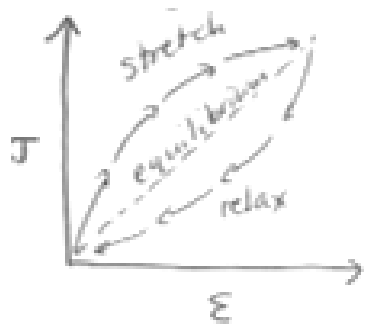
BIOEN 326 2013 LECTURE 18: VISCOELASTIC MODELS

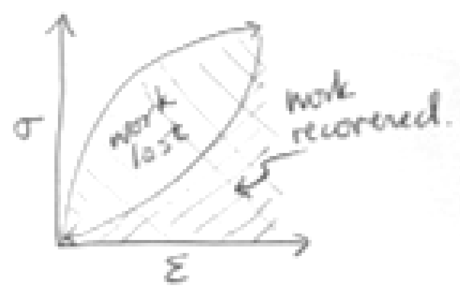
Definition of Viscoelasticity.

Until now, we have looked at time-independent behaviors. This assumed that materials were purely elastic in the conditions tested, so that the current strain depended only on the current stress (or vice-versa), and not on past conditions. This allows us to use algebraic equations to relate stress to strain. However, many materials are actually **viscoelastic**, meaning that they have both elastic and viscous properties. Recall that liquids can change shape, but resist doing so. The forces involved in resisting movement or shape changes increase with both viscosity and speed. For viscoelastic materials, the stress-strain curves are history-dependent, meaning the stress is not just a function of the current strain, but of the rate of change in strain. Thus, we can use differential equations to predict viscoelastic behavior. This week, we will limit our discussion to **linear viscoelastic models**, meaning that the differential equations are linear, just as we have used linear elastic models for most of the class.

Behaviors of Viscoelastic Materials

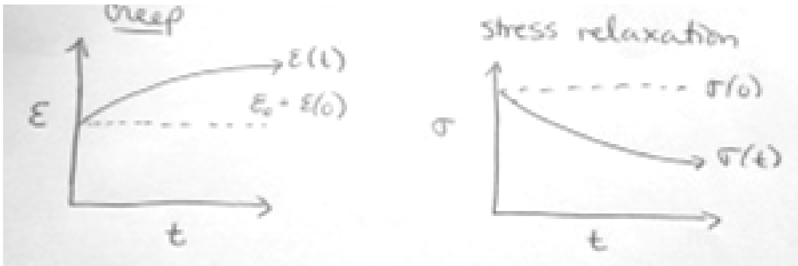
We can obtain a stress-stress diagram by stretching and relaxing a material. For a pure elastic material, the two legs of this experiment (stretching and relaxation, or loading and unloading) overlap. However, viscoelastic materials follow a different curve in the stretch and relaxation stages. This is called **hysteresis**, and is illustrated below, with the arrow heads indicating the direction of the movement. However, if the material is stretched and relaxed more slowly, the hysteresis is reduced until the stress-strain curves overlay each other (within experimental error). This line, shown as the dotted line in the figure here, is the stress-strain curve at thermodynamic equilibrium.





The area under the stress-strain curve is the work applied to stretch the material and the work recovered during relaxation. When the two legs do not overlap, the work recovered is less than the work applied, so the area inside the cycle is lost work, which takes the form of thermal energy, heating the material. Because the amount of hysteresis increases with speed, so does the amount of work lost into heat, which can cause overheating. Thus, one test for viscoelasticity is the appearance of hysteresis in a **stretch-relaxation (or compression-relaxation**) experiment.

Another way that viscoelastic materials exhibit time-dependent behavior is in the **creep test**. It this test, a constant stress σ_0 is applied and held, while the strain is measured. The strain in a purely elastic material will remain constant, but will increase over time in a viscoelastic material, which is referred to as creep.

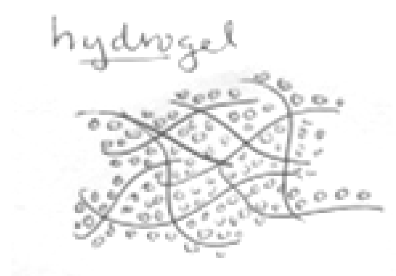


A **stress relaxation** test is the opposite; the material is held at a constant strain ϵ_0 and the stress is measured over time. The stress in a purely elastic material will remain constant, but will decrease over time in a viscoelastic material, which is referred to as stress relaxation.

Viscoelastic systems also have characteristic responses to periodic inputs such as sinusoids, which we can address once we develop mathematical models.

Molecular Basis of Viscoelasticity

Materials are viscoelastic if they are composed of elastic molecules that are relatively slow to stretch or reorganize. For example, hydrogels are almost always viscoelastic, because the liquid surrounding the elastic molecules needs to move when the fibers stretch or realign. If a material is made of polymers that can slide along each other, this sliding my create friction that slows the response, so that the material exhibits viscous properties even if there is no fluid.



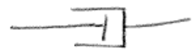
Mathematical Models of Viscoelasticity

We model viscoelastic materials as combinations of elastic and viscous elements. Building quantitative engineering models is a critical engineering skill, so we will learn model building skills here, and then develop three common models of viscoelasticity in this class. The models we build here are designed to capture the **systems properties** of the material, but not its geometry, so we build the models using stress and strain as dependent variables, and time as the independent variable. Since there is only one independent variable, these models are ordinary differential equation (ODE) models. However, we usually use an experimental set up to constrain either stress or strain so that one of these becomes the forcing function and the other the variable. <u>Thus, the ODE depends on the material property, but the initial conditions and forcing function depend on the experimental conditions in which the material is used or tested.</u>

You can compare the mechanical systems models to those you may have seen of electrical circuits. You will see that we have analogies to resistors, capacitors, impedence, voltage and current as well as Kirchoff equations (conservation equations). However, be aware of many differences between mechanical and electrical models.

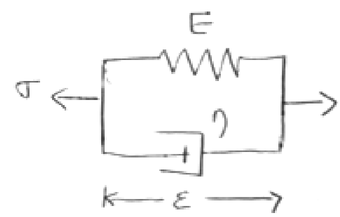
Viscoelastic materials models are built by combining elastic elements and viscous elements.

Each **elastic element** is shown as a **spring**  with parameter E = Young' modulus, in Pa. An elastic element follows Hooke's law ($\sigma = E\epsilon$). An elastic element is like a capacitor in an electrical circuit, because it stores potential energy. Elastic elements return to their original shape (zero strain) if there is no stress. Also, since the current stress only depends on current strain (and vice versa), response is instantaneous within the elastic element.

Each **viscous element** is shown as a **dashpot**  with parameter ('eta') η = viscosity in Pa*sec. A viscous element is like a resistor in an electrical circuit, because it resists movement (strain vs electron flow). The viscous element follows the equation: $\sigma = \eta \frac{d\epsilon}{dt}$. That is, the stress is proportional to the rate of change in strain.

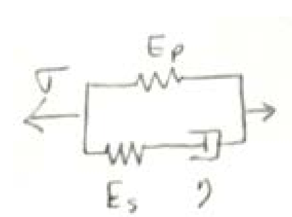
Note that because we are describing materials properties, we use stress and strain rather than force and distance as the dependent variables. However, if you are doing an experiment where you measure force F and change of length δ, then you need to divide force by cross-sectional area and change in length by unstressed length. Alternatively, you could use the same form of equations as we develop here, but use F and δ as variables, and spring constant k in N/m and damping coefficient b in Ns/m, as parameters instead of Young's modulus and viscosity.

The **Voigt** model has one elastic and one viscous element in parallel, as shown on the right. While they are shown as two distinct elements, you should think of them as two functional properties of the material that act in parallel, rather than as two physical elements laying side by side. For example, a simple hydrogel may be modeled as a Voigt model, with the combination of all cross-linked fibers making up the elastic element and the viscous water plus any fiber friction making up the viscous element. The elements are in parallel because both must deform the same amount for the material to stretch or relax. That is, you cannot stretch the fibers without the water moving to allow the new shape.



The **Maxwell** model has the elastic and viscous elements in series, as shown here. This could represent a material made up of elastic polymers that are not cross-linked so they can slide along each other. This is appropriate since the elements are in series because either fiber sliding or stretching can lead to a shape change. When we refer to the strain in the individual elements, we are referring to the change of length of this element relative to the initial unstressed length of the whole material, rather than relative to the original length of this element. This is appropriate since the two elements are usually functional properties, not two separate pieces of the material.

The **Kelvin** model is a Maxwell model in parallel with an elastic element. This is also sometimes called the **Standard Linear Model**. The Kelvin model has properties of both the Maxwell and Voigt models. Most real viscoelastic materials exhibit behavior that are best described by the Kelvin model. However, you can also think of the Voigt or Maxwell models as a Kelvin model in which one of the two elastic elements is so soft or stiff



(depending on model structure) that its effect is insignificant. Not surprisingly then, some viscoelastic materials are described by the Voigt or Maxwell models with only small errors, even though an exact Voigt or Maxwell is rarely observed.

While these are the three most common models for materials, it is also possible to build models that include active elements. For example, muscle cells can create a tensile stress within tissue. Thus, a model for muscle tissue can include a stress creating element, for which $\sigma_M = M(t)$, independently of the strain or rate of strain.

Today, we will learn how to **build a mathematical systems model** of a mechanical system. This means you start with a written description or a diagram and develop an equation that relates stress to strain. In the next lecture, we will apply Laplace transforms to predict how one of these models responds to specific conditions, or experiments. This will help us to understand hysteresis, creep and stress relaxation.

Building Viscoelastic Models

It is best to learn a rigorous system for building systems models, to avoid making errors. What follows is Wendy's step-by-step "DIESE" process to build an accurate systems model of viscoelasticity.

STEP 1: Diagram the system.

Diagram the system to draw the individual elements and how they relate to each other. The three cartoons above are examples of diagrams. This can be challenging to do from a written description or understanding.

STEP 2: Identify parameters and variables

Recall that a **parameter** is a constant like the Young's modulus, that does not change during the experiment, or at least does not depend on the rest of the system. In contrast, **variables** are the values like stress and strain that change during the experiment. Define the parameter name for each element (e.g. $E_P, E_S, and\ \eta$ in the Kelvin model), the variable name for stress and strain of each element, and the name of the overall stress and strain on the material. Finally, identify the **variables of interest**, which are the variables you want to have in the final equation: usually, this is stress and strain of the whole material. By convention, I use subscripts like σ_1 and ϵ_1 for the stress and strain on specific elements, and plain σ and ϵ as the stress and strain on the whole material, to help me remember which are the variables of interest.

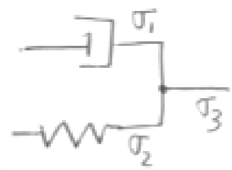
STEP 3: Equations.

List all the equations that you can use to derive your equation for the whole system. There are three types of equations:

E**lement equations** relate stress and strain over a single element. So far we learned $\sigma_i = E_i \epsilon_i$ for an elastic element and $\sigma_i = \eta_i \frac{d\epsilon_i}{dt}$ for a viscous element (and $\sigma_i = M_i(t)$ for a stress creating element). We are about to derive the equations for the Voigt, Maxwell, and Kelvin models, which themselves can be elements in other models.

Distance Constraints. Any two elements in parallel must have the same strain. The strain across elements in series adds together to equal the strain over the combined material. Thus, **strains in parallel are equal, while strains in series add**.

Newton's second law states that force equals mass times acceleration $\sum F = ma$. If we are stretching a material slowly enough to neglect inertia, we can assume that we acceleration is zero. Similarly, if we are considering an infinitesimally small point, we assume the mass is zero. In either case, Newton's second law reduces to the statement that the sum of forces on a point is zero: $\sum F = 0$. We apply Newton's laws to all nodes in a diagram, which are the points where the different elements connect together. When we apply this, keep in mind that we have used stress instead of force in our equations. Remember that stresses acting on opposite faces of an object or point are the same, while forces would have been equal in magnitude but opposite in direction. So, $\sum F = 0$ translates to the assumption that the sum of stresses on each side of a point must be equal. The simplest way to think of this is that **stresses in parallel add, while stresses in series are equal**. Example: in the figures below, $\sigma_3 = \sigma_1 + \sigma_2$ and $\sigma_4 = \sigma_5$.



For a shortcut, you can simply use the same variable name where the laws say two stresses or strains must be equal. But, make sure you don't do this unless you can do it error-free. I will often do this in the derivations here, but will justify the choice, to avoid mistakes.

<u>STEP 4: Simplify</u>

Combine the equations to eliminate the unknown variables that are not of interest, until you an equation relating the unknown variable(s) of interest to the known ones. If you follow the convention described above, and you are interested in the stress-strain relationship of the material as a whole, then this means you need to remove all the <u>**S**ubscripted</u> variables.

This step is simply algebra, but can take time to figure out the right path in a complex problem. Some time saving tips are:

1. Chose one equation that has at least one of the variables of interest (σ or ϵ).
2. Remember that your goal is to eliminate the subscripted variables, so find an equation that replaces one of these in your chosen equation.
3. Use each equation only once, or you will go in circles.
4. Try not to reintroduce an unwanted variable you already eliminated in a previous step. Instead, try to replace unwanted variables with wanted variables or unwanted variables that are still in the equation or have not been introduced yet. Remember that parameters are always fine.

<u>STEP 5: Error Check</u>

There are several checks you can do to make sure you got the right answer. Unless instructed to do so, you are not required to do this on homeworks and exams for Bioen 326. **However, in all**

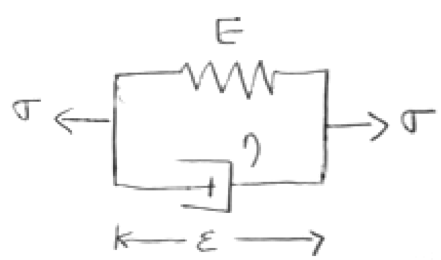
my classes, if you make an error that would have been caught with error checking, you will get more points off than if you make other errors, and conversely will recover some lost points if you perform error checking and note that you know you have a mistake. You should remember that in real engineering research and design, no one knows the answer, so you can't simply compare answers with the key or colleagues. Moreover, mistakes on the job are more serious than in class; they can cost money and lives. Therefore, you need to learn to catch your own mistakes. Here are some tests that can catch errors in engineering models in general.

1. Do a dimensional analysis on your final equation. You know the units for each variable and parameter. (E and σ are Pa, ϵ is unitless, η is Pa*s and taking a time derivative is dividing by seconds. Check that the units for all terms on both sides of the equation are the same. (remember: a **term** is anything separated by +, – , and = signs.)
2. Did you use all the parameters? If not, does this make sense in retrospect, that the model behavior is not affected by one of the parameters?
3. Find the equilibrium equation for stress versus strain by setting all derivatives to zero. Does this behavior make sense intuitively given the diagram or description of the system? (When the strain and stress are no longer changing, a viscous element creates zero stress, so this knocks out entire branches of a system.)

Example 1: Voigt Model

We will apply this technique to derive the stress-strain relationship of the Voigt model.

Step 1. Diagram



Step 2. Identify Parameters and Variables

- The parameters of the system are E and η.
- Define σ_1 and ϵ_1 in the elastic element, σ_2 and ϵ_2 in the viscous element
- σ and ϵ are stress and strain in the overall model, which are the variables of interest.

Step 3. Equations

- Element equations are (Eq 1) $\sigma_1 = E\epsilon_1$ and (Eq. 2) $\sigma_2 = \eta \frac{d\epsilon_2}{dt}$
- Stresses add in parallel, so (Eq. 3) $\sigma_1 + \sigma_2 = \sigma$
- Strains are equal in parallel, so (Eq. 4) $\epsilon_1 = \epsilon$ and (Eq. 5) $\epsilon_2 = \epsilon$

Step 4. Simplify (with my logic added to help you see the motivation)

- I will start with Eq. 3, since it has one of my variables of interest: $\sigma_1 + \sigma_2 = \sigma$
- I need to replace the subscripted stresses. There is only one choice, which is to use Eqs 1 and 2: $E\epsilon_1 + \eta \frac{d\epsilon_2}{dt} = \sigma$

- That may seem like it was no better, but I can replace the subscripted strains using Eqs 4 and 5. Thus, the equation for the Voigt model is:

$$E\epsilon + \eta\frac{d\epsilon}{dt} = \sigma$$

- We are done, since we have an equation that relates stress to strain using the parameters and no subscripted variables.

Step 5. Error check

- ☑Each term has units Pa.
- ☑Both parameters are used.
- ☑Equilibrium behavior is $E\epsilon = \sigma$. This makes sense, because the viscous element is irrelevant once the material is at equilibrium.

Note: we have now added an element equation to our arsenal. If we see the Voigt model within another model, we don't need to rederive this; we can simply include $E\epsilon + \eta\frac{d\epsilon}{dt} = \sigma$ in our list of element equations, and represent the Voigt model as an element within the larger model.

Example 2: Maxwell Model

Step 1, Diagram

Step 2. Identify Parameters and Variables

- The parameters of the system are E and η.
- Define ϵ_1 in the elastic element and ϵ_2 in the viscous element. Note that all have same σ, since stresses in series are the same. (I'm short-cutting by using the stress equation in my identifications).
- σ and ϵ are stress and strain in the overall model, which are the variables of interest.

Step 3. Equations

- Element equations are (Eq 1) $\sigma = E\epsilon_1$ and (Eq. 2)$\sigma = \eta\frac{d\epsilon_2}{dt}$
- Strains add in series, so (Eq. 3) $\epsilon_1 + \epsilon_2 = \epsilon$.
- I already used the stress equation in step 2 shortcut.

Step 4. Simplify (with my logic added to help you see the motivation)

- I will start with Eq. 2, since I will need to remove ϵ_2 and I want derivatives, not integrals. $\sigma = \eta\frac{d\epsilon_2}{dt}$
- To remove the ϵ_2 rewrite Eq 3: $\epsilon_2 = \epsilon - \epsilon_1$ and substitute into chosen equation to get: $\sigma = \eta\frac{d\epsilon - d\epsilon_1}{dt}$
- To remove the ϵ_1 rewrite Eq 1: $\epsilon_1 = \sigma/E$ and substitute into chosen equation to get: $\sigma = \eta\frac{d\epsilon}{dt} - \frac{\eta}{E}\frac{d\sigma}{dt}$

- Rearrange to get stress and strain on opposite sides to get the Maxwell model equation:

$$\sigma + \frac{\eta}{E}\frac{d\sigma}{dt} = \eta\frac{d\epsilon}{dt}$$

Step 5. Error check

- ☑Each term has units Pa
- ☑Both parameters are used.
- ☑Equilibrium behavior is $\sigma = 0$. This makes sense because the stress on a viscous element is zero at equilibrium, and the stress on the whole system is the same as that on the viscous element.

Example 3: Kelvin Model.

You will derive in your homework that the equation for the Kelvin Model is:

$$\sigma + \frac{\eta}{E_S}\frac{d\sigma}{dt} = E_P\epsilon + \eta\left(\frac{E_P + E_S}{E_S}\right)\frac{d\epsilon}{dt}$$