BIOEN 326 2014 LECTURE 5: STRESS AND STRAIN ANALYSIS

Reviewing Plane Stress

We addressed stress analysis for plane stress, which is where $\tau_{yz} = 0, \tau_{xz} = 0, \sigma_z = 0$.

Assumptions we made. This analysis method is based on equilibrium equations only, meaning that it did not depend on the mechanical properties of the material. This means it applies to linear, nonlinear, plastic, elastic and even viscoelastic materials. However, we did need to calculate the areas of faces to write the equilibrium equations, so we implicitly assumed that these areas did not change. Thus, we assumed small deformations, and our analysis is invalid for large deformations. The allowable deformations depend on the accuracy required for the question being asked.

To review:

- We learned the transformation equations to calculate the normal stress and shear stress along an angle θ relative to the original x-axis.
  - $\sigma_{x\theta} = \sigma_{av} + \frac{\sigma_x - \sigma_y}{2}\cos(2\theta) + \tau\sin(2\theta)$
  - $\tau_\theta = -\frac{\sigma_x - \sigma_y}{2}\sin(2\theta) + \tau\cos(2\theta)$
  - $\sigma_{y\theta} = \sigma_{x(\theta+\frac{\pi}{2})}$
- We defined the average normal stress as: $\sigma_{av} = \frac{\sigma_x + \sigma_y}{2}$, and the amplitude of oscillations in normal and shear stress as $R = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau^2}$.
- The principal stresses are the maximum and minimum normal stresses, $\sigma_{1,2} = \sigma_{av} \pm R$, and occur at the principal angles, $\theta_{p1}, \theta_{p1} = \frac{1}{2}\operatorname{atan}\left(\frac{2\tau}{\sigma_x - \sigma_y}\right)$. To determine which principal angle goes with which principal stress, you can plug θ_{p1} into the transformation equations to obtain σ_1.
- The maximum shear stress is: $\tau_{MAX} = R = \frac{\sigma_1 - \sigma_2}{2}$. These occur at angle on the diagonal from the principal angles: $\theta_s = \theta_p \pm 45°$, or $\theta_s = \frac{1}{2}\operatorname{atan}\left(-\frac{\sigma_x - \sigma_y}{2\tau}\right)$.
- Out-of-plane stresses. You should also remember that $\sigma_z = 0$ is the third principal stress in plane stress, and that there are two more maximum shear stresses that occur as we rotate out-of-plane between σ_1 and σ_z and between σ_2 and σ_z, which are $\tau_{MAX1z} = \frac{\sigma_1}{2}$ and $\tau_{MAX2z} = \frac{\sigma_2}{2}$.

Since we have not yet learned how to calculate the internal stresses on elements in an object from the external forces and moments, we will only solve problems this week where we are told that the stresses are the same throughout the object, so stress simply equals force divided by area, or where we are simply considering an element.

Three dimensions and the Cauchy Stress Tensor. (Not covered in Gere)

Today we consider the more general case of three dimensions, not limited to plane stress. A tensor is a physical property that is in matrix, rather than vector or scalar form. The Cauchy stress tensor is a tensor that completely defines the state of stress at a point:

$$\sigma = \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_y & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_z \end{bmatrix}$$

Note that the Cauchy stress tensor is symmetrical, so only has six unique elements.

If we want to rotate the coordinate axis, we identify a specific rotation matrix A, that allows us to transform the stress tensor with the following equation: $\sigma' = A\sigma A^T$.

As in plane stress, there exists a transformation (rotation, or matrix A) that expresses all stress as normal stress, with all shear stress being zero at this orientation. The normal stresses in this situation are the principal stresses, $\sigma_1, \sigma_2, \sigma_3$. Thus, the Cauchy stress tensor at the principal angles, which we will call the principal Cauchy stress tensor, σ_p, is a diagonal matrix:

$$\sigma_p = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix}$$

We are interested in the matrix A_p such that: $\sigma_p = A_p \sigma A_p^T$. You might realize then that A_p diagonalizes the Cauchy stress tensor. Thus, the principal stresses are simply the eigenvalues of the stress tensor, and the rotation matrix that transforms the original orientation to one aligned with the principal angles is simply the matrix of eigenvectors, A_p.

In MATLAB, you can obtain the principal stresses and rotation matrix from the Cauchy stress tensor with the simple command: [PA, PS] = eig(Cauchy), where you have already defined Cauchy to be the stress tensor. Then, PS is a diagonal matrix of eigenvalues, so the principal stresses are listed on the diagonal, and PA is the matrix where each column is an eigenvector. You can think of PA two ways. First, it is the rotation matrix that transforms the state of stress into the principal stresses. Second, each column provides the axis direction for the associated principal stress.

For example, we earlier considered plane stress, and rotated the coordinate system by an angle θ counter clockwise around the z-axis. The rotation matrix for this rotation is:

$$A_{z\theta} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

You can confirm for yourself that the following equation provides the same transformation equations for plane stress that we derived in the previous lecture.

$$\begin{bmatrix} \sigma_{x\theta} & \tau_\theta & 0 \\ \tau_\theta & \sigma_{y\theta} & 0 \\ 0 & 0 & 0 \end{bmatrix} = \sigma' = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \sigma_x & \tau_{xy} & 0 \\ \tau_{xy} & \sigma_y & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

This can help you to solve more complicated problems using MATLAB.

Solving Failure Analysis Problems for Plane Stress

Example: An element is exposed to uniaxial stress, $\sigma_x = 10\ Pa$ (σ_x is the only nonzero stress). The material has UTS = 12 Pa, UCS = 3 Pa, and USS = 4 Pa. Will the element fail, and if so, along what angle?

Answer:

We solve this problem in three steps:

1) calculate the principal stresses σ_1, σ_2 and the maximum shear stress, τ_{MAX}.
2) check if $\sigma_1 > UTS$, $-\sigma_2 > UCS$, or $\tau_{MAX} > USS$. (The negative sign in the UCS inequality is needed since UCS is positive by convention while a compressive stress is negative).
3) if one of these occurs, identify the angle that corresponds with that stress.

Now we begin:

1) You can calculate σ_{av} and R to get σ_1, σ_2 using the equations above. However, you can solve this more simply by recognizing that uniaxial stress is already oriented in the direction of the principal angles, since there is no shear stress. That is, $\sigma_1 = \sigma_x = 10\ Pa$, and $\sigma_y = \sigma_2 = 0$.
   Again, you can calculate τ_{MAX} from R, but you can solve it more quickly using $\tau_{MAX} = \frac{\sigma_1 - \sigma_2}{2} = \frac{\sigma_x}{2} = 5\ Pa$.
2) Now compare these values to the ultimate stresses:

   $\sigma_1 = 10\ Pa < 12\ Pa = UTS$, so no failure due to tension.

   $-\sigma_2 = 0\ Pa < 3\ Pa = UCS$, so no failure due to compression.

   $\tau_{MAX} = 5\ Pa > 4\ Pa = USS$, so fails due to shearing.

   Thus yes, the element fails.

3) Finally, we calculate the angle of the principal shear stress, since this is the only reason for failure.
   This occurs at angle $\theta_s = \frac{1}{2}\text{atan}\left(-\frac{\sigma_x - \sigma_y}{2\tau}\right) = \frac{1}{2}\text{atan}(-\infty) = \frac{1}{2}90° = 45°$ Again, we could solve this more quickly by recognizing that we must rotate 45° from the principal angle, which was 0.

(If there is more than mechanism of failure, then you would need to identify which occurs 'first'. This means you realize that the load cannot increase instantaneously, so you ask which mode of failure will occur at the lower value of the indicated load.)

Thus, the answer is: "yes, the material will fail due to shearing at an angle of 45 degrees with respect to the direction of uniaxial stress."

Strains: Hooke's Law in 3D. (Gere covers Hooke's law for Plane stress; this is more general)

In week 1, we only considered the strains arising from uniaxial stress for linear isotropic materials. We now need to consider strains that arise in the general case, when stress is defined by a Cauchy stress tensor, but we still limit our analysis to linear isotropic materials. This requires us to learn the following simple rules:

1. It doesn't matter which coordinate system you use to calculate strains. The resulting deformation should be the same, although the way the strain is described will of course depend on the coordinate system.
2. Normal strains resulting from normal stresses add together. For example, the strain in the x-direction is the sum of the axial strain resulting from stress in that same direction, and the lateral strains resulting from stress in the y- or z-directions.
3. Normal strains are not affected by shear stress.
4. Shear strains result only from shear stress in that same direction.

To derive Hooke's law in 3D, we recall that we learned the following rule for lateral strain: $\epsilon' = -\nu\epsilon$. Thus, if we have uniaxial stress with σ_x and all other stresses are zero, the strains are:

$$\epsilon_x = \frac{1}{E}\sigma_x$$

$$\epsilon_y = -\frac{\nu}{E}\sigma_x$$

$$\epsilon_z = -\frac{\nu}{E}\sigma_x$$

Thus, for triaxial stress, $\epsilon_x = \frac{1}{E}\sigma_x - \frac{\nu}{E}\sigma_y - \frac{\nu}{E}\sigma_z = \frac{1}{E}(\sigma_x - \nu\sigma_y - \nu\sigma_z)$, etc.

When shear stress is also present, each shear stress contributes only to the shear stress in the same axes. Recall that $\gamma_{xy} = \tau_{xy}/G$ and that $G = \frac{E}{2(1+\nu)}$ so $\gamma_{xy} = \frac{2(1+\nu)}{E}\tau_{xy}$. This form of the shear strain formula uses the same materials properties (Young' modulus and Poisson ratio) as used for normal strain.

Hooke's law in 3D simply combines all these statements. It is common to express Hooke's law in 3D using the alternative form of the Cauchy stress tensor:

$$\vec{\sigma} = \begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{xz} \\ \tau_{yz} \end{bmatrix}$$

Thus Hooke's law can then be written in matrix form, which can simplify things when writing code for numerical solutions.

$$\vec{\epsilon} = \begin{bmatrix} \epsilon_x \\ \epsilon_y \\ \epsilon \\ \gamma_{xy} \\ \gamma_{xz} \\ \gamma_{yz} \end{bmatrix} = \frac{1}{E} \begin{bmatrix} 1 & -\nu & -\nu & & & \\ -\nu & 1 & -\nu & & 0 & \\ -\nu & -\nu & 1 & & & \\ & & & 2(1+\nu) & 0 & 0 \\ & 0 & & 0 & 2(1+\nu) & 0 \\ & & & 0 & 0 & 2(1+\nu) \end{bmatrix} \begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{xz} \\ \tau_{yz} \end{bmatrix}$$

This can be inverted to obtain an expression for $\vec{\epsilon}$ as a function of $\vec{\sigma}$. Recall that inverting a matrix is simply solving simultaneous equations. We therefore skip the algebra here, but you can confirm for yourself if you are interested; because the matrix is sparse, the algebra is not so bad.

$$\vec{\sigma} = \begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{xz} \\ \tau_{yz} \end{bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & \nu & & & \\ \nu & 1-\nu & \nu & & 0 & \\ \nu & \nu & 1-\nu & & & \\ & & & (1-2\nu)/2 & 0 & 0 \\ & 0 & & 0 & (1-2\nu)/2 & 0 \\ & & & 0 & 0 & (1-2\nu)/2 \end{bmatrix} \begin{bmatrix} \epsilon_x \\ \epsilon_y \\ \epsilon \\ \gamma_{xy} \\ \gamma_{xz} \\ \gamma_{yz} \end{bmatrix}$$

Volumes:

Since any stress tensor can be transformed into the principal stresses with no shear stress, considering the case of triaxial stress is of general interest. In the case of triaxial stress, and where Hooke's law applies, let's consider how volume changes.

If an element has sides of length a, b, c for x, y, z directions, then $V_0 = abc$ is initial volume. As in a previous homework, we see that the final volume is

$$V = a(1+\epsilon_x)b(1+\epsilon_y)c(1+\epsilon_z)$$

which can be rewritten as:

$$V = V_0\left(1 + \epsilon_x + \epsilon_y + \epsilon_z + O(\epsilon^2)\right)$$

where $O(\epsilon^2)$ means "order strain squared" meaning terms with two or more strains multiplied together. If we assume small deformations, then strain << 1 for all three strains, so we can neglect these higher order terms. Thus, $\Delta V = V - V_0 = V_0(\epsilon_x + \epsilon_y + \epsilon_z)$. Finally, we define the **dilatation** is the fractional change in volume: $e = \frac{\Delta V}{V_0} = \epsilon_x + \epsilon_y + \epsilon_z$. We can use Hooke's law to calculate the dilatation for various types of stress, such as uniaxial and biaxial stress. Note that the shear stress does not affect the volume of the material, at least for small deformations, which is what we assume for Hooke's law in 3D.