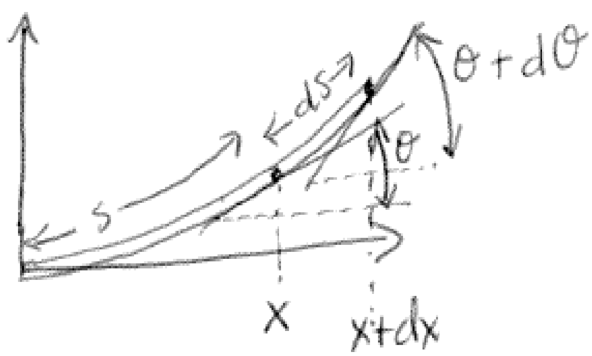
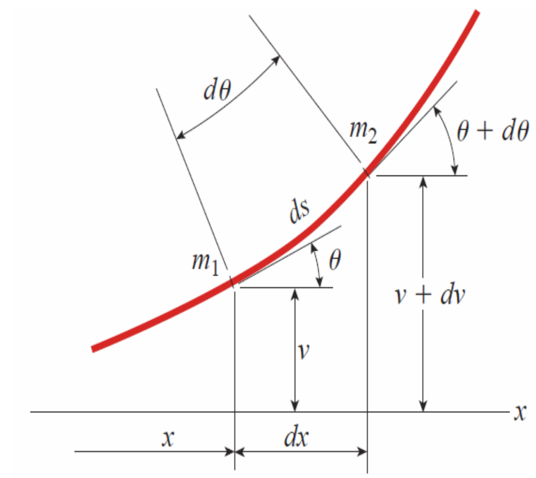
BIOEN 326 2013 LECTURE 11: BEAM DEFLECTIONS

Also read Gere chapter 8

Today we consider how the strains in beams accumulate over the length to change the shape of the beam. Specifically, we are interested in deflections of the beam. Even when every element remains within the proportional limit and has only small deformations, the beam may bend so extremely that it can bend 90 degrees or even bend into a full circle.





As in the diagram above, we consider a beam with tangent angle θ at length s along the beam, with the neutral plane at position (x, v). At length $s + ds$ along the curve, the neutral axis lies at position $(x + dx, v + dv)$, and the tangent angle is $\theta + d\theta$.

Thus, v is the **deflection** of the neutral plane of the beam above its initial position. We use v here to distinguish it from y, the height within the beam above the neutral plane.

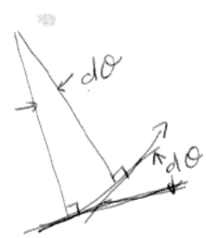
Now we consider geometry to determine how the deflection, v, depends on V(x), M(x), or other things we have already learned to calculate. We note that $d\theta$ is identical to the $d\theta$ that we considered when we asked how the curvature changes, so that $\kappa = \frac{d\theta}{ds}$.

We also note that θ is the angle to the x-axis, so $\tan\theta = \frac{dv}{dx}$, and $\theta = \text{atan}\left(\frac{dv}{dx}\right)$.

Combining these gives $\kappa = \frac{d\theta}{ds} = \frac{d\left(\text{atan}\left(\frac{dv}{dx}\right)\right)}{ds}$, since s is the same as x for small angles.

Now we recall the derivative of arc tangent: $d(\text{atan}\, x) = \frac{1}{1+x^2}$, and we apply the chain rule twice, to get

$$\kappa = \frac{1}{1 + \left(\frac{dv}{dx}\right)^2} \cdot \frac{d^2v}{dx^2} \cdot \frac{dx}{ds}$$



Now recall that ds is the hypotenuse and dv and dx the sides of a right triangle. Thus,

$$\frac{ds}{dx} = \frac{\sqrt{dx^2 + dv^2}}{dx} = \sqrt{1 + \left(\frac{dv}{dx}\right)^2}$$

When we substitute this into the equation for κ, and also substitute in $\kappa = \frac{M(x)}{EI}$ to get:

$$\frac{M(x)}{EI} = \left(1 + \left(\frac{dv}{dx}\right)^2\right)^{-3/2} \cdot \frac{d^2v}{dx^2}$$

Which we rearrange to be a nonlinear second-order differential equation referred to as the beam equation:

$$\frac{d^2v}{dx^2} = \frac{M(x)}{EI}\left(1 + \left(\frac{dv}{dx}\right)^2\right)^{3/2}.$$

To solve the beam equation, you will need two initial or boundary conditions. For example, if the beam is held at a solid support, you may know $v(0)$ and $dv/dx(0)$, the position and angle at one end of the beam. Alternatively, for a beam supported at pins at two ends, you will know v(0) and v(L).

If we can assume the tangent angle θ(L) at the end of the beam is small, then we can make approximations that allow us to solve the beam equation. For small angles, we can assume:

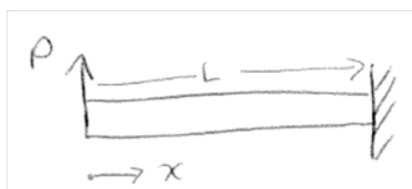
$$\frac{dv}{dx} \ll 1, so \left(1 + \left(\frac{dv}{dx}\right)^2\right)^{3/2} \sim 1$$

Thus

$$\frac{d^2v}{dx^2} = \frac{M(x)}{EI}$$

<u>Example 1. Simple cantilever</u>

For the simple cantilever of previous examples, we learned that M(x) = xP.



Thus, the ODE is:

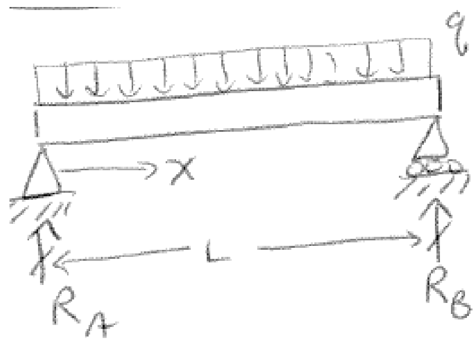
$$\frac{d^2v}{dx^2} = \frac{P}{EI}x$$

With initial conditions $v(L) = 0$, and $dv/dx(L) = 0$, since there is a solid support at x = L.

We can solve this by integrating twice:

$$\frac{dv}{dx} = \frac{P}{2EI}x^2 + C_1$$

$$v = \frac{P}{6EI}x^3 + C_1 x + C_2$$



Then we use the initial conditions to solve for the constants of integration:

$$\frac{dv}{dx}(L) = 0 = \frac{P}{2EI}L^2 + C_1, or\ C_1 = -\frac{PL^2}{2EI}$$

Then

$$v(L) = 0 = \frac{P}{6EI}L^3 - \frac{PL^2}{2EI}L + C_2, or\ C_2 = \frac{PL^3}{3EI}$$

So

$$v = \frac{P}{6EI}x^3 - \frac{PL^2}{2EI}x + \frac{PL^3}{3EI}$$

In particular, we can ask about the deflection at x = 0, where the force is applied, a distance L from the base:

$$v(0) = \frac{PL^3}{3EI}, or\ P = \frac{3EI}{L^3}v$$

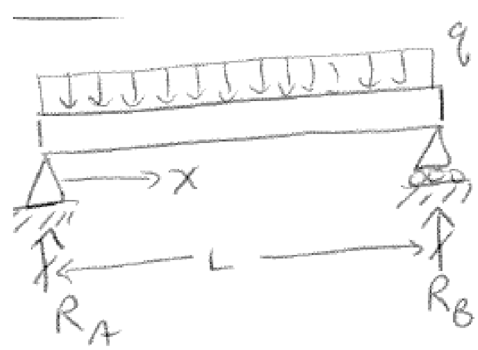
$k_c = \frac{3EI}{L^3}$ is called the **cantilever spring constant**.

Note that the beam is bending upward, since M(x) = xP is positive, and deflects upward. This makes sense.

Example 2: beam supported at either end, with uniform distributed load.

Again, we solved for the bending moment already:

$$M(x) = \frac{qx(L-x)}{2}$$



Now we apply the small angle beam equation:

$$\frac{d^2v}{dx^2} = \frac{qx(L-x)}{2EI} = \frac{qL}{2EI}x - \frac{q}{2EI}x^2$$

And integrate twice:

$$v(x) = \frac{qL}{3*2*2EI}x^3 - \frac{q}{4*3*2EI}x^4 + C_1 x + C_2$$

The two supports are pins so do not constrain the angle, dv/dx, but do keep the deflection of both ends at x = 0.

$$v(0) = 0 = C_2$$

$$v(L) = 0 = \frac{qL}{3*2*2EI}L^3 - \frac{q}{4*3*2EI}L^4 + C_1 L$$

Thus

$$C_1 = -\frac{q}{12EI}L^3 + \frac{q}{24EI}L^3 = -\frac{qL^3}{24EI}$$

So

$$v(x) = \frac{qL}{12EI}x^3 - \frac{q}{24EI}x^4 - \frac{qL^3}{24EI}x$$

We can check this answer:

$$v(0) = 0$$

$$v(xL) = \frac{2qL^4}{24EI} - \frac{qL^4}{24EI} - \frac{qL^4}{24EI} = 0$$

By symmetry, we expect v(x) to have a minimum value at L/2, so we should find that $\frac{dv}{dx}\left(\frac{L}{2}\right) = 0$

$$\frac{dv}{dx}(L/2) = \frac{3qL}{12EI}\left(\frac{L}{2}\right)^2 - \frac{4q}{24EI}\left(\frac{L}{2}\right)^3 - \frac{qL^3}{24EI} = \frac{qL^3}{16EI} - \frac{qL^3}{48EI} - \frac{qL^3}{24EI} = \frac{qL^3}{48EI}(3-1-2) = 0$$

Thus, we are pretty sure we have not made any mistakes.

If we had values for L, q, E and I, we would also confirm at this point that the small angle approximation was valid, by seeing whether $v \ll L$.

<u>Solve the beam deflection equation numerically</u>.

If we cannot assume small deflections, the beam equation rarely has an analytical solution, but we can calculate it numerically, for example using MATLAB ode solvers. ***However, we need to remember that when we calculate M(x), we usually need to assume that the lateral forces are still acting perpendicular to the beam.*** If the beam bends so the tangent angle θ is not close to zero, then what was originally a lateral force becomes a combination of a lateral force and an axial force, and the M(x) and V(x) calculations using the original geometry are incorrect.