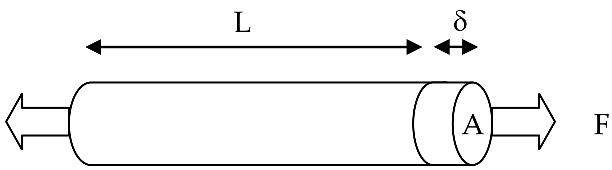
BIOEN 326 LECTURE 3, SEPT 30, 2013

Linear Elasticity (Gere Ch. 1)

When external forces are applied to an object, this creates internal stresses on any point in the object. We will spend much of unit 2 calculating these. However, for some simple situations, the internal stresses are simple to understand and calculate because they are uniform throughout the body, and we will consider these today.

Normal stress and strain

Consider a bar or block to which we apply a tensile or compressive force, F uniformly across opposite faces and **normal** (perpendicular) to the faces, as illustrated in the figure below. The bar has a cross-sectional area, A, and a length L. When the force is applied, the bar stretches or compresses by length δ.



Extrinsic vs Intrinsic

The **extrinsic** properties of this system are F, A, L and δ. These are things that can be directly measured through external observations. We can use these to measure the spring constant of the bar, which is also an extrinsic property: k = F/δ.

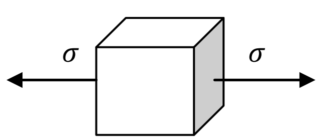
However, if we want to use this measurement to determine a fundamental property of the material making up the bar, we need to calculate intrinsic properties.

Normal Stress

We define the **normal stress** (often just called stress), $\sigma = F/A$.

Note that the units are in $/m^2$, also called a pascal (Pa).

We chose this example carefully so that the stress within the bar is uniform, so the equation above applies to any point in the bar. Thus, if we chose any element within the bar, with arbitrarily small length a in all three dimensions, it will have this same stress σ.



Note that the stress on opposite faces is identical at equilibrium. To understand this and other equivalences in this lecture, we apply the assumptions of equilibrium, which is that all forces and all moments must sum to zero. We can replace the uniform stress on each face with a force $f = a^2\sigma$ at the centroid of that face. To have the sum of forces in the x-direction be zero, we need to have these two forces be equal and opposite in direction, as shown in the figure above.

Sign of stress. By convention, $\sigma < 0$ for compression, and $\sigma > 0$ for tension. You can remember this because the sign reflects the effect the stress with have on the length (e.g. lengthens for tension, $\sigma > 0$, as illustrated in the figure above.) Note that when $\sigma > 0$, the stress points in the positive direction on the positive face, and in the negative direction on the negative face (+/+ and -/-). In contrast, if $\sigma < 0$, you have compression, and stress points in the negative direction on the positive face and vice-versa (-/+ and +/-). So, if you multiply the signs of the direction of the face and the direction of the stress, you get the sign of the stress.

Normal strain

Now we consider the resulting deformation. We define the **normal strain** (often just called strain) on the object to be $\epsilon = \delta/L$. Note that strain is unitless and measures the fractional elongation. In this case of uniform stress, we also have uniform strain, and each element has the same strain.

Hooke's Law.

If the stress and strain on a material are small enough, σ/ϵ remains constant.

The range over which Hooke's Law holds is referred to the regime for **linear elasticity**, and depends on the material. Even nonlinear elastic materials will display linear properties over sufficiently small strains.

We define the **Young's modulus** of elasticity, or **elastic modulus**, to be $E = \sigma/\epsilon$, also in units of Pa. The Young's modulus is the intrinsic form of the spring constant, just as the strain is the intrinsic form of the deformation, and stress of the applied force.

Lateral strain

When we stretch the bar in the x-direction, it usually changes size in the y-direction as well. We define the **lateral strain** $\epsilon' = \Delta W/W$, where ΔW is the change in width in a direction perpendicular to the applied force, and W is the original width. Lateral strain is also unitless. For all but some very strange materials, when $\epsilon > 0$, we observe that $\epsilon' < 0$. The lateral strain thus acts to partially maintain the volume of the material.

Poisson ratio.

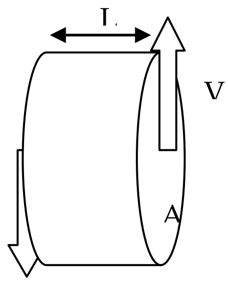
We define the **Poisson ratio** is $\nu = -\epsilon'/\epsilon$, as the ratio of lateral to axial strain, with the negative sign included so that the Poisson ratio is defined to usually be positive.

Both ν and E are intrinsic properties of the material. We will see on Friday and next week how the molecular structure of the materials determines these properties.

When $\nu = 0$, there is no lateral change in size. When $\nu = 0.5$, the material shrinks laterally in a way that maintains a constant volume. Except for very unusual materials, the Poisson ratio is between these two values, and for most materials, it has been observed that ν = ~1/3.

Shear stress and strain

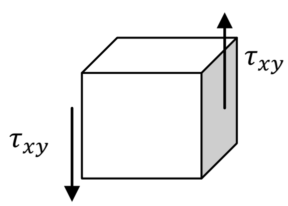
Now consider instead that we apply a force tangential to a face of the block. We call this a **shear** force. We often label a shear force as V instead of F, but you should always consider the geometry rather than the name to differentiate shear from normal forces.



<u>Shear stress</u>

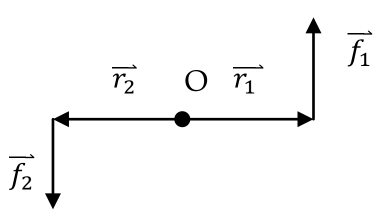
We define the **shear stress** to be the shear force per unit area: $\tau = V/A$. Note that the units are again Pa. Again, we have picked a geometry where the shear stress is uniform throughout the object (as long as the force is distributed uniformly on the faces of the object). Also, there is no normal stress, so we call this a situation of **pure shear**.

We can again consider a small volume element of dimensions a in all directions, anywhere inside the block, and represent the shear stress as a force on the centroid of the face.

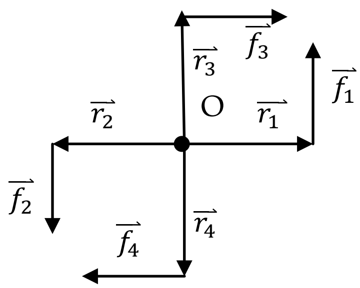


By convention, we call the shear stress acting on the x-face in the y-direction to be τ_{xy}. Again, applying the equilibrium assumptions, we see that the shear stress on opposite faces must be identical to avoid having a net force in the y-direction. Like in the normal stress, a stress in the positive direction on a positive face (+/+) must be balanced by a force in a negative direction on a negative face (-/-), and so we again use the same sign convention and say that in this situation, $\tau_{xy} > 0$. So, if you can remember the logic for the normal stress, and remember that shear stress is identical, you can remember the sign conventions for stress and strain.

Now we ask about the shear stress on the y-face in the x-direction, τ_{yx}, and how it relates to τ_{xy}. To do this, we again use the equilibrium assumptions but now we consider the issue of moments. In the case of normal strain, the force $f = a^2\sigma$, which acted on the centroid of the face, created no moment, since the line of force also went through the centroid of the element, but this is not true for shear stress, so we will sketch the moments. Since everything is in the same plane in the z-direction, I will sketch in the plane instead of in 3D.



This figure shows the direction of the direction and force vectors. Note that $r = a/2$ is the magnitude of both direction vectors, and $f = a^2\tau_{xy}$ is the magnitude of both force vectors. The sign of both moments is clearly positive by the right hand rule, so they do not cancel. Therefore, we need to balance these moments with the forces resulting from the two τ_{yx} shear stresses.

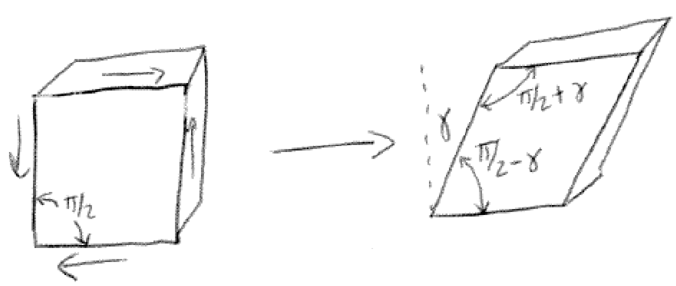


This requires that these two forces also have magnitude $f = |a^2\tau_{xy}|$, so τ_{xy} and τ_{yx} are equal in magnitude. They must point in the direction shown above in order to be negative in sign. Finally, we note that this is applying a positive force to a positive face (+/+) for one and (-/-) for the other, so the sign must be positive. Thus, $\tau_{yx} = \tau_{xy}$.

Note that there are three stresses on each face. For example, on the positive x-face, there are σ_{xx}, τ_{xy}, and τ_{xz}. This makes 18 stresses total. But, at equilibrium, sets of two normal stresses and four shear stresses are all the same, so there are only 6 unique stresses: are σ_{xx}, σ_{yy}, σ_{zz}, τ_{xy}, and τ_{xz} and τ_{yz}, or for planar stress, just σ_{xx}, σ_{yy}, and τ_{xy}. While we derived these observations for cases of pure normal stress and pure shear, it is true for more complicated conditions as well.

Deformation due to shear stress:

In pure shear ($\sigma = 0$), the size of the faces stay the same, but the object changes shape. The square volume element becomes a parallelogram



We define **shear strain** = γ to be the decrease in the angle between two positive faces. Note that when $\tau > 0$, this means $\gamma > 0$.

The shear strain causes a displacement of the corners of the element, which we again refer to as δ. If the volume element has height H, then the displacement is $\delta = H \tan\gamma$. Note that when $\gamma > 0$, the displacement is in the positive direction, so $\delta > 0$ according to both the sketch and the equation above. This same calculation can be used to calculate the displacement over the original body, using the height of that body.

If the shear strain is sufficiently small, we can use the small angle assumption, which is that $\tan\theta \sim \theta$ for $\theta \ll 1$. Thus, $\delta \sim H\gamma$, or $\gamma \sim \delta/H$. This is reminiscent of the equation we saw for normal strain: $\epsilon = \delta/L$.

<u>Hooke's Law in Shear</u>

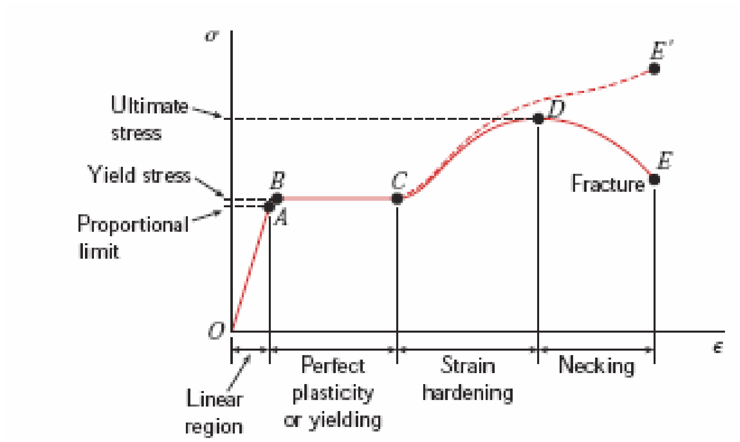
Analogous to the situation for normal strain, the ratio of shear stress to shear strain is constant within the linear range. The constant of proportionality is called the **shear modulus** or the **modulus of rigidity**: $G = \tau/\gamma$ or $\tau = G\gamma$. Note that it is also in Pascal.

The three materials properties, the Young's modulus, Poisson ratio and shear modulus, are related by the following equation, which we accept without proof for now: $G = \frac{E}{2(1+\nu)}$. Note that if $\nu = 0$, then $G = E/2$ and if $\nu = 0.5$, then $G = E/3$. Recall that we said that $0 \leq \nu \leq 0.5$ for almost all materials, so $E/2 \leq G \leq E/3$ for most materials.

<u>Limitations</u>.

Note that this entire discussion has assumed that the material is isotropic, meaning the elastic modulus, shear modulus, and Poisson ratio are all independent of the orientation of the material. If a material is anisotropic, these properties will depend on how force is applied relative to the orientation of fibers or other nanostructures in the material. For example, wood has a different Young's modulus when you stretch it along the grain versus across the grain. In this case, the material also has many different shear moduli, depending on orientation. If you know the shear or elastic modulus for the orientation in question, you can still apply Hooke's law. However, understanding the relationship between E, G, and ν for anisotropic materials is beyond the scope of this class.

Another limitation is that this entire discussion has assumed that the materials are linear, and in fact has defined linear elasticity. However, most materials become softer (**yield**) or harder (**strain hardening**) above the '**proportional limit**'. Other materials are **nonlinear** everywhere, but for sufficiently small deformations, the nonlinearity can be ignored since deviations may be within a few percent. Thus, one needs to determine whether linear elasticity can be assumed before applying equations such as Hook's law. However, many other definitions, such as the definition of normal stress and strain, apply regardless of linearity. We will consider nonlinear materials later. Below is a stress- strain diagram for structural steel showing these behaviors.



Finally, all materials fail above critical values. Different materials fail in different ways. **Brittle** materials are stiff and remain linear until they fail by sudden fracture, while ductile materials undergo an irreversible plastic deformation

The **yield stress** (or strength) of the material is the stress at which the material becomes plastic and yields. The **ultimate stress** (or strength) of a material is the highest stress withstood by a material. The ultimate stress is different for compressive, tensile and shear conditions, so even isotropic materials have three critical values that must be considered when calculating conditions that will cause a device to fail mechanically: the ultimate shear stress (USS), ultimate tensile stress (UTS) and ultimate compressive stress (UCS).

An important skill for a bioengineer working with materials is thus to calculate internal stresses from external loads, and to compare these to USS, UTS and UCS to determine what level of external loads will cause material failure. Of course, we then use working conditions significantly below these levels to ensure that the device does not fail. The working