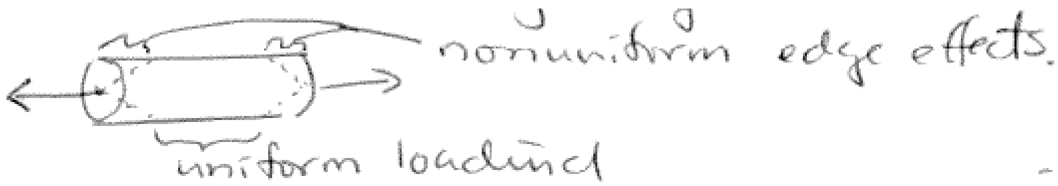
BIOEN 326 2014 LECTURE6: AXIAL LOADS

Optional reference: Gere chapter 2.1 to 2.3, 2.6

Here we consider pure normal force on a bar or a cable, applied at the ends, with no shear force. If the force is distributed evenly over the end areas, then it is equally distributed across the cross-section of the bar or cable at any point along the length, and the normal stress is uniform throughout the beam. Even if the force is not uniformly distributed at the end, then the stress will spread so that it is uniformly distributed throughout most of the bar or cable:



In a few weeks, we will learn that long thin bars can buckle under compression, like when you press the two ends of a ruler together, and it bends, as in the figure below.

We are not ready to calculate buckling effects, so we will only consider tensile loads on bars or cables, or compressive loads on bars that are not too long and thin.

With these assumptions, we consider pure **axial loads** that are uniform across the cross-sectional area of the bar or cable.

Calculating internal forces. To calculate the stress and thus strain, we need to calculate the total tension or compression at any point of interest in a bar or cable. We refer to this tension or compression as an internal force, and we have a new sign convention for it:

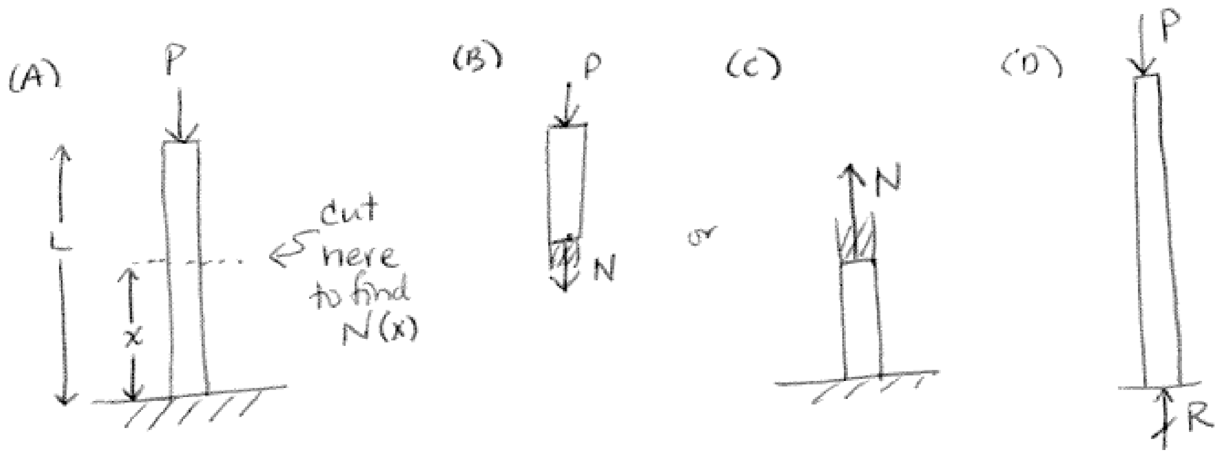
- **Deformation, or Internal sign conventions**. The sign conventions for the internal forces is the same as for normal stress; tension is positive (positive direction on positive face or negative direction on negative face) and compression is negative (negative direction on positive face or positive direction on negative face).
- **Cartesian or External sign conventions**: In contrast, the external forces acting on the bar are positive if facing in the positive axis direction, and negative if facing in the negative axis direction. We have been and will continue to use the external sign conventions to write the equilibrium equations.

To determine the internal forces and thus the internal stresses on a bar or cable, we take the following steps, which are illustrated in the example that follows.

1. Make an imaginary cross-sectional cut across the object at a position of interest.
2. Replace the object on one side of the cut with a solid support that applies a support reaction to the remainder of the object. You will get the same answer regardless of which

section of the object you keep versus replace with a solid support, but one approach will usually be faster.

3. Define the internal force, N, as the support reaction applied by this solid support to the remainder of the object. Use the deformation sign convention, so normal internal force is always pointing away from the object, pulling tension. (plus face, plus direction or minus face, minus direction). If we always do this, then once we solve for N, its sign will tell us whether the internal force is tensile or compressive, without having to return to the diagram for the correct interpretation.
4. Use this FBD to write the equilibrium equations (in one direction for axial loads) to solve for N, using the external sign conventions as always for FBD and equilibrium equations.



Simple example of calculating internal forces. What is the internal force on a short column that is attached to a solid support and has a downward force P applied uniformly across the top surface? This is illustrated in panel A of the figure above. We draw a cut at an arbitrary position x between the two ends. Since this was arbitrary, the answer we get will be true for any position x between the two ends. The value x will appear in our equations only if the internal force varies with the axial position x. We can keep either end after we cut.

If we keep the upper section, we have panel B in the figure. The equilibrium equation is simple: $-P - N = 0$. Note that both arrows point downward, so they are negative using the external sign convention as always for equilibrium equations. Thus, $N = -P$. Since this is negative, the column is under compression, as we can see from the figure.

If we keep the lower half, we have panel C, and we see that we first need to solve for the support reaction from the lower surface. This will obviously be slower, but we can do it anyway to confirm that we get the same answer. In panel D, I draw the FBD showing the support reaction R, pointing upward using the external sign convention. For the forces to balance, R = P. Now panel C has R upward as well, and the equilibrium equation for the lower section in panel C is: $R + N = 0$, or $N = -R = -P$. Yes! This is the same.

Calculating stress and strain from internal forces.

If the bar is prismatic, meaning the cross-sectional geometry is constant along the length, and if the internal force N is also constant, then the area A and internal force N are constants with respect to x (the position along the bar o cable). In this case, the stress is:

$$\sigma = \frac{N}{A}$$

If the materials properties are the same along the length, and we remain within the proportional limit for linear elasticity, then there is a constant young's modulus E, and the strain is:

$$\epsilon = \frac{\sigma}{E} = \frac{N}{EA}$$

We refer to EA as the axial rigidity of the bar or cable.

The cross-sectional area, young's modulus, or internal force can change with x along the length of the bar. For a segmented bar, we can divide the bar into n segments, each of which is uniform, so we refer to the properties of the ith segment as A_i, E_i, N_i. In this case,

$$\sigma_i = \frac{N_i}{A_i}, \epsilon_i = \frac{N_i}{E_i A_i}$$

If instead the bar changes continuously along the length (or within one of the segments), we refer to the properties at position x as $A(x), E(x), N(x)$. In this case,

$$\sigma(x) = \frac{N(x)}{A(x)}, \epsilon(x) = \frac{N(x)}{E(x)A(x)}$$

Checking for failure.

To determine if a beam will fail, we need to identify the position of maximum stress, and compare the state of stress in this position to the ultimate stresses. For many applications, engineering standards define the **maximum allowable stress**, which is a much more conservative value, such as one third of the ultimate stress, in order to provide a degree of safety that addresses unpredictability in operating conditions or material properties. Thus, we will often compare stresses in an object to the ultimate or maximum allowable stresses.

An object will fail if the state of stress at any position of the object surpasses any of the three ultimate stresses (UTS, UCS, and USS). A similar requirement exists for maximum allowable stress. We thus need to identify the positions in the object with the highest state of stress, and then apply stress analysis to identify the principle stresses and the maximum shear stress at these positions.

Note that $\sigma(x) = \frac{N(x)}{A(x)}$ and $\sigma_i = \frac{N_i}{A_i}$, so the highest axial stress occurs where the force is highest and/or the cross-sectional area is smallest. If it is obvious where this occurs, you do not need to check the stress at all positions, but instead can focus on that critical position.

Calculating deformations.

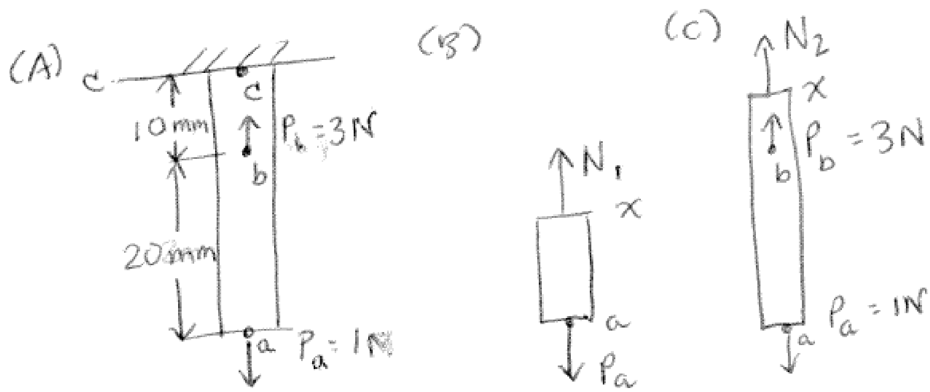
To determine the elongation or shrinkage of a bar or cable, we need to integrate the axial strain over the entire length, L, of the bar. Thus,

$$\delta = \int_0^L \epsilon(x)dx = \int_0^L \frac{N(x)}{E(x)A(x)}dx$$

For any segment with uniform conditions, this means that the deformation, $\delta_i = \epsilon_i L_i = \frac{N_i L_i}{E_i A_i}$.

For a bar of n segments that each deforms by δ_i, the total deformation is $\delta = \sum_{i=1}^{n} \delta_i$.

Example 1:



A bar is attached at point c, the top, to a solid support, and is pulled by $P_b = 3N$ upward at point b, 10 mm from the top, and downward by $P_a = 1N$, at point a, 33 mm from the top.

1) If UTS = 1 MPa, UCS = 1 MPa, and USS = 0.5 MPa, how large must the cross-sectional area A be to prevent failure?
2) What is the total change of length δ in the bar if $A = 10^{-5}\, m^2$ and $E = 2\, GPa$?

We first calculate the internal forces. This is a segmented bar, with segments ab and bc. In panel B, we show the internal force N_1 in segment ab, by making a cut between points a and b. We draw N_1 so that it applies tension to the beam, to make it positive in the deformation convention. Equilibrium then requires that $N_1 - P_a = 0, or\ N_1 = 1N$, which is tension. Next in panel C we consider the internal force N_2 in segment bc, by making a cut between points b and c. We draw N_2 so that it applies tension to the beam. Equilibrium then requires that $N_2 + P_b - P_a = 0$, or $N_2 = P_a - P_b = -2N$, which is compression.

1) Note that segment ab is under the most (the only) tension, so we compare $\sigma_1 = \frac{N_1}{A_1}$ to UTS. Similarly, we compare $-\sigma_2 = -\frac{N_2}{A_2}$ to UCS. Finally, we do stress analysis to address USS. This is uniaxial load, and we learned last week that the maximum shear stress for uniaxial load is half the absolute value of the axial stress, so $\tau_{Max} = \sigma_1/2$, $\tau_{Max} = -\sigma_2/2$, and the maximum observed for this is: $\max(\tau_{Max}) = \max(\sigma_1, -\sigma_2)/2$.

The beam has the same cross-section, so $A_1 = A_2 = A$, is what we seek to find. We find the cross-section that will cause each type of failure:

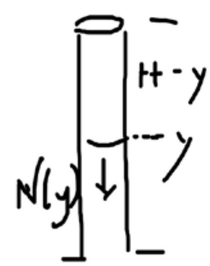
- $\frac{N_1}{A} = UTS$, so $\frac{1N}{A} = 1MPa$, or $A = \frac{1N}{1MPa} = 10^{-6}m^2$.
- $-\frac{N_2}{A} = UCS$, so $\frac{2N}{A} = 1MPa$, or $A = 2 * 10^{-6}m^2$
- $\max(\tau_{Max}) = \max(\sigma_1, -\sigma_2) = USS$, so $\max\left(\frac{1N}{2A}, \frac{2N}{2A}\right) = \frac{1N}{A} = 0.5MPa$, or $A = 2 * 10^{-6}m^2$.

Failure occurs for all values of A smaller than any of these. This means that the bar will fail if the cross-section is as small as $2 * 10^{-6}m^2$ (assuming it doesn't buckle first; this is about 1mmX1mm, and the part under compression is 10 mm long; it is not that thin, but we would need to check this later when we learn about buckling). It will fail in the upper segment that is under compression, either due to compressive stress or shear stress. The lower section would not fail unless the cross-sectional area is half that big, but at this point, the upper section would have already failed.

2) calculate the change of length of each segment and add them together. $\delta_1 = \frac{N_1L_1}{EA} = \frac{1N*20mm}{EA}$ and $\delta_2 = \frac{N_2L_2}{EA} = -\frac{2N*10mm}{EA}$, so $\delta = \frac{20Nmm-20Nmm}{EA} = 0$. The bar does not change length.

<u>Example 2</u>:

Consider a column of cross-sectional area A, height H, and density ρ. What is the stress on the column at an arbitrary height y from the ground?



In this case, we are given the column density, so we do not neglect the mass of the column itself. The force due to this mass varies continuously with height, so we need to find N(y). E and A are constants with respect to y.

We can draw a cut at height y, and replace the lower part of the column with a solid support that applies an internal normal force N(y). We draw this pointing downward to be tensile and thus positive by the deformation sign convention. We can replace the distributed load of gravity with a resultant force at the centroid of this column section. Since the distributed load is uniform, we can multiply density by volume to get mass, and this by gravitational acceleration to get force, which is the same as integrating over the volume. Thus, the resultant force is $F_R = A(H-y)\rho g$, and it points downward. The location of the centroid doesn't matter to this calculation, so we will skip that.

Now, the equilibrium equation for this situation is $-F_R - N(y) = 0$, or $N(y) = -A(H-y)\rho g$. The negative sign indicates compression, which makes sense for this problem.

To calculate stress, we divide internal force by cross-sectional area: $\sigma(y) = \frac{N(y)}{A(y)} = -(H-y)\rho g$.

Error check: does this make sense? This means that at y = H, the top of the column, there is no stress, while at y = 0, the bottom of the column with the whole column pressing down, is $-H\rho g$. That makes sense. We can also check units: $m * \frac{kg}{m^3} * \frac{m}{s^2} = \frac{N}{m^2} = Pa$, which is correct for stress.