BIOEN 326 2013 LECTURE 19: BEHAVIOR OF VISCOELASTIC MATERIALS

Today we will consider how a viscoelastic material responds to specific tests, such as the creep test, the stress relaxation test, and a linear load/unload, or hysteresis test. To do this, we need to interpret the test to identify the initial conditions and forcing function, and then use these to solve the differential equation for the remaining system variable. We will use Laplace Transforms to solve these, since they are the most straightforward for systems models like this. One of your learning outcomes in this course is ability to use Laplace Transforms to solve engineering problems, so we will not be using any other methods, and **you MUST use Laplace Transforms to solve these problems to get credit**. We do this because Laplace transforms are the most versatile way to solve linear systems, and you will need to be proficient in them for bioen 336. You have all learned these in your ODE prereq class, but if you need a refresher on the use of Laplace transforms, see the essential prior knowledge notes. Here, we will jump straight to applying them.

These models are used to predict how a material will respond to a particular input. For example, in the hysteresis and stress relaxation tests, we will prescribe the strain and want to know the stress that occurs in response, so the strain is the input and the stress is the output. In the creep test, we use the stress as the input and the strain as the output. This is comparable to determining how voltage responds to current or vice-versa in an electrical system.

Overview of Model Solving.

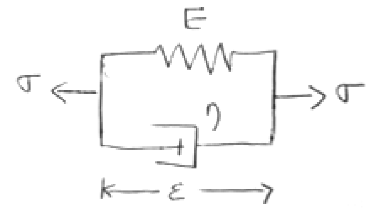
Overview of model solving process: ("ETSIE")

1) **E**quations: Identify the correct ODE model for the material, and the correct input function and initial conditions to represent the experimental condition or test.
    a. A stress relaxation test jumps instantly to a new nonzero constant strain, and measures the stress response. Thus, the forcing function is $\epsilon(t) = \epsilon_0\phi(t)$, where $\phi(t)$ is the step function that goes from 0 to 1 at time t = 0, and we are solving for $\sigma(t)$. The material was not stressed before the test started, so the initial condition is $\sigma(0) = 0$. We do not need an initial condition for the strain, because that is incorporated into the forcing function, which defined it as zero.
    b. A creep test jumps instantly to a new nonzero stress, and measures the strain response. Thus, the forcing function is $\sigma(t) = \sigma_0\phi(t)$, we are solving for strain, and the initial condition is $\epsilon(0) = 0$.
    c. A hysteresis test increases strain linearly to a value, then decreases again. This can be done with various types of cycles, such as a triangular shape (linear increase and linear decrease) or a sinusoidal shape.
2) **T**ransform the systems model, IC, and input functions and combine to obtain a single transformed equation. I always let $X(s) = L[\epsilon(t)]$, and $F(s) = L[\sigma(t)]$, since strain relates to distance X and stress relates to force F. (I don't want to use capital S or epsilon or sigma as they get confusing).
3) **S**implify: Algebraically rearrange it to express the transformed output as a ratio of polynomials. If you have both nonzero initial conditions and nonzero forcing function,

you will have an answer that is the sum of two ratios of polynomials, one for the particular and one for the complementary solution.

4) **I**nverse Transform the solution using partial fractions, lookup tables, and/or method of residues.
5) **E**rror check:
    a. Does your equation answer the question? Did you solve for the correct system property as the response, in terms of things that were provided in the question, and not anything you introduced? It is fine to introduce a new parameter as a function of others to simplify the way you provide your answer, but you need to include this definition as part of the answer.
    b. Check the units for the equation for the response. Is it appropriate for what the response is supposed to be? (eg, if the response is stress, it should be Pa).
    c. Does your equation match the ODE and IC? Plug your answer back into the ODE. Also, find the value at t = 0. Check that these match the problem.
    d. Identify characteristic time constants and check each time constant to make sure it has units of time. (e.g. if the equation includes "$exp(-at)$", then make sure that a has units of 1/sec, so the time constant $\tau = 1/a$ will have units of time. Another way to think of this is that the argument of an exp, sin or similar function should be unitless. You should also ask if the time constants make sense. Eg, if the viscosity η is increased, you expect the response to be slower, so the time constant should get bigger. This means that in "$exp(-at)$", a should have the eta in the denominator so 1/a has it in the numerator.
    e. Calculate the behavior at t = 0, and $t = \infty$. The first should be the IC, and the second the equilibrium behavior (if the forcing function is not constant). Also, sketch the full response, with the time constant. You may also want to sketch the input, and even the structure of the model. Then, ask if the relationship between these are logical. While you may at first struggle, your intuition and understanding will grow with practice. First make sure you know how a single viscous or elastic element should act for the input in question. Then consider how the response should be if these elements are in parallel vs in series. Check your answers against the many responses we already worked out. Also, see the logic in the examples here.

Example 1: Stress Relaxation of Voigt



How will a Voigt model respond in a stress relaxation test? Recall that the Voigt model is where the elastic and viscous elements are in parallel.

Equations:

We showed in the previous lecture that the system response is: $E\epsilon + \eta\frac{d\epsilon}{dt} = \sigma$.

For stress relaxation, we set $\epsilon(t) = \epsilon_0 \emptyset(t)$. We need to find $\sigma(t)$. The IC is $\sigma(0) = 0$.

Transform:

Let $X(s) = L[\epsilon(t)]$, and $F(s) = L[\sigma(t)]$. This means that $L\left[\frac{d\epsilon(t)}{dt}\right] = sX(s) - \epsilon(0)$.

Therefore, the equation transforms to $EX(s) + \eta\big(sX(s) - \epsilon(0)\big) = F(s)$.

Recall that $L\big(\phi(t)\big) = 1/s$, so the transform of $\epsilon(t)$ is $X(s) = \epsilon_0/s$, and $\epsilon(0) = 0$.

Simplify:

We are trying to find the output $F(s)$ so we rearrange to get $F(s)$ as function of $X(s)$:

$$F(s) = X(s)(E + \eta s) - \eta\epsilon(0)$$

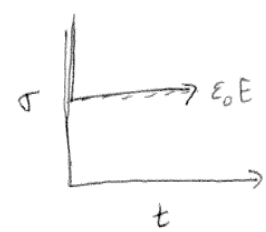
We combine this to give: $F(s) = \frac{\epsilon_0}{s}(E + \eta s) - 0$, which becomes $F(s) = \frac{\epsilon_0 E}{s} + \epsilon_0\eta$

Invert:

This can be transformed quickly since each term is a constant times something that can be found in a look-up table such as http://en.wikipedia.org/wiki/Laplace_transform: $L\big(\delta(t)\big) = 1$ and $L\big(\phi(t)\big) = 1/s$. Thus, $\sigma(t) = E\epsilon_0\phi(t) + \epsilon_0\eta\delta(t)$

Error check:

1) The equation solves for stress, as required. In a stress relaxation test, the strain is held to ϵ_0; E and η are model parameters. $\phi(t)$ and $\delta(t)$ are standard defined functions. So everything on the right is given.



2) Units: each term has units Pa (since E has units Pa, ϵ_0 is strain so unitless, $\delta(t)$ has units 1/sec (recall that $\int \delta(t)dt = 1$) and η is Pa*sec).

3) Plug back in. To do this, plug in $\epsilon(t) = \epsilon_0\phi(t)$ and recall that $\frac{d\phi(t)}{dt} = \delta(t)$, so $E\epsilon + \eta\frac{d\epsilon}{dt} = E\epsilon_0\phi(t) + \eta\epsilon_0\delta(t) = \sigma(t)$, so check yes.

4) In this case, we have no characteristic time constant; response is instant.

5) Sketch this to understand the behavior. Note that there is an infinitely high impulse function as the material is first stretched, at t = 0. This makes sense, because the viscous element must stretch infinitely fast in the stress relaxation test, and the stress is proportional to this speed. Next note that the stress is simply $E\epsilon_0$ for all t > 0. Again, this makes sense; the material is no longer changing shape, so the viscous element does not hold any stress, and the stress is simply determined by Hooke's law over the elastic element.

Example 2: Creep Test of Voigt

Equations

In a creep test, we set $\sigma(t) = \sigma_0\phi(t)$ which also means $\sigma(0) = 0$, and we need to find $\epsilon(t)$. Before t = 0, the material was relaxed, so $\epsilon(0) = 0$.

Transform.

We already derive the transform of the ODE: $EX(s) + \eta\big(sX(s) - \epsilon(0)\big) = F(s)$.

The transform of the input is $F(s) = \sigma_0/s$.

Simplify:

Since $\sigma(t)$ is the input and $\epsilon(t)$ is the output in a creep test, we want to rearrange to get $X(s)$ as function of $F(s)$: $X(s)(E + \eta s) = F(s) + \eta\epsilon(0)$, so $X(s) = \frac{1/\eta}{(E/\eta+s)}F(s) + \frac{1}{(E/\eta+s)}\epsilon(0)$.

We then replace the initial and input conditions with the expressions indicated above:

Thus $X(s) = \frac{1/\eta}{(E/\eta+s)}\frac{\sigma_0}{s} = \frac{\sigma_0/\eta}{(E/\eta+s)s}$

Invert.

The easiest way to invert this is to remember the following Laplace transform: $L[(1 - e^{-at})\emptyset(t)] = \frac{a}{s(s+a)}$, which you will derive in your homework.

To use this, we can just pull out the right coefficient to leave the "a" in the numerator :

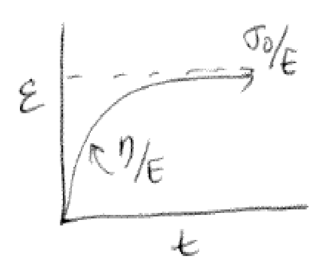
$$X(s) = \frac{\frac{\sigma_0}{\eta}}{\left(\frac{E}{\eta} + s\right)s} = \frac{\sigma_0}{E}\frac{\frac{E}{\eta}}{\left(\frac{E}{\eta} + s\right)s}$$

And then take the inverse transform, to get:

$$\epsilon(t) = \frac{\sigma_0}{E}\left(1 - e^{-\frac{E}{\eta}t}\right)$$

Error check.

1. Answer: we solved for strain as a function of the stress applied, as needed for creep test.
2. The units are Pa/Pa, is unitless.
3. Plug the solution into the ODE: $E\epsilon + \eta\frac{d\epsilon}{dt} = \sigma$ with $\sigma(t) = \sigma_0$ and $\epsilon(t) = \frac{\sigma_0}{E}\left(1 - e^{-\frac{E}{\eta}t}\right)$ gives $E\frac{\sigma_0}{E}\left(1 - e^{-\frac{E}{\eta}t}\right) + \eta\frac{\sigma_0}{E}\left(-\frac{E}{\eta}\right)\left(-e^{-\frac{E}{\eta}t}\right) = \sigma_0$, or $\sigma_0\left(1 - e^{-\frac{E}{\eta}t}\right) + \sigma_0 e^{-\frac{E}{\eta}t} = \sigma_0$, check.
4. The time constant is $\tau = \frac{1}{a} = \eta/E$. This has units of Pa*s/Pa, so units time, and gets bigger (slower) with more viscosity.
5. To sketch this, note that it goes from 0 at $t = 0$ to $\epsilon = \frac{\sigma_0}{E}$ at $t = \infty$. When the viscous element doesn't matter, this looks like a spring and we just have hooks law. Check.



If we hadn't found that transform, we could have used other methods:

We need to separate this into the sum of two fractions:

$$X(s) = \ X_P(s) = \frac{A}{(E/\eta + s)} + \frac{B}{s}$$

We can find A and B using the method of partial fractions, in which we multiply top and bottom of each term by the denominator of the other term(s), so we can add them together to get the original expression. We can always use this method. When we do this, we get one equation:

$$As + B\left(\frac{E}{\eta} + s\right) = \ \sigma_0/\eta$$

But this is essentially two equations, one for the constant terms and one for the s-terms, because this needs to be true for all s:

$$\frac{BE}{\eta} = \frac{\sigma_0}{\eta} \ and \ As + Bs = 0$$

This quickly gives us:

$$B = \frac{\sigma_0}{E} and \ A = -\ \frac{\sigma_0}{E}$$

Alternatively, we can find A and B using the method of residues, also called the Heaviside cover-up method. This method is easy to use here because the roots are distinct. If you don't remember or haven't seen this, see http://en.wikipedia.org/wiki/Heaviside_cover-up_method. You evaluate X(s) at each root after multiplying by the term that will prevent the denominator from going to zero when you multiply by that root:

$$A = \ \left(\frac{E}{\eta} + s\right)X(s) \ evaluated \ at \ s = \ -\frac{E}{\eta}$$

$$B = \ sX(s) \ evaluated \ at \ s = \ 0$$

That is:

$$A = \frac{\sigma_0\eta}{s} \ @ \ s = \ -\frac{E}{\eta}, so \ \ A = -\frac{\sigma_0\eta}{\frac{E}{\eta}} = -\frac{\sigma_0}{E}$$

$$B = \ \frac{\sigma_0/\eta}{(E/\eta + s)} \ @ \ s = \ 0, so \ B = \frac{\sigma_0\eta}{\frac{E}{\eta}} = \frac{\sigma_0}{E}$$

Reassuringly, the two methods gave the same answer, and we plug these back in:

$$X(s) = \frac{-\frac{\sigma_0}{E}}{(E/\eta + s)} + \frac{\frac{\sigma_0}{E}}{s}$$

Now take the inverse transform, recalling that $L^{-1}\left(\frac{1}{s+a}\right) = e^{-at}$ and $L^{-1}\left(\frac{1}{s}\right) = \emptyset(t)$:

$$\epsilon(t) = \frac{\sigma_0}{E}\left(1 - e^{-\frac{E}{\eta}t}\right)$$

Nonzero initial conditions

We may also ask what happens to the material when we remove the stress in a creep test, after waiting until the strain had reached equilibrium. This means that the initial condition will be $\epsilon(0) = \frac{\sigma_0}{E}$, since that was the value in the previous calculation at infinite time. Thus, we have nonzero initial conditions, so we need to keep the initial condition in the equation:

$$X(s) = \frac{1/\eta}{(E/\eta + s)} F(s) + \frac{1}{(E/\eta + s)} \epsilon(0)$$

Recall that a linear response can be separated into the Particular solution that is the response to just the inPut, and the Complementary that is just the response to the Initial Conditions. Thus, the equation above can be solved as two separate equations:

$$X_P(s) = \frac{1/\eta}{(E/\eta + s)} F(s)$$

$$X_C(s) = \frac{1}{(E/\eta + s)} \epsilon(0)$$

We can also write this as $X_P(s) = H(s)F(s)$, where $H(s)$ is the system response:

$$H(s) = \frac{1/\eta}{(E/\eta + s)}$$

This time, however, the stress is zero, so $F(s) = 0$, so the particular solution is 0. Thus,

$$X(s) = X_C(s) = \frac{1}{(E/\eta + s)} \epsilon(0) = \frac{\sigma_0/E}{(E/\eta + s)}$$

This can be recognized at the transform of the exponential decay, so:

$$\epsilon(t) = \frac{\sigma_0}{E} e^{-\frac{E}{\eta}t}$$

This time, the equilibrium strain is zero, which makes sense because we have removed all stress. The initial value is indeed the initial condition, and the characteristic time is the same.

Response of the three models to Creep and Stress Relaxation tests:

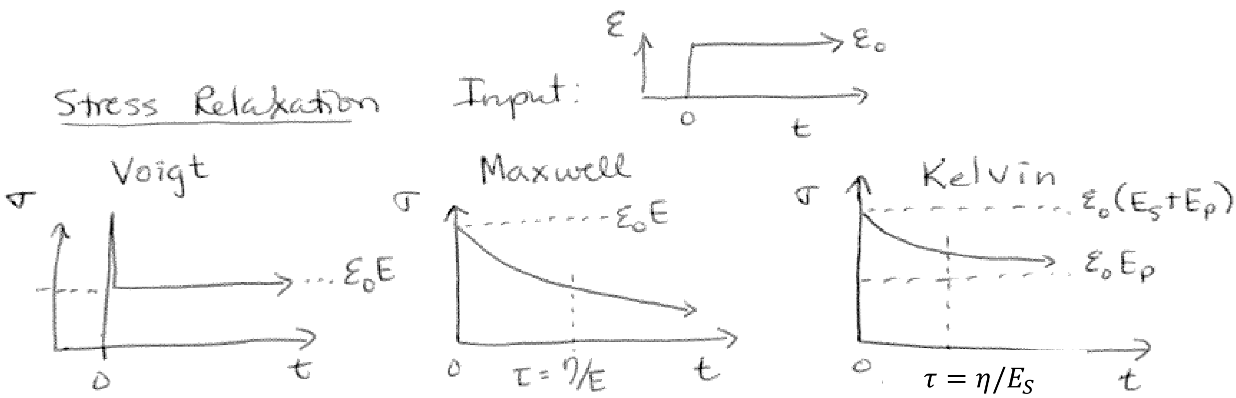
The derivations of the response of the Maxwell and Kelvin models are left to the reader or as homework, but here we summarize the different behaviors of the three models:

<u>Stress relaxation test</u>:

Voigt: $\sigma(t) = E\epsilon_0\emptyset(t) + \epsilon_0\eta\delta(t)$

Maxwell: $\sigma(t) = E\epsilon_0 e^{-\frac{E}{\eta}t}$

Kelvin: $\sigma(t) = \epsilon_0 E_p + \epsilon_0 E_s e^{-\frac{E_s}{\eta}t}$



<u>Creep Test</u>:

Voigt: $\epsilon(t) = \frac{\sigma_0}{E}\left(1 - e^{-\frac{E}{\eta}t}\right)\phi(t)$

Maxwell: $\epsilon(t) = \frac{\sigma_0}{E} + \frac{\sigma_0}{\eta}t$

Kelvin: $\epsilon(t) = \frac{\sigma_0}{E_p} - \frac{\sigma_0 E_s}{E_p(E_s+E_p)}e^{-at}$, or $\epsilon(t) = \frac{\sigma_0}{E_p}(1 - e^{-at}) + \frac{\sigma_0}{E_s+E_p}e^{-at}$ where $a = \frac{E_s E_p}{\eta(E_s+E_p)}$