STRESS ANALYSIS (Ch. 6, Gere)

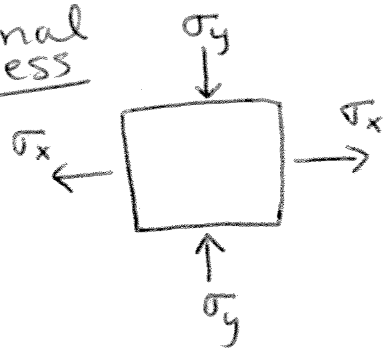
(Gere 6.2) Transformation of Plane stress.

Plane stress = no stress in z ($\sigma_z = \tau_{zy} = \tau_{zx} = 0$)

So just σ_x, σ_y, τ_{xy}, which we will call τ here.

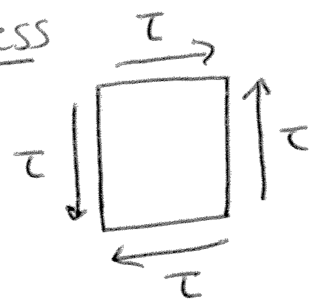
Recall from week 1 we defined normal & shear stress:

normal stress



here, $\sigma_x > 0 \Rightarrow$ tension
(+/+ & −/−)
$\sigma_y < 0 \Rightarrow$ compression
(+/− & −/+)

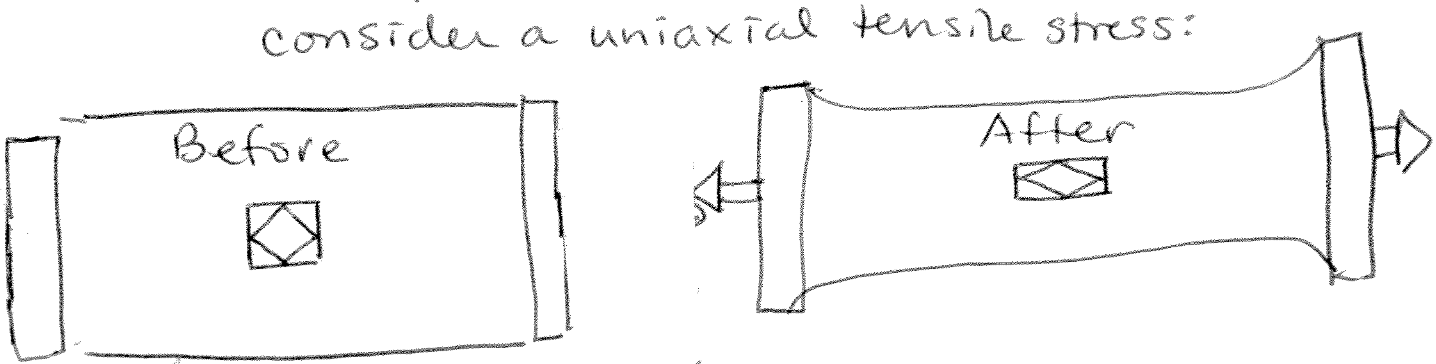
shear stress



here, $\tau > 0$
since +/+ & −/−

But what happens if you consider elements in the same position that are rotated?

consider a uniaxial tensile stress:



□ & ◇ are rotated 45° (& different size)

Regardless of how we defined the element, it is under the same stress conditions & responds with the same strain.

Thus, □ → ▭, which we called <u>normal stress</u>,

& ◇ → ⬦, which clearly includes some <u>shear stress</u>,

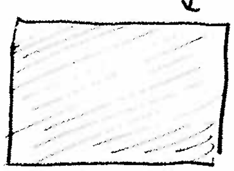
Are the same.

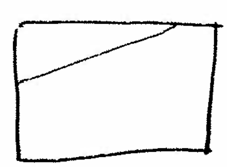
We refer to the "state of stress" as the physical condition.

But our <u>representation</u> of this state depends on our <u>coordinate system</u>.

By rotating the coordinate system by 45°, we <u>transformed</u> the pure normal stress to something that includes shear stress.

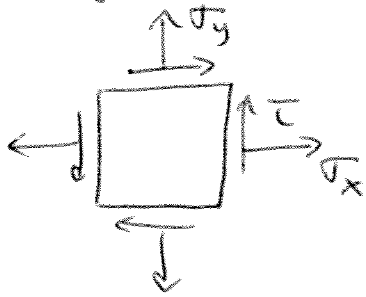
We can use a transformation when the material has an inherent angle. (e.g. <u>anisotropic</u> material, or material with a <u>seam</u>.)

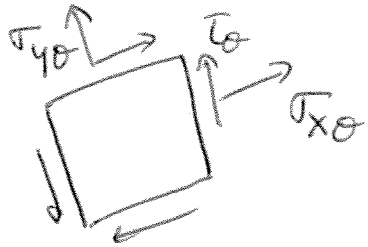


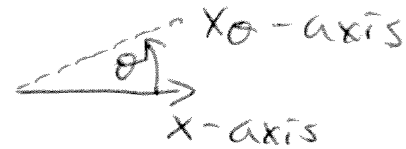


The transformations also allow us to identify maximum & minimum normal stress & shear stress, which are fundamental properties of the state of stress, independent of coordinates. We need these to predict when materials will fail.

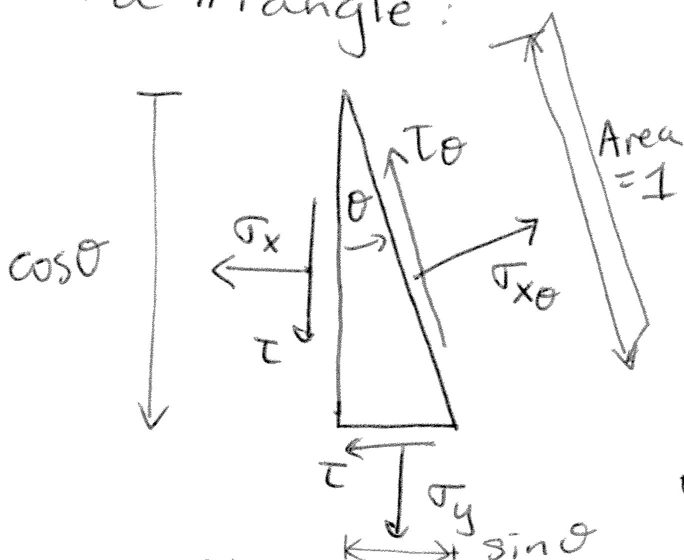
Derive the transformation equations to get $\sigma_{x\theta}$, $\sigma_{y\theta}$, τ_θ in terms of σ_x, σ_y, τ, and θ:







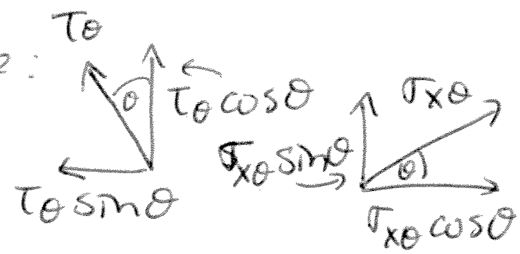
We do this by using Equilibrium Equations for a triangle:



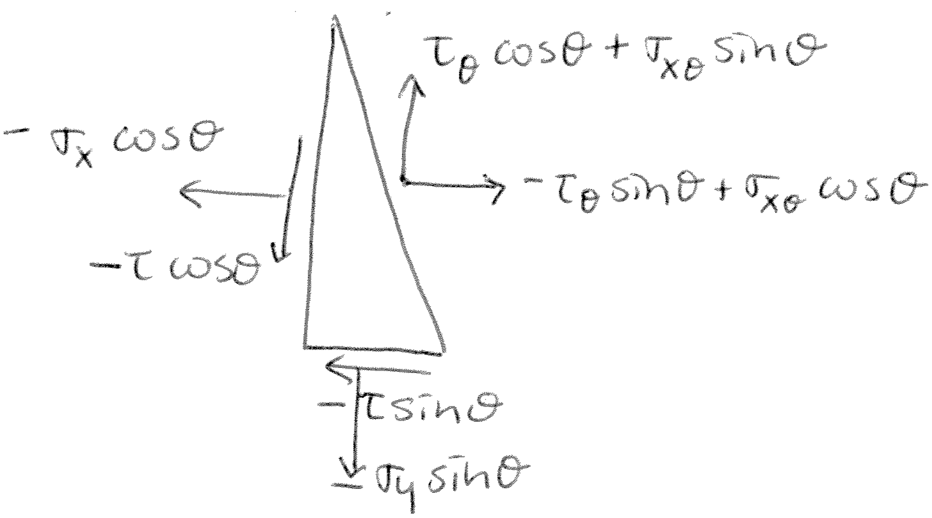
define units so that area of x_θ-face = 1
then area of ~~x~~ y-face = $\sin\theta$
area of ~~y~~ x-face = $\cos\theta$

(We need forces to do force balance)
using $f = \sigma \cdot A$ & $V = \tau \cdot A$.

We will also need to decompose:



Thus force balance:



Thus we have:

$$\Sigma F_x = 0 = -\sigma_x \cos\theta - \tau \sin\theta - \tau_\theta \sin\theta + \sigma_{x\theta} \cos\theta$$

$$\Sigma F_y = 0 = -\sigma_y \sin\theta - \tau \cos\theta + \tau_\theta \cos\theta + \sigma_{x\theta} \sin\theta$$

combine to get τ_θ & $\sigma_{x\theta}$ on one side, & prepare to cancel

$\Sigma F_x * \cos\theta$: $\sigma_x \cos^2\theta + \tau \sin\theta\cos\theta = -\tau_\theta \sin\theta\cos\theta + \sigma_{x\theta}\cos^2\theta$

$\Sigma F_y * \sin\theta$: $+\ \underline{\sigma_y \sin^2\theta + \tau \sin\theta\cos\theta = \tau_\theta \sin\theta\cos\theta + \sigma_{x\theta}\sin^2\theta}$

$$\sigma_x \cos^2\theta + \sigma_y \sin^2\theta + 2\tau \sin\theta\cos\theta = \sigma_{x\theta}$$

Similarly, use $\Sigma F_x * \sin\theta$ & $\Sigma F_y * (-\cos\theta)$ to cancel $\sigma_{x\theta}$ and keep τ_θ, to get expression for τ_θ

Then, substitute $\sigma_{y\theta} = \sigma_{x(\theta+90°)}$ to get expression for $\sigma_{y\theta}$.

But first, use trig identities:

$\cos^2\theta = \frac{1}{2}(1+\cos 2\theta)$,

$\sin^2\theta = \frac{1}{2}(1-\cos 2\theta)$

$\sin\theta\cos\theta = \frac{1}{2}\sin 2\theta$

so $\sigma_{x\theta} = \frac{1}{2}\sigma_x(1+\cos 2\theta) + \frac{1}{2}\sigma_y(1-\cos 2\theta) + \tau \sin 2\theta$

$$\boxed{\begin{aligned} \sigma_{x\theta} &= \frac{\sigma_x+\sigma_y}{2} + \frac{\sigma_x-\sigma_y}{2}\cos 2\theta + \tau\sin 2\theta \\ \sigma_{y\theta} &= \frac{\sigma_x+\sigma_y}{2} - \frac{\sigma_x-\sigma_y}{2}\cos 2\theta - \tau\sin 2\theta \\ \tau_\theta &= -\frac{\sigma_x-\sigma_y}{2}\sin 2\theta + \tau\cos 2\theta \end{aligned}}$$

Transformation Equations = Tx Eqs

Sum of normal stresses:

We see immediately that $\sigma_{x\theta} + \sigma_{y\theta} = \sigma_x + \sigma_y$.

∴ Regardless of coordinate (θ), normal stress sum is constant.

Principal Stresses

$\sigma_{x\theta}$, $\sigma_{y\theta}$, & τ_θ oscillate with θ.

We are interested in their max & min values.

since $\sigma_{y\theta} = \sigma_{x(\theta+90)}$, we know $\sigma_{x\theta}$ & $\sigma_{y\theta}$ have same min & max values. We call these principal stresses.

We find them by setting $\frac{d\sigma_{x\theta}}{d\theta} = 0$.

$$\frac{d\sigma_{x\theta}}{d\theta} = 0 = \frac{\sigma_x - \sigma_y}{2}(-2\sin 2\theta) + \tau(2\cos 2\theta)$$

rearrange: $\tan 2\theta = \frac{2\tau}{\sigma_x - \sigma_y}$

There are many θ that satisfy this.

$\tan\theta$ repeats every $180°$, so $\tan 2\theta$ every $90°$.

The principal angles are θ_p between 0 & 90, & between 90 & 180, that satisfy $\boxed{\theta_p = \frac{1}{2}\tan^{-1}\left(\frac{2\tau}{\sigma_x - \sigma_y}\right)}$

Now we calculate $\sigma_{x\theta}$, and τ_θ at each θ_p.

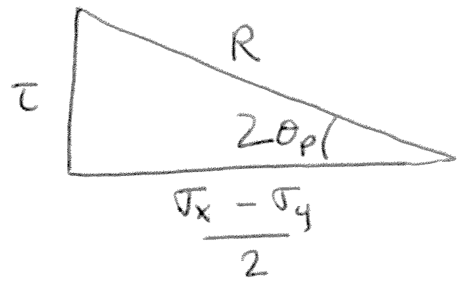
Note that the equation above for $\frac{d\sigma_{x\theta}}{d\theta} = 2\tau_\theta$.

Since $\frac{d\sigma_{x\theta}}{d\theta} = 0$ @ θ_p, so does $\tau_{\theta_p} = 0$.

Thus, at principle angles, there is no shear stress (at least not in-plane)

To calculate the principal stresses, need to calculate $\cos 2\theta_p$ & $\sin 2\theta_p$.

To do this, note that $\tan 2\theta_p = \left(\frac{2}{\sigma_x - \sigma_y}\right)(\tau)$ so:



thus $R = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau^2}$

& $\cos 2\theta_p = \frac{\sigma_x - \sigma_y}{2R}$

$\sin 2\theta_p = \tau / R$

If we sub this into the Tx Eqs, they simplify to:

$$\boxed{\sigma_{1,2} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau^2}}$$

can also use the decomposition of $\tan 2\theta_p$ to $\sin 2\theta_p$ to note that:

$$\cos 2\theta_{p_1} = \frac{\sigma_x - \sigma_y}{2R} \quad \& \quad \sin 2\theta_{p_1} = \frac{\tau}{R}$$

θ_{p_1} corresponds to $\sigma_1 = \frac{\sigma_x + \sigma_y}{2} + R$

then rotate 90° to get $\theta_{p_2} = \theta_{p_1} + 90°$ (or $\theta_{p_2} = \theta_{p_1} - 90°$), which corresponds to $\sigma_2 = \frac{\sigma_x + \sigma_y}{2} - R$

Now we ask about the θ_s angle where shear stress is maximum.

We set $\frac{d\tau_\theta}{d\theta} = 0$; $-(\sigma_x - \sigma_y)\cos 2\theta - 2\tau \sin 2\theta = 0$

$\Rightarrow \tan 2\theta_s = -\frac{\sigma_x - \sigma_y}{2\tau}$ There is a θ_s every 90° since tan θ ~~cs~~ repeats c̄ 180°.

Note than shear stress ~~since~~ signs are just a convention, but $\tau > 0$ & $\tau < 0$ are same physically.

We can show (see Gere) that $\theta_s = \theta_p \pm 45°$.

Like before, we use geometry to realize that:

$$\cos 2\theta_s = \frac{\tau}{R}, \quad \sin 2\theta_s = -\frac{\sigma_x - \sigma_y}{2R}$$

& we substitute these into Eqn for τ_θ to get:

$$\tau_{\theta_s} = -\left(\frac{\sigma_x-\sigma_y}{2}\right)\left(-\frac{\sigma_x-\sigma_y}{2R}\right) + \tau\left(\frac{\tau}{R}\right) = \frac{1}{R}\left[\left(\frac{\sigma_x-\sigma_y}{2}\right)^2 + \tau^2\right] = \frac{R^2}{R} = R$$

thus, max shear stress (τ_{max}) occurs at

$$\boxed{\tau_{max} = R = \sqrt{\left(\frac{\sigma_x-\sigma_y}{2}\right)^2 + \tau^2}}$$ in terms of original coordinate system

so both the shear stress τ & difference in normal stresses contribute to τ_{max}.

Recall that $\sigma_{1,2} = \frac{\sigma_x + \sigma_y}{2} \pm R$,

so $\sigma_1 - \sigma_2 = 2R$, so $R = \frac{\sigma_1 - \sigma_2}{2}$, so we can also say:

$$\boxed{\tau_{max} = \frac{\sigma_1 - \sigma_2}{2}}$$ in terms of principle stresses.

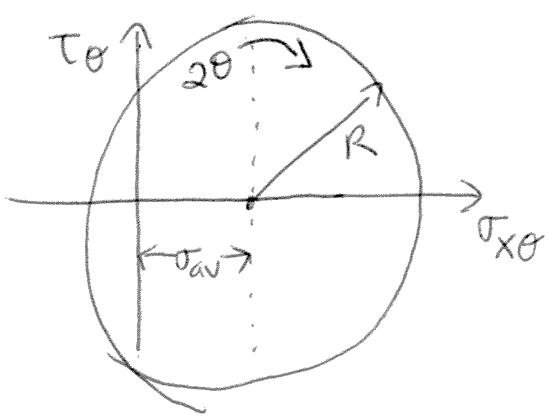
Putting it together, we see that principle stresses are:

(average normal stress $\boxed{\sigma_{av} = \frac{\sigma_x + \sigma_y}{2}}$) $\pm R$:

$\sigma_1, \sigma_2 = \sigma_{av} \pm R$.

$\tau_{max} = R$.

Indeed, we can think of plotting $\sigma_{x\theta}$ vs: τ_θ; which is called Mohr's circle:



so $\sigma_{x\theta}, \sigma_{y\theta}$ ~~are offset~~ oscillate with amplitude R & offset σ_{av}.

τ_θ just oscillates w/ amplitude R.