BIOEN 326 2013 LECTURE 14: REVIEW FOR MIDTERM

The Midterm will cover the basic principles of statics and mechanics of linear isotropic materials with small deformations.

Specifically, we learned the following knowledge and skills.

Week 1: Free Body Diagrams and Equilibrium Equations

- We remembered how to calculate forces and moments.
- We learned that a **distributed load** could be represented as a **concentrated load $\boldsymbol{F_R}$** acting at position (x_R, y_R, z_R).
  - The concentrated load is calculated by integrating the distributed load over one dimension for a linearly distributed load, two dimensions for a surface distributed load and three dimensions for a body force. This can generally be represented as $F_R = \int F_d(\vec{r})d\vec{r}$, where $\vec{r}$ has one, two or three dimensions. In one dimension it simplifies to $F_R = \int F_d(x)dx$, which is the form we usually encountered in this unit.
  - The position of the concentrated load is calculated by $x_R = \frac{\int x F_d(\vec{r})d\vec{r}}{F_R}$, etc. For one dimension, this is $x_R = \frac{\int x F_d(x)dx}{F_R}$.
- We learned about four types of unknown support reactions, and learned that we solve for them using free body diagrams.
  - **a fixed support** can support forces and moments in all directions. In a plane, this is just F_x, F_y and M_z.
  - **Pins or Hinges** can support forces in two directions (F_x, F_y), but cannot prevent rotation.
  - **Cables** can support tension only in the direction of the cable, so have one unknown, the magnitude of tension, F. Assuming you know the angle of the cable, θ, then you also know $F_x = F cos\theta$ and $F_y = F sin\theta$, but realize that F is still the only unknown.
  - **Rollers** can support compression perpendicular to the surface. It is like a cable, but only supports compression.
- We learned to draw free body diagrams and write equilibrium equations.
  - make sure you have as many equations as unknowns. (The unknowns usually arise from the support reactions.)
  - we write equations for forces by summing all forces acting on the body in the direction of interest. (if a force is present that is not aligned with an axis, decompose the force into the x- and y- components ($F_x = F\cos\theta$, $F_y = F\sin\theta$.)
  - we write equations for moments by summing all moments acting on the body in the direction of the moment (e.g. all M_z in a plane), and also all moments resulting from forces that do not go through the point chosen to calculate the moment (e.g. all $\vec{r} \times \vec{F}$, where $\vec{r}$ is the vector from the point chosen to the point where the force is applied.)
  - solve for the unknowns.

Week 2: Stress and Strain

- Definitions of stress, strain and materials properties
  - **Normal stress** is $\sigma = F/A$ (in Pa) where F is perpendicular to the surface. Stress is positive when positive force on positive face.
  - **Normal strain** is $\epsilon = \delta/L$, the fractional change in length (unitless), and is positive when the object gets longer.

- **Young's modulus** of elasticity, or **elastic modulus**, is $E = \sigma/\epsilon$, also in units of Pa. This only applies within the **proportional limit**, where this value is constant.
- **lateral strain** $\epsilon' = \Delta W/W$, where ΔW is the change in width in a direction perpendicular to the applied force, and W is the original width
- **Poisson ratio** is $\nu = -\epsilon'/\epsilon$.
- S**hear stress** is the shear force per unit area: $\tau = V/A$ (in Pa), and is positive when a positive force is applied to a positive face.
- **shear strain** = γ is the decrease in the angle between two positive faces (in radians, so unitless)
- the **shear modulus** or the **modulus of rigidity** is $G = \tau/\gamma$ or $\tau = G\gamma$ (in Pa).
- The three materials properties are related: : $G = \frac{E}{2(1+\nu)}$

- Stress Analysis
  - We learned that asymmetric normal stress ($\sigma_x \neq \sigma_y$) converts to shear stress when we rotate the orientation of our axes. Stress analysis is a skill that allows us to calculate the normal and shear stresses at various orientations.
  - We learned the transformation equations to calculate the normal stress and shear stress along an angle θ relative to the original x-axis.
    - $\sigma_{x\theta} = \sigma_{av} + \frac{\sigma_x - \sigma_y}{2}\cos(2\theta) + \tau\sin(2\theta)$
    - $\tau_\theta = -\frac{\sigma_x - \sigma_y}{2}\sin(2\theta) + \tau\cos(2\theta)$
    - $\sigma_{y\theta} = \sigma_{x(\theta+\frac{\pi}{2})}$
  - We defined the average normal stress as: $\sigma_{av} = \frac{\sigma_x + \sigma_y}{2}$, and the amplitude of oscillations in normal and shear stress as $R = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau^2}$.
  - The principal stresses are the maximum and minimum normal stresses, $\sigma_{1,2} = \sigma_{av} \pm R$, and occur at the principal angles, $\theta_{p1}, \theta_{p1} = \frac{1}{2}\text{atan}\left(\frac{2\tau}{\sigma_x - \sigma_y}\right)$.
  - The maximum shear stress is: $\tau_{MAX} = R = \frac{\sigma_1 - \sigma_2}{2}$. These occur at angle on the diagonal from the principal angles: $\theta_s = \theta_p \pm 45°$, or $\theta_s = \frac{1}{2}\text{atan}\left(-\frac{\sigma_x - \sigma_y}{2\tau}\right)$.
  - Out-of-plane stresses. You should also remember that $\sigma_z = 0$ is the third principal stress in plane stress, and that there are two more maximum shear stresses that occur as we rotate out-of-plane between σ_1 and σ_z and between σ_2 and σ_z, which are $\tau_{MAX1z} = \frac{\sigma_1}{2}$ and $\tau_{MAX2z} = \frac{\sigma_2}{2}$.
  - We learned that we can use the Cauchy stress tensor to indicate the state of stress of the system, $\sigma = \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_y & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_z \end{bmatrix}$, and that we can then find the principle stresses and angles from the eigenvalues and eigenvectors of the stress tensor.
- Material failure
  - The **tensile strength** or **ultimate tensile stress (UTS)** is the positive normal stress at which the material fails.
  - the **compressive strength** or **ultimate compressive stress (UCS)** is the negative normal stress at which the material fails.
  - The **shear strength** or **ultimate shear stress (USS)** is the shear stress at which the material fails.
  - Since we design with a margin of safety, there may be engineering standards that indicate the **allowable** stresses for materials for specific types of applications. These are several fold lower than the ultimate stresses (strength) of the material.

- To test for material failure, we compare the principal stresses and maximum shear stress to these values.
- Hooke's Law in 3D:
  - We learned that the strain in any direction is the sum of the normal and lateral strains for triaxial stress: $\epsilon_x = \frac{1}{E}(\sigma_x - \nu\sigma_y - \nu\sigma_z)$.
  - Inverting that provides the equation for stress if you know the triaxial strains: $\sigma_x = \frac{E}{(1+\nu)(1-2\nu)}\left((1-\nu)\epsilon_x + \nu\epsilon_y + \nu\epsilon_z\right)$.
  - Shear stress is only affected by shear strain in the same direction, and vice versa: $\tau_{xy} = G\gamma_{xy} = \frac{E}{2(1+\nu)}\gamma_{xy}$.
  - For isotropic materials, volume is only affected by normal stress, with $\Delta V = V_0(\epsilon_x + \epsilon_{yx} + \epsilon_z)$.

Week 3 and 4: Rods and Beams

- General Problem –solving approach
  - If you have **combined loads**, (more than one load or type of load on a rod or beam) you can divide and conquer, finding the solutions to individual loads and then superimposing (adding) the stresses and strains to get the total effect.
  - Identify the **internal, or deformation, forces and moments** caused by the external loads **– N(x), T(x), V(x) and M(x).** These use the internal sign conventions, in which normal force N(x) is the same as normal stress, shear force V(x) is opposite of shear stress, M(x) is positive for upward bending, and we didn't worry about the sign for T(x).
    - **Cut and Draw the FBD**. Cut the rod or beam at an arbitrary x, identify which part of the beam is considered the new body, and draw N, V, T, or M in the positive deformation direction to the free body diagram, since that is what you want to find.
    - **Write the equilibrium equations**
      - Use the static (external) sign conventions for the forces and moments illustrated in your FBD
      - If the new body includes any supports, you need to calculate support reactions from the FBD for the intact body first.
      - If the new body includes any distributed loads, calculate the resultant force and location ($F_R and\ x_R$) only for the part left after the cut.
    - **Solve** the equilibrium equations to find N, V, T, or M
    - **Repeat** for each segment for which the free body diagram is different.
  - **Calculate stresses** from the deformation forces and moments. Note similarity in form and that units always work out.
    - Axial force: $\sigma = \frac{N}{A}$.
    - Torsional moment: $\tau(r) = r\frac{T}{I_p}$; $I_p = \frac{\pi}{2}r^4$
    - Bending moment: $\sigma_x(x,y) = -\frac{M(x)}{I}y$; $I = \frac{\pi}{4}r^4$ or $I = \frac{H^3W}{12}$.
    - Shear force for rectangular cross-section $\tau(x,y) = \frac{V(x)}{2I}\left(\frac{H^2}{4} - y^2\right)$; $\max(\tau) = \frac{3V(x)}{2A}$; for a circular cross section, $\max(\tau) = \frac{4V(x)}{3A}$; for hollow cylinder, , $\max(\tau) = \frac{4V(x)}{3A}\frac{(r_2^2+r_2r_1+r_1^2)}{r_2^2+r_1^2}$
  - If you have **combined loads**, superimpose the stresses from the individual types to get overall stresses.
  - To test for failure.

- **perform stress analysis** on the locations of maximum overall stress, if you can find where this is. If multiple candidate locations, apply to all candidates. Then compare principal stresses to UTS and UCS (or ATS and ACS if appropriate) and τ_{MAX} to USS or ASS.
- If the material has a seam, use stress analysis to determine the normal and shear stresses along the orientation of the seam, and compare this to the strengths of that seams.

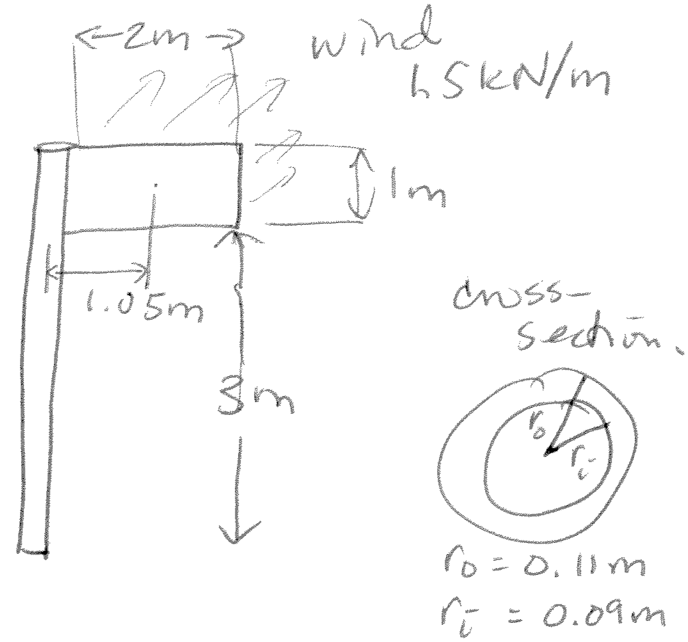
- To predict **deformations** of the bar or beam:
  - **calculate strains from stresses** using elastic or shear moduli
  - **integrate the strains** over a segment to get the deformation of that segment.
    - for axial force: $\delta = \frac{NL}{EA}$, for constant conditions, or $\delta = \int_0^L \frac{N(x)}{E(x)A(x)} dx$ for conditions that vary over the length.
    - for torsional moment: $\phi(L) = \frac{TL}{GI_p}$ for constant conditions.
    - we didn't address deflection of beams, which is deformation due to shear force and bending moment, until after week 4.
  - **add the deformations** of each segment to get total deformation.

Hints for exam:

1. **Don't waste time!** Because of short time period, I try to make the problem simple to solve in terms of the calculations, but may bury the simple problem in excess information to make sure I'm testing your ability to figure out what you need to solve the problem. Don't just try calculating everything possible; instead, figure out what you need to do to solve the problem, then do it, ignoring the excess information.
2. **Provide a complete answer!** Make sure your final answer is in terms of the information given; if you define or introduce something, like M(x) or σ_x that is not given, substitute it out in your final answer.
3. **Check your work.** If you have time at the end of the exam, check your answers by asking things like "does it make sense that the bar got shorter?" Or "Does the answer have the right units?". If your answer fails this test, comment on this (extra credit may compensate for your mistake) and try again on the extra pages at the end. Don't erase what you already did! It might be 99% right.

Example Problem:

Sign In Wind:



Q: what is τ_{max}?

Replace wind force with concentrated load:

$$F_R = (1m)(2m)(1.5\,kN/m) = 3\,kN.$$

location is 3.5m high and 1.05m from pole.

This force adds a shear force to the pole (F_R) and a twisting moment ($T = 1.05 F_R$).

Thus, we can solve this with combined loads.

Twisting Moment, T:

produces no normal stress, but $\tau(r) = \frac{T}{I_p} r$

this is in the $x\theta$ direction

& is maximum @ $r = r_o$.

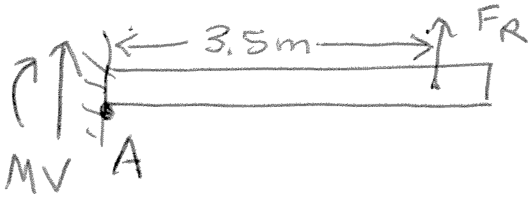
also note $I_p = \frac{\pi}{2}(r_o^4 - r_i^4)$

so

$$\tau = \frac{F_R (1.05)\, 2 r_o}{\pi(r_o^4 - r_i^4)}$$ at periphery.

(direction is $\tau_{x\theta}$)

. Lateral Load



The highest M(x) will be at the base. (x=0)
V(x) will be uniform from x=0 to x = 3.5.

calculate V(x) & M(x) @ base.

draw positive w/ deformation conventions

Eq. Eqs: (using static sign convention)

$$V(0) + F_R = 0 ; \quad V(0) = -F_R.$$

$$-M(0) + 3.5\text{m}\, F_R = 0 ; \quad M(0) = +3.5\text{m}\, F_R.$$

Calculate τ_{xy} & σ_x from these:

$$\max(\tau_{xy}) = \frac{3V}{2A}\left(\frac{r_o^2 + r_o r_i + r_i^2}{r_o^2 + r_i^2}\right) = +\frac{3(F_R)(r_o^2 + r_o r_i + r_i^2)}{2(r_o^2 - r_i^2)\pi(r_o^2 + r_i^2)} = \frac{3F_R(r_o^2 + r_o r_i + r_i^2)}{2\pi(r_o^4 - r_i^4)}$$

this should be 4/3 not 3/2; the mistake carries through, but does not affect the final conclusion

$$\max(\sigma_x) = +\frac{M(0)}{I}(r_o) = \frac{3.5\text{m}\, F_R \cdot r_o}{\pi/4\,(r_o^4 - r_i^4)} = \frac{14\text{m}\, F_R r_o}{\pi(r_o^4 - r_i^4)}$$

we can see quickly that $\max(\sigma_x) >> \max(\tau_{xy})$
since $14\text{m} * r_o >> \frac{3}{2}(r_o^2 + r_o r_i + r_i^2)$ since $14\text{m} >> r_o, r_i$.

so we just consider point A on diagram (or opposite, under compression). We have σ_x & $\tau_{x\theta}$

$$\tau_{max} = R = \sqrt{\left(\frac{\sigma_x}{2}\right)^2 + \tau_{x\theta}^2}. \quad \sigma_x = 14\text{m} \cdot \frac{F_R r_o}{\pi(r_o^4 - r_i^4)}$$

$$\tau_{x\theta} = 2.1\text{m} \cdot \quad "$$

$$\text{so } R = \sqrt{\left(\frac{14}{2}\right)^2 + 2.1^2} \cdot \frac{F_R r_o}{\pi(r_o^4 - r_i^4)}$$

$$\sim 7.3\text{m} \cdot \frac{F_R r_o}{\pi(r_o^4 - r_i^4)} \qquad \text{units: } \frac{\text{mNm}}{\text{m}^4} = \text{N/m}^2 \ \checkmark \ \ddot{\smile}$$