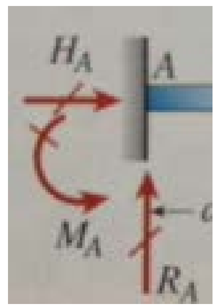
BIOEN 326 2014 LECTURE 2: FREE BODY DIAGRAMS

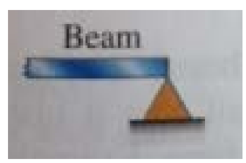
Equilibrium. An object is at equilibrium if it has no linear or angular acceleration, including the common case where the object is not moving. When an object is at equilibrium, all forces and moments are balanced. If all forces act in the same plane, this requires three equations, $\sum F_x = 0, \sum F_y = 0, \sum M_z = 0$. In three dimensions, there are three more: $\sum F_z = 0, \sum M_x = 0, \sum M_y = 0$.

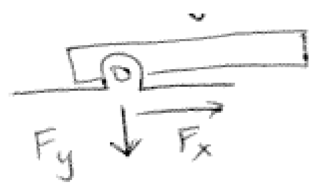
Unknown Support Reactions are the reaction forces applied by supports touching the object. These supports apply whatever forces or moments are necessary to prevent movement. The Gere textbook indicates support reactions with crossed arrows.

- **fixed support** can support forces and moments in all direction. In 2D (everything lies in a plane), this has three unknowns: F_x, F_y and M_z. In 3D, it has 6 unknowns. Gere indicates these with a solid contact as shown.

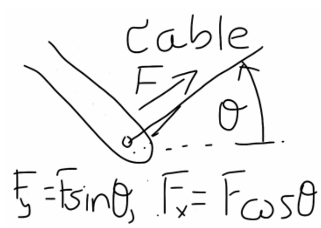


- **Pins or Hinges** are relevant in 2D and have two unknowns: they can prevent movement (thus apply forces) in both directions (F_x, F_y), but cannot prevent rotation so apply no moment. Gere indicates these with a triangle support, others as a pinned hinge.

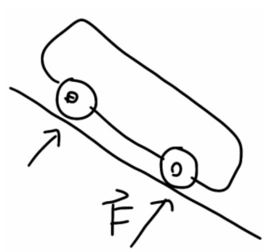




- **Cables** can support tension only in the direction of the cable, so have one unknown, the magnitude of tension, F. Assuming you know the angle of the cable, θ, then you also know $F_x = F cos\theta$ and $F_y = F sin\theta$, but realize that F is still the only unknown.



- **Rollers** can support compression perpendicular to the surface. Like the cable, this has one unknown, F, which decomposes into F_x and F_y based on the angle of the surface. However in this case, the force is only compressive, while the cable is only tensile. Gere indicates these with a triangle support on rollers a shown below.



Method: To find the forces on an object at equilibrium: (DICESE)

1. **D**raw a free body diagram of the object, indicating distances and angles of interest, whether known or unknown.
2. **I**ndicate external forces acting on the object, including the known forces and moments as well as the unknown support reactions. (more on both of these below).
3. **C**larify the knowns and unknowns.
4. Write the **E**quilibrium equations.
    - At equilibrium, the forces in each direction must sum to zero. In 2D, $\sum F_x = 0$, $\sum F_y = 0$. It doesn't matter what point you pick to sum forces, as long as all forces are applied to the same rigid object. That is, each force applies the same force to any point in the object.
    - In addition, the moments at any point in the object must sum to zero. $\sum M_z = 0$. Recall that a force applied to one point causes a moment at any other point (see definition of moments above). **In addition, a moment applied to one point causes the same moment at any point in the object (just like forces cause same forces)**. This is not usually taught in Phys 121.
    - You should have one equation for each degree of freedom (eg 2D translation and 1D rotation is 3 equations for a plane). However, you could pick two points to sum the moments. If you do this, you will get four equations that are not linearly independent, and you should pick the three that are simplest to solve. This could mean that you replace out one of the force equations with a moment equation, but it is usually simpler to use only one moment equation (for a planar problem).
5. **S**olve for the unknowns. You should have one independent equation for every unknown, or the problem is not solvable. Remember that you can write equations for more than one point on the object. In some cases, these will result in dependent (the same) equations, others independent equations.
6. **E**rror check your answer. You should always ask, does your answer make sense? That is, does the sign and magnitude seem reasonable, when the problem is simple enough that you trust your intuition. You may also want to do simple tests like plugging the answer back into the equilibrium equations to make sure that they come to zero.

The rigorous method for signs.

For simple examples, it is easiest to set up the equations using intuition and the right hand rule to determine the signs of the forces and moments. However, for more complex problems, you want to use a rigorous method to ensure that your signs are correct, so you should use this for easy problems too, to practice where you can test against your intuition to make sure you are doing it correctly. For example, it is a good idea to always draw the support reactions facing in the positive direction, since the actual direction is unknown before solving. This means that the sign of your answer tells the direction, without needing to refer back to the diagram. If you define a positive value to be anything else, you need to clarify the orientation when you state the final answer, since the sign itself no longer provides that information. You may want first identify a coordinate axis that you think will make the problem easiest to solve, and represent the angle of all forces relative to that axis (θ_F). Then, for each force, use

$$F_x = F cos\theta_F$$

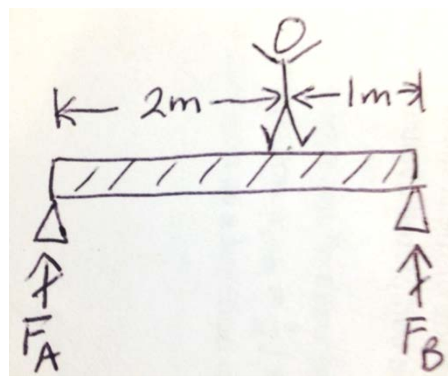
$$F_y = F sin\theta_F$$

$$M_F = rF sin(\theta_F - \theta_r)$$

where θ_r is the angle and r is the length of $\vec{r}$, the direction vector from the point O to where force is applied (see diagram for moments above.). Make sure you include your unknowns the same way. Then, for the equilibrium equations, add all of these up and set to zero for each direction. If your solution results in a negative value for any of your unknowns, this means that the force or moment is in the opposite direction from that you drew. You may prefer to draw unknowns in the direction you think is correct, or you may prefer to draw all in a positive direction. Either way, consider how it was drawn and the sign when you interpret the result.

Hints for how to start: If you don't know where to start on a question, write down the known variables and then unknown variables you need to find. Then, look at the equations at your disposal to find a set of equations that could connect the two sets. In the A section, you may need to use prior knowledge or previous weeks. Make sure you are considering the meaning of all terms, not just the letter used. For example, in problem 1, note that X is what we called L in the notes when we looked at normal stress.

Hint for dealing with units: put everything into SI units, using scientific notation. That is, convert 0.5 um to 0.5e-6 m, or 5e-5 m. If the substitution is not obvious to you, then do the following: remember that 1e-6 m = 1um, so you can multiply or divide by $\frac{1e{-}6\text{ m}}{1\text{um}}$ since that equals one. So here, we say $0.5\text{ um} = 0.5\text{um} * \frac{1e{-}6\text{ m}}{1\text{um}} = 0.5\text{e} - 6\text{m}$ since the two um cancel. Also, remember to convert derived SI units as appropriate. As an engineer, you need to be able to solve problems with real units without making mistakes (which cost $ and lives), so some of our problems will have units and you will be penalized if you make errors in the units.



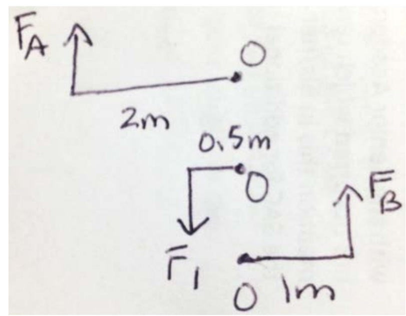
Example 1. Person standing on a beam.

A person with mass 50 kg stands on a beam that is 3 m long and weighs 30 kg. She stands two meters from the left end of the beam, as illustrated in the figure. *What are the support reactions on the two ends?*

1. Draw FBD – this is already done in the problem statement.
2. Indicate all external forces and moments, including both known forces and support reactions.

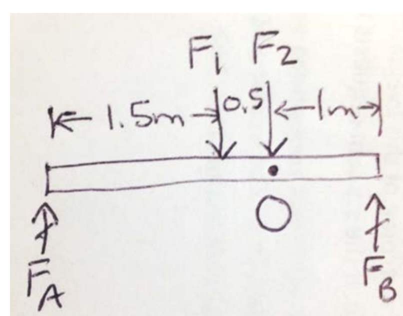
- Calculate the force applied by the person, which we will call F2. $F = ma$, and $m = 50\,kg$ and $a = g = 10\,m/s^2$ so $F2 = 500\,kg * \frac{m}{s^2} = 500N$. F2 is applied 2 m from the left because we are told that.
- Calculate the resultant force from the weight of the beam, which we will call F1 = 300N, using the same conversion from kg to N as above. By symmetry; F1 must be in the center, 1.5 m from each end.
- Define the support reactions. We define FA and FB as the support reactions at the two ends.

3. Clarify the unknowns. We are solving for FA and FB in terms of F1, F2, and the distances indicated.
4. Write the equilibrium equations.



- There are no forces in the x-direction
- $\sum F_y = 0 = -F1 - F2 + F_A + F_B$. Here, I use the standard x-y coordinate system to determine the signs of each. That is, F1 points in the negative y direction, so is negative. Note that you could also have written that the y-forces cancel each other, so $F1 + F2 = F_A + F_B$. This is the same equation.
- $\sum M_z = 0$. To calculate the moments, I need to determine what I'm using as the origin for this equation. I will define the point where the person is standing (see O in figure above). There is no moment applied externally, and no moment for F2, since it point through O, but the moments created on O by F1, F_A, and F_B must sum to zero. For this simple problem, I will determine signs with the right-hand rule; that is, all distances and forces are positive, and the direction of the arrows tells me if the moment is positive or negative. In the drawing to the right, I have sketched the three direction vectors and forces, with the direction vector pointing at the base of the force. The first is negative (thumb points into paper), and the other two are positive. Thus, $M_B = -2m * F_A$ $M1 = 0.5m * F1$, $M_B = 1m * F_B$. Combining these gives $-2m * F_A + 0.5m * F1 + 1m * F_B = 0$

5. Solve for the unknowns.



- We have two equations (from Fy and Mz) and two unknown, F_A and F_B, so this is a well defined problem. To solve, we need to get rid of one of the unknowns.
- Rewrite the $\sum F_y$ equation as: $F_A = F1 + F2 - F_B$.
- Substitute this into the $\sum M_z$ equation

$$-2m * (F1 + F2 - F_B) + 0.5m * F1 + 1m * F_B = 0$$
$$-2mF1 - 2mF2 + 2mF_B + 0.5mF1 + 1mF_B = 0$$
$$-1.5mF1 - 2mF2 + 3mF_B = 0$$

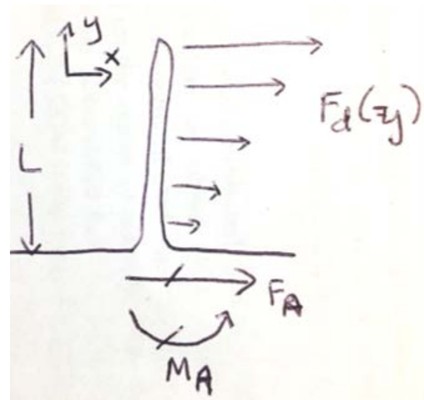
$$F_B = \frac{1}{2}F1 + \frac{2}{3}F2 = \frac{300}{2} + 2 * \frac{500}{3} = 150 + 333 = 483N$$

- Put this into the first equation to solve for FR1:

$$F_A = F1 + F2 - \frac{1}{2}F1 - \frac{2}{3}F2$$
$$F_A = \frac{1}{2}F1 + \frac{1}{3}F2 = 150 + 167 = 317N$$

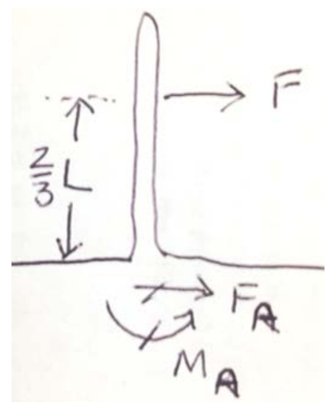
6. Check your answer. The support reaction forces are positive. This makes sense, because the supports should be applying an upward force on the beam. Also, by symmetry, each support should be supporting half the beam, so the ½F1 term in each makes sense. Finally, it makes sense that the reaction closer to the person supports more weight, by the same ratio as the distances, so the 2/3F2 vs 1/3F2 terms also make sense.

Example 2: Cilia in flow.



Cilia are organelles in many endothelial and epithelial cells that consist of a plasma membrane stretched over a bundle of microtubules. They help cells sense mechanical force. Hair cells in the ear have specialized stereocilia that appear in bundles and are involved in hearing. Here we are interested in cells that express a single cilia, which are found in kidneys, blood, etc, where they help regulate cell functions that must respond to fluid shear stress. Mutations in signaling molecules localized near the base of these cilia cause polycystic kidney disease, and these patients often also display vascular abnormalities. Here we ask a simple mechanics question about cilia: *What are the forces and moments at the base of a cilia?*

- Cilia has length L
- We assume: the wall shear rate is S, meaning fluid velocity is V = Sy at a distance y from the surface.
- D = the per length drag coefficient of the cilia. (so force per unit length is D*V)



1. Draw a free body diagram.
2. Indicate external forces and moments. The cilia is exposed to a known force due to fluid drag, with force density $F_d(y) = DV(y) = DSy$. The base is exposed to two support reactions, F_A and M_A, which are in the x-direction and the moment respectively. There are no forces in the y-direction, so we can neglect that support reaction. We can replace the linearly distributed load with a resultant force, F, applied to the cilia at a height Y_R from the wall. This problem is harder to solve with symmetry, so we can use the integral definitions: $F_R = \int F_d(x)dx$, and $X_R = \frac{\int xF_d(x)dx}{F_R}$, but with y instead of x. The integrals will be calculated over the length of the cilia, so from y = 0 to y = L.

$$F_R = \int_0^L DSydy = \frac{DSy^2}{2}\bigg|_0^L = \frac{DSL^2}{2}$$

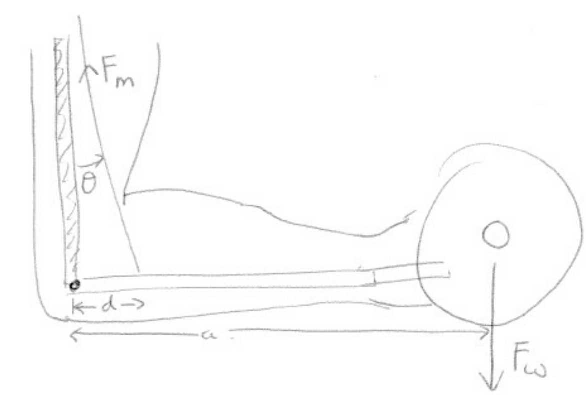
$$Y_R = \frac{\int yDSydy}{F} = \frac{DSy^3}{3F}\Big|_0^L = \frac{DSL^3}{3} * \frac{2}{DSL^2} = \frac{2}{3}L$$

We now redraw the FBD with this resultant force, as indicated in the figure to the right.

3. Clarify the knowns and unknowns. We are solving for M_R and FR in terms of D, S, and L. (FR, YR and y are intermediate variables that must be removed).
4. Equilibrium equations. There are no forces in the y-direction, and the problem is 2D, not 3D, so we have two equations. We will calculate the moments around the base of the cilia (y = 0).
   - $\sum F_x = F + F_A = 0$. Note that all are defined in the +x direction, since we are using the rigorous sign method.
   - $\sum M_z = M_A + M = 0$, where M is the moment due to the drag force FR. This time we will use the rigorous method to calculate M. With $\theta = 0$ on the x-axis, $r = \frac{2}{3}L$, $F = \frac{DSL^2}{2}$, $\theta_F = 0$ and $\theta_r = +90$, so $M = rFsin(\theta_F - \theta_r) = \frac{2}{3}L\frac{1}{2}DSL^2\sin(-90) = -\frac{1}{3}DSL^3$. You should be able to confirm that you would get the same thing if you simply used the right-hand rule.
5. Solve for the unknown. In this case, solving is trivial.

$$F_A = -F = -\frac{DSL^2}{2}$$
$$M_A = -M = \frac{1}{3}DSL^3$$

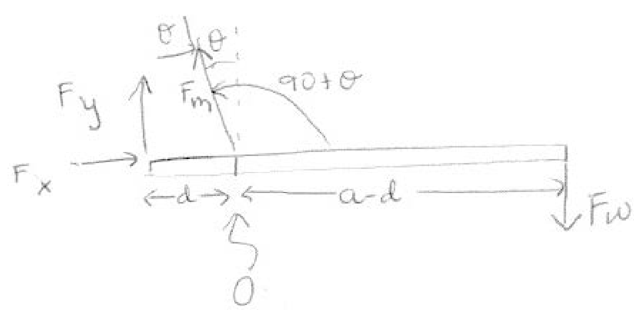
6. Error check your answer. It makes sense that the support reaction force is negative, since the surface must push the cilia in the opposite direction as the drag force. It makes sense that the support moment is positive since the MA as drawn will indeed keep the cilia from rotating due to the drag force.



Example 3. Arm Curls

Assume you curl a weight, Fw, and your lower arm has length a, and your bicep muscle attaches to a position at distance d from the elbow, at an angle θ from the upper arm, as illustrated in the figure to the right. Also assume that you are holding the weight so it does not move. *What is the force on the bicep muscle, Fm?*

1. To draw a free body diagram, we will write the forces on the lower arm. The upper arm is attached at a hinge to the lower arm at the elbow.
2. The external forces are Fw and Fm. A hinge has two support reactions, Fy, and Fx, so we also define these.



3. Our unknown is F_m. Our knowns are a, d, Fw, θ. We also introduced the unknowns F_x and F_y, but we don't need these, so will want to remove them from the equations.
4. To write the equilibrium equations, we need to pick a origin. We will choose the point at which the bicep muscle attaches, as shown in the diagram. Another good option would be the elbow as the origin. You may want to repeat the problem that way as an exercise. This problem is more complex, so we will use the rigorous method for signs.
   - F_m is at angle $\theta_m = 90 + \theta$. It will put no moments on the origin.
   - F_w is at angle $\theta_w = 270$. It acts at a vector $rw = (a - d)$ at angle $\theta rw = 0$.
   - F_x is at angle 0, and F_y is at angle 90. Both act at a vector $r = d$ at angle 180.

   To write the equilibrium equations, recall that you sum $Fcos\theta$ for $\sum F_x$, $Fsin\theta$ for $\sum F_y$, and $rFsin(\theta_F - \theta_r)$ for $\sum M_z$.
   - $\sum F_x = 0 = F_m \cos(90 + \theta) + F_w \cos(270) + F_x \cos(0) + F_y \cos(90)$
   - $\sum F_y = 0 = F_m \sin(90 + \theta) + F_w \sin(270) + F_x \sin(0) + F_y \sin(90)$
   - $\sum M_z = 0 = (a - d)F_w \sin(270 - 0) + dF_x \sin(0 - 180) + dF_y \sin(90 - 180)$

   Which can be simplified by calculating the trig functions:
   - $\sum F_x = 0 = F_m \cos(90 + \theta) + F_x$
   - $\sum F_y = 0 = F_m \sin(90 + \theta) - F_w + F_y$
   - $\sum M_z = 0 = -(a - d)F_w - dF_y$
5. Finally, solve the equilibrium equations to find $F_m$in terms of a, d, Fw, θ.

$$F_y = -\frac{(a - d)F_w}{d}$$

$$F_m \sin(90 + \theta) = F_w - F_y = F_w + \frac{a - d}{d} F_w = \frac{d + a - d}{d} F_w = \frac{a}{d} F_w$$

   Thus,

$$F_m = \frac{a}{d} \frac{F_w}{\sin(90 + \theta)}$$

   Interestingly, we never needed to use the equilibrium equation for forces in the x-direction.
6. Interpret and Error check. Since θ is small, $\sin(90 + \theta)$ is about 1, so the whole value is positive, and indeed the muscle is under tension, matching our experience that the bicep must contract during arm curls. More specifically, this tells us that $F_m \sim \frac{a}{d} F_w$. Since d is close to the elbow, $a \gg d$, and $F_m \gg F_w$. That is, the muscle is applying a much larger force than the weight that is being lifted, because the bicep attaches close to the elbow. Note also that $F_y < 0$. This means that the upper arm bone is pressing downward onto the lower arm bone at the elbow. One might have assumed $F_y > 0$, so that the upper arm would be pulling upward on the lower arm, since something needs to lift the weight. We realize that the muscle more than compensates for the weight.