappa = M/EI$
- $\sigma_x = N/A$
- $\tau(r) = \frac{T}{I_p}(r)$
- $\sigma_x(x,y) = -\frac{M(x)}{I}y$
- $\tau_{xy}(x,y) = \frac{V(x)}{2I}\left(\frac{H^2}{4} - y^2\right)$, $\max\left(\tau_{xy}(x)\right) = \frac{3V(x)}{2A}$
- $\max\left(\tau_{xy}(x)\right) = \frac{4V(x)}{3A}$
- $\max\left(\tau_{xy}(x)\right) = \frac{4V(x)}{3A}\frac{(r_2^2+r_2r_1+r_1^2)}{r_2^2+r_1^2}$
- $\frac{d^2v}{dx^2} = \frac{M(x)}{EI}$
- $\sigma = \sigma_L = \frac{pr}{2t}$
- $\sigma_C = \frac{pr}{t}$
- $F_c = \frac{EI\pi^2}{L^2}$ (2 pins); $F_c = \frac{EI\pi^2}{4L^2}$ (solid support); $F_c \cong \frac{2EI\pi^2}{L^2}$ (pin and solid support); $F_c = \frac{4EI\pi^2}{L^2}$ (2 solid supports);
- $r = \sqrt{I/A}$; L/r is slenderness ratio
- Voigt $E\epsilon + \eta\frac{d\epsilon}{dt} = \sigma$
- Maxwell $\sigma + \frac{\eta}{E}\frac{d\sigma}{dt} = \eta\frac{d\epsilon}{dt}$
- Kelvin $\sigma + \frac{\eta}{E_S}\frac{d\sigma}{dt} = E_P\epsilon + \eta\left(\frac{E_P+E_S}{E_S}\right)\frac{d\epsilon}{dt}$
- Transforms:
  - $L[\delta(t)] = 1$
  - $L[\phi(t)] = 1/s$
  - $L[t \cdot \phi(t)] = 1/s^2$
  - $L[e^{-at}\phi(t)] = \frac{1}{s+a}$
  - $L[(1-e^{-at})\emptyset(t)] = \frac{a}{s(s+a)}$
  - $L[\sin(\omega t)\phi(t)] = \frac{\omega}{s^2+\omega^2}$
  - $L[\cos(\omega t)\phi(t)] = \frac{s}{s^2+\omega^2}$
- $k_BT = 4.1e-21\,J$ at room temp.
- $f(r) = \frac{k_BT}{L_p}\left(\frac{1}{4\left(1-\frac{r}{L_0}\right)^2} - \frac{1}{4} + \frac{r}{L_0}\right)$
- $f(r) \approx \frac{3k_BT}{2L_p}\frac{r}{L_0}$
- $r_{rms} = \sqrt{2L_0L_p}$
- $E = \frac{\frac{d^2U}{dr^2}(r_0)}{r_0}$
- $U_0[(r_0/r)^{12} - 2(r_0/r)^6]$
- $K_{eq} = \frac{P_2}{P_1} = \exp\left(\frac{-\Delta G^0}{k_BT}\right)$
- $P_2 = \frac{K_{eq}}{1+K_{eq}}$
- $\Delta G(f) = \Delta G^0 - f \cdot \Delta x(f)$
- $\Delta G(f_{eq}) = 0$
- $\Delta x(f) = x_2(f) - x_1(f)$
- $\Delta x_{1t}(f) = x_t(f) - x_1(f)$
- $K_{eq}(f) = K_{eq}^0 \exp\left(\frac{f\Delta x(f)}{k_BT}\right)$
- $k_{12}(f) = k_{12}^0 \exp\left(\frac{f\Delta x_{1t}^0(f)}{k_BT}\right)$

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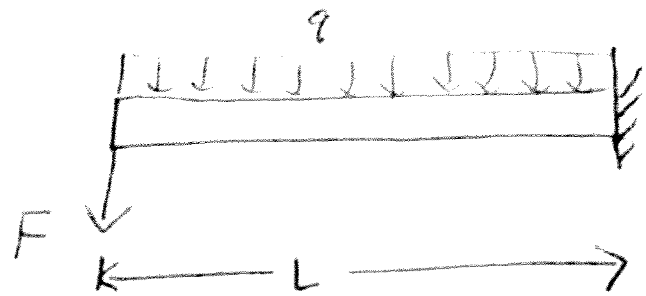
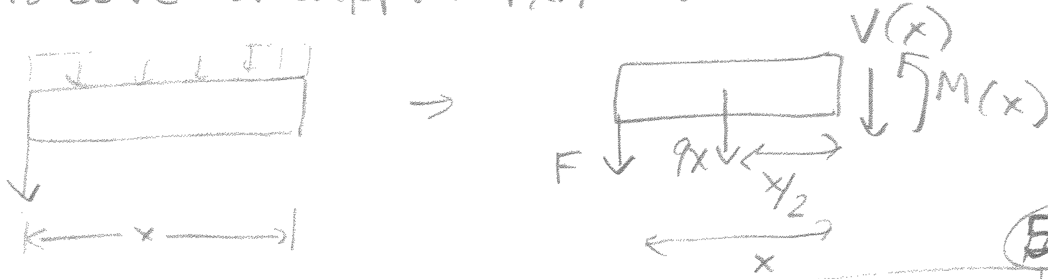
1. **(30 points)** A beam of length L is subjected to a force F as shown in the diagram, and to a uniform load $q = F/L$. The cross section of the beam is rectangular with height H and width W. The beam fails at USS = U, UCS = 2U, and UTS = 3U. What is the maximum force F that the beam can withstand before failure? Assume small deflections, and the beam is slender; that is, L >> H,W.



Need to find M(x) and V(x) to find σ_x & τ, then stress analysis

Don't need to solve for support rxn if cut:



5pts

$M(x) + qx\frac{x}{2} + xF = 0$; $M(x) = -Fx - qx^2/2 = \boxed{-Fx - \frac{F}{L}\frac{x^2}{2}}$

$-V(x) - qx - F = 0$, $V(x) = -F - qx = -F - \frac{F}{L}x$

All terms negative, so extreme value at largest x : L

$M(L) = -FL - \frac{F}{L}\frac{L^2}{2} = -FL - FL/2 = \boxed{-\frac{3}{2}FL}$ ← 5pts!

$V(L) = -F - \frac{F}{L}L = -2F$

5pts

$\max/\min(\sigma_x) = -\frac{M}{I}y = \pm\frac{3}{2}FL \cdot \frac{H}{2}\frac{1}{I}$; $I = \frac{H^3W}{12}$; $\boxed{\sigma_x = \pm\frac{3}{4}\frac{FLH\,12}{H^3W}}$

5pts

$\sigma_x = \pm 9\frac{FL}{H^2W}$; $\sigma_x = 3\frac{L}{H}\tau$, & $\boxed{L >> H, \text{ so } \sigma_x >> \tau}$

$\max(\tau) = \frac{3V}{2A} = \frac{6F}{2HW} = 3\frac{F}{HW}$

$\boxed{\tau_{max} = \frac{\sigma_x}{2} = \frac{9}{2}\frac{FL}{H^2W}}$ 5pts

Fails when $\frac{9}{2}\frac{FL}{H^2W} = U$; or $\frac{9FL}{H^2W} = 2U$ or $\cancel{3U}$.

$F = \frac{2UH^2W}{9L}$ from shear or compression.

5pts

5pts for M(x)
5pts for max(M/x) = M(L) = $\frac{3}{2}$
5pts for σ_x = correct
5pts to exclude τ
5pt for $\tau_{max} = \sigma_x/2$ = correct

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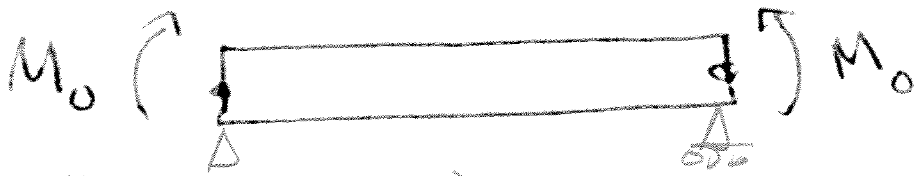
2. (30 points) A beam has length L, Young's modulus E, and a circular cross-section with radius r. What is the maximum deflection, v, of the beam in the situation below, and at what position x does this deflection occur? (you can assume small deflections)



for deflection, need $M(x)$.

5 pts for $M(x)$

$M(x) = M_0$, $V(x) = 0$ for all x. (Don't need FBD to see this.)

3 pts for $\frac{d^2v}{dx^2}$

$$\frac{d^2v}{dx^2} = \frac{M(x)}{EI} = \frac{M_0}{EI}$$

3 pts for first integration

$$\frac{dv}{dx} = \frac{M_0}{EI}x + C_1$$

(by symmetry, $\frac{dv}{dx}\left(\frac{L}{2}\right) = 0$,) 4 pts for assumption 1 & solve for C_1

so $\frac{M_0}{EI}\frac{L}{2} + C_1 = 0$; $C_1 = -\frac{M_0 L}{2EI}$

$$\frac{dv}{dx} = \frac{M_0}{EI}x - \frac{M_0 L}{2EI}$$

$$v(x) = \frac{M_0}{EI}\frac{x^2}{2} - \frac{M_0 L}{2EI}x + C_2$$

; $\boxed{v(x) = 0}$ since ~~beam~~ hinges (4 pts for assumption 2)

$\Rightarrow C_2 = 0$.

3 pts for 2nd integration

$$v(x) = \frac{M_0}{2EI}(x^2 - xL).$$

This is max where $\frac{dv}{dx} = 0$, which was @ $\frac{L}{2}$. 5 pts

$$v\left(\frac{L}{2}\right) = \frac{M_0}{2EI}\left(\frac{L^2}{4} - \frac{L^2}{2}\right) = +\frac{M_0}{2EI}\left(-\frac{L^2}{4}\right) = -\frac{M_0 L^2}{8EI}$$

$I = \frac{\pi r^4}{4}$, so $v\left(\frac{L}{2}\right) = \frac{4 M_0 L^2}{\pi 8 E r^4} = \boxed{-\frac{M_0 L^2}{2\pi E r^4}}$ $\frac{N \cdot m \cdot m^2}{N/m^2 \cdot m^4}$ is m ✓ since distance

3 pts

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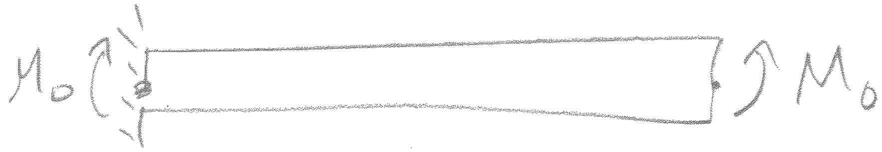
See alternative on page 1

Problem 2: assume deflection is relative to position & orient. of one end.



Still get $\frac{d^2v}{dx^2} = \frac{M_0}{EI}$,

$\frac{dv}{dx} = \frac{M_0}{EI}x + C_1$ but here this $= 0$,

so $C_1 = 0$.

$\frac{dv}{dx} = \frac{M_0}{EI}x$

$v = \frac{M_0^2}{EI}\frac{x^2}{2} + C_2$. this is also 0,

so

$v(x) = \frac{M_0^2}{EI}\frac{x^2}{2}$. max @ $x = L$

$v(L) = \frac{M_0^2 L^2}{2EI}$. $I \frac{\pi R^4}{4}$,

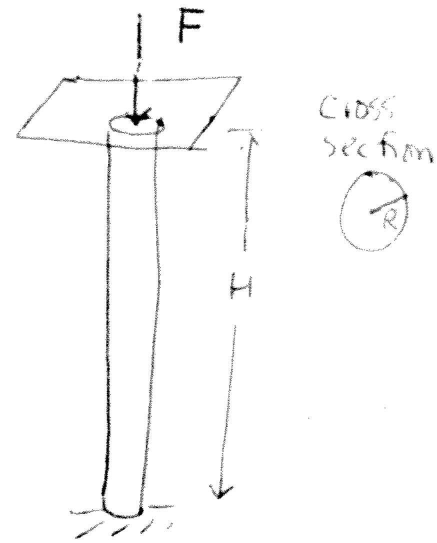
so $V_{max} = \frac{4M_0^3 L^2}{\pi r^4 2EI} = \boxed{\frac{2M_0 L^2}{\pi E r^4}}$

Problem 2 point summary

5 pts to find $M(x) = M_0$
3 to find d^2v/dx^2
7 to find $\frac{dv}{dx}$: 3 integrate 4 for assumption, C_1, & complete.
7 to find $v(x)$
5 to find $\max(v(x))$
3 to plug in with I & ~~sol~~ combine.

continued on page ______________________

3. **(10 points).** A solid metal pole with a circular cross section of radius R is anchored upright into the ground. The top of the pole is at a height H above the ground and has a small platform. A weight causing a downward force, F, is placed on the platform directly above the pole. What is the highest allowable weight without buckling the pole? & $\underline{E}$ = young's modulus.



This is one solid support only) so

$$F_c = \frac{EI\pi^2}{4L^2}$$ (5) for right F_c equation.

here, $I = \frac{\pi R^4}{4}$ (2); $L = H$ (2)

so $\boxed{F_c = \frac{E\pi^3 R^4}{16H^2}}$ (1)

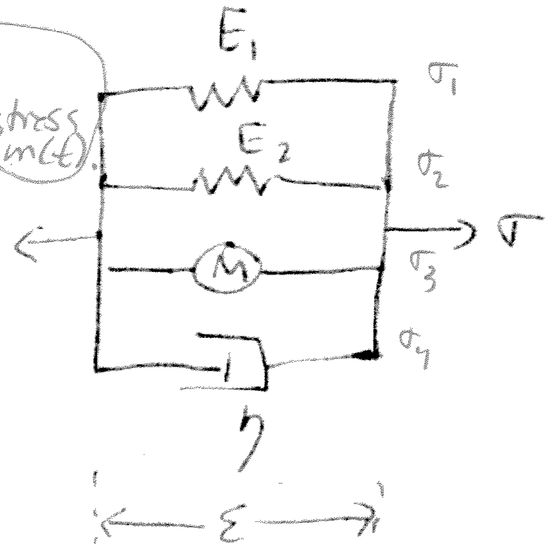
- 5 for realizing 1 solid support (2 for F_c equation, but wrong one)
- 2 for right I
- 2 for $L = H$
- 1 for combine.

continued on page ______________________________

4. **(40 points).**

15 a) Derive a differential equation (ODE) relating stress, σ, to strain, ϵ, for the muscle model diagramed here. (Also muscle creates tensile stress m(t))

25 b) ~~If the muscle creates a tensile stress of $m(t) =$~~ $M\phi(t)$, the material is at rest before the test starts, and there is no external stress on the material ($\sigma(t) = 0$), then what is the strain, $\epsilon(t)$?



(a) I want σ & ε; remove $\sigma_1 \to \sigma_4$
so need 5 Eqns.

5 for elements
$\sigma_1 = E_1 \varepsilon$
$\sigma_2 = E_2 \varepsilon$
$\sigma_3 = m(t)$
$\sigma_4 = \eta \frac{d\varepsilon}{dt}$

5 for $\to \sigma = \sum \sigma_i$

$$\boxed{\sigma = (E_1 + E_2)\varepsilon + m(t) + \eta \frac{d\varepsilon}{dt}}$$

5 to combine.

(b) $\overset{m(t)=}{M\phi(t)}$. IC = 0, $\sigma(t) = 0$; find $\varepsilon(t)$.

$$0 = F(s) = (E_1 + E_2)X + \frac{M}{s} + \eta s X$$

$$0 = (E_1 + E_2 + \eta s)X + \frac{M}{s}$$

$$X(E_1 + E_2 + \eta s) = -\frac{M}{s}$$

$$X = -\frac{M}{s(E_1 + E_2 + \eta s)} \cdot \frac{1/\eta}{1/\eta} = -\frac{M/\eta}{s\left(\frac{E_1 + E_2}{\eta} + s\right)}$$

5 to transform ODE
5 to transform σ & $m(t)$

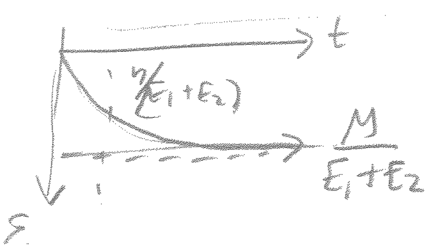
10 simplify $\to$

$$X = \frac{-M}{E_1 + E_2} \cdot \frac{\frac{E_1 + E_2}{\eta}}{s\left(s + \frac{E_1 + E_2}{\eta}\right)}$$

5 invert correctly

Invert: $\boxed{\varepsilon(t) = -\frac{M}{E_1 + E_2}\left(1 - e^{-\frac{E_1 + E_2}{\eta}t}\right)}$

✓: units good. makes sense $\to$



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5. **(20 points; molecular biomechanics)**. Several people have developed a fluorescent force sensing domain that can be genetically introduced into proteins to detect the amount of force applied to the proteins in vivo. The domain has two states (A and B) with different fluorescent spectra. You design such a domain in which state A and state B have the same length with no force ($x_A^0 = x_B^0$), but state A is stiffer ($\kappa_A > \kappa_B$). State A has energy $G_A^0 = 0$, and state B has energy $G_B^0 > 0$. **What is the equilibrium force, f_{eq}, at which the domain is equally likely to be in either state?**

$\Delta G(f_{eq}) = 0$. 5

$\Delta G(f) = \Delta G^0 - f\,\Delta x(f)$ so find f to solve that.

$\Delta x(f) = x_B(f) - x_A(f)$ 3, & $\Delta G^0 = G_B^0 - G_A^0 = G_B^0$ 3

(or could define both as opposite; will get same answer.)

$x_B(f) = x_B^0 + f/\kappa_B$
$x_A(f) = x_A^0 + f/\kappa_A$ } by definition of spring constant. 3

so $\Delta x(f) = x_B^0 - x_A^0 + f\left(\frac{1}{\kappa_B} - \frac{1}{\kappa_A}\right)$. +1 to combine.

$x_B^0 = x_A^0$, so $\boxed{\Delta x(f) = f\left(\frac{1}{\kappa_B} - \frac{1}{\kappa_A}\right)}$

so need $G_B^0 - f \cdot f\left(\frac{1}{\kappa_B} - \frac{1}{\kappa_A}\right) = 0$ 3 to combine

$f^2\left(\frac{1}{\kappa_B} - \frac{1}{\kappa_A}\right) = G_B^0$

$f^2 = \dfrac{G_B^0}{\frac{1}{\kappa_B} - \frac{1}{\kappa_A}}$; $\boxed{f_{eq} = \sqrt{\dfrac{G_B^0}{\frac{1}{\kappa_B} - \frac{1}{\kappa_A}}}}$

or $f_{eq} = \sqrt{\dfrac{G_B^0\,\kappa_B\kappa_A}{\kappa_A - \kappa_B}}$

9 for $\Delta x(f)$
3 for ΔG^0
5 for $\Delta G^0 = f\,\Delta x(f)$
3 to combine

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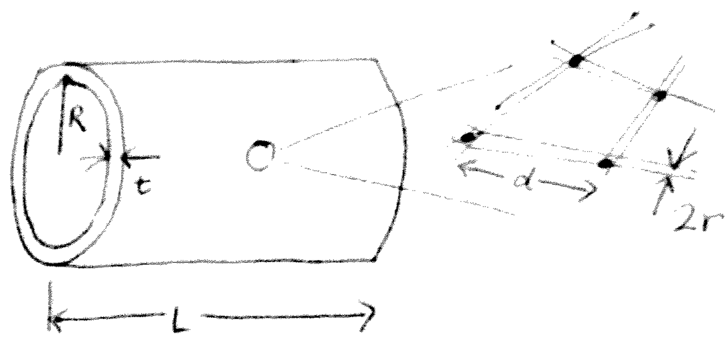
6. **(40 points)** You can cast a hydrogel scaffold of cross-linked collagen fibers, where the individual fibers have a circular cross-section of radius $r = 0.1\,\mu m$ and an average distance $d_{av} = 10\,\mu m$ between cross-links. The Young's modulus of collagen fibers is $E_C = 1\,GPa$. You cast this scaffold into a cylindrical tube that is L = 100 mm long, R = 10 mm in diameter, with t = 1 mm thick walls, clip the ends, and then pump liquid inside. How much pressure P should you use to pressurize the cylindrical tube if you want to cause an estimated 10% increase in radius of the cylinder? (*Hint: since we said 'estimated', you can make approximations if they are appropriate to this problem.*)



assume $\nu = 0$

To get estimated 10% increase in radius, I can neglect effects of change in R & t on σ_c, σ_L. (or see p. 8)

so $\sigma_c = \frac{PR}{t}$ ⑤, $\sigma_L = \frac{PR}{2t}$.

5 pts to get $\varepsilon_c = \sigma_c/E$

$\varepsilon_c = \frac{1}{E}\sigma_c - \frac{\nu}{E}\sigma_L$, but $\nu = 0$, so

5 pts for σ_c

5 pts for $P = \frac{\varepsilon_c E t}{R}$ or $\frac{\varepsilon_c E t}{1+\varepsilon_c R}$

$\boxed{\varepsilon_c = \frac{1}{E}\sigma_c}$ ⑤ $= \frac{PR}{Et}$; $P = \frac{\varepsilon_c E t}{R}$ ⑤

But we are not given E of material, just E of fibers.

$E = \frac{\frac{d^2U}{dr^2}(r_0)}{r_0}$ ③ here $r_0 = d_{av} = 10\,\mu m$. ⑤

assume rod, not WLC, since $r/d_{av} = 1/100$. See page 8

8 pts for $E = k_a/r_0$.

⑤ $\frac{d^2U}{dr^2}(r_0) = k_a$, axial spring constant of fiber. {since $U = \frac{1}{2}k(r_0 - r)^2$ for spring)

5 pts to get k_a

$k_a = \frac{E_c A}{L} = \frac{E_c \pi r^2}{d_{av}}$ ⑤ so $E = \frac{E_c \pi r^2}{d_{av}^2}$ ②

5 pts for $r_0 = d_{av}$

2 for combine for E

combine to get: $\boxed{P = \frac{0.1 \cdot E_c \pi r^2 t}{R\, d_{av}^2}}$ ② $= 0.1\pi \frac{t}{R}\left(\frac{r}{d_{av}}\right)^2 E_c$

2 pts to combine, 2 plug in #s, ① to simplify

$= 0.1 \cdot 3.14 \cdot 10^9\,Pa \cdot \left(\frac{1}{10}\right)\left(\frac{0.1}{10}\right)^2$ ② $= 3.14 \cdot 10^8 \cdot 10^{-1} \cdot 10^{-4}$

$= 3.14 \cdot 10^3 \approx \boxed{3\,kPa}$ P ①

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See notes on p. 8

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Problem 6, cont.

you might instead use a more accurate version:

$$\sigma_\theta = \frac{PR(\sigma_c)}{t(\sigma_c)} = \frac{PR(1+\varepsilon_c)}{t} \text{ for } \ldots$$

$\sigma_c = E\,\varepsilon_c$ for $\nu = 0$, so $\varepsilon_c = \frac{PR}{Et}(1+\varepsilon_c)$

$$\frac{\varepsilon_c}{1+\varepsilon_c} = \frac{PR}{Et} \; ; \; P = \frac{Et}{R}\left(\frac{\varepsilon_c}{1+\varepsilon_c}\right)$$

$$P = \frac{Et \cdot \left(\frac{0.1}{1.1}\right)}{R}$$

at the end, will have $\boxed{P = \frac{3.14}{181} \times 10^3}$

for a more exact answer.

also, may have wanted to check that collagen fiber is a rod, not a WLC, over dav.

$\frac{1}{L_p} = \frac{k_B T}{E_c I}$. $E_c = 10^9\,Pa$

$I = \frac{\pi}{4} \cdot r^4 \sim r^4 = (0.1E-6)^4$:

$E-7 \times 4 = E-28$

$$\frac{1}{L_p} = \frac{4.1E-21\ N \cdot m}{10^9 N/m^2 \cdot E^{-28}\ m^4} = \frac{4.1E-21}{E-19\ m}$$

$L_p = \frac{1}{4}\frac{E-19}{E-21}\ m = 0.25E-2\ m = 2.5\ mm$

$\gg 10\,\mu m$

$= dav.$

so yes, justified to not use WLC.

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7. (30 points) In the abstract of the del Rio paper (attached after this), the authors state that "**Application of physiologically relevant forces caused stretching of single talin rods that exposed cryptic binding sites for vinculin.**" Analyze the appropriate level of certainty of this statement <u>if they had only performed the magnetic tweezers experiments (Figure 3 and the accompanying text)</u>.

Note that we are saving time in your answer by not requiring the following: You do not need to discuss the atomic force microscopy experiment in Figure 4, nor the author's asserted level of certainty.

In fig 3, they show that # of photobleaching increases when they turn on magnet. To get that this is due to Talin stretching, the have negative control w/o Talin. Vinculin is only fluorescent thing, but they don't show that binding is specific to vinculin (missing control)
they don't do statistical analysis. (But do have 2 expts, with single & double talins so maybe don't need;
they don't support that 12pN is physiological very well - just cite paper showing integrins can withstand 20pN "without breaking".

∴ certainty should only be moderate?
(but argue for high or low OK too; based on how balance ~~[illegible]~~

5 pts - note or analyze statistics ~~& controls~~ reproducibility (mention is ok)
5 pts - note controls (")
5 pts - note whether 12pN is physiological ~~[illegible]~~ "
5 ~~[illegible]~~ pts - ~~[illegible]~~ conclude on level of certainty
10 pts - whole argument makes sense.

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