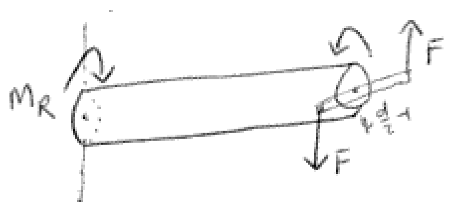
BIOEN 326 2014 LECTURE 7: TORSION

Optional: For additional examples and explanations, read Gere chapter 3.l to 3.3

Here we consider bars that are subjected to twisting, which is a moment that tends to produce a rotation around the longitudinal axis. We also call a twisting moment a **torque**. For example, a twisting moment arises when we turn a screwdriver. To provide pure torsion, we need to avoid net lateral forces on the bar, which would tend to bend rather than twist the bar. While our hand can do this through a complicated set of biomechanical motions, the simplest way to provide pure torsion is through a set of equal and opposite forces, as in the sketch below.



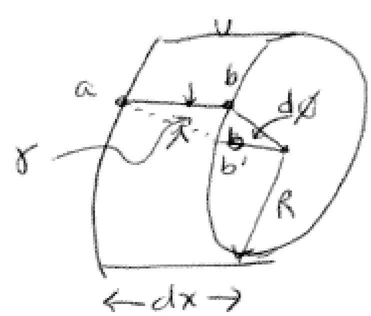
In this case, we have two moments, $M = \frac{d}{2}F$, which are both positive since our thumb points in the positive x-axis direction when applying the right hand rule. Therefore, the solid support must have a support reaction that balances the sum of these two moments. We have already sketched the moment support reaction to be in the negative direction, so do not need to repeat the negative sign, and $M_R = Fd$. The net force due to the concentrated forces is zero, since the forces are equal and opposite. If we had applied only one of the two forces, then we would be applying both a twisting moment and a bending moment to the bar, and we will not address bending moments until the next chapter. We do not worry about sign conventions for pure torsion.

Here we consider solid or hollow bars with circular cross-sections; that is, bars that are radially symmetric. We also require that the bar is prismatic, which means every cross section is identical. In this case, a twisting moment creates pure torsion.

<u>Strains in Torsion</u>

We now derive the torque/stress/strain relationships for pure torsion. We do this by starting with the deformations, or strains.

Imagine that one end of the bar twists by an angle $\phi(x)$, at any position x. Because the bar is prismatic, each section along the bar behaves the same, so we find the strain of a section if length dx, which twists by angle $d\phi$. We call $\frac{d\phi}{dx}$ the rate of twist.



- due to the twist, point b moves to point b'.
- the shear strain on the surface (at radius R) is γ(R).
- recall that $\tan(\gamma) = \delta/H$, where δ is the displacement and H the height of an element, and that for small angles, this approximates as $\gamma = \delta/H$.
- Here the displacement is the length bb' and H is ab=dx, so $\gamma(R) = bb'/dx$.

- However, we also see that $bb' = R\sin(d\phi)$, or since we can make $d\phi$ arbitrarily small by making dx small, that $bb' = Rd\phi$.
- We can set the two expressions for bb' equal, to get: $dx\gamma(R) = Rd\phi$, or $\gamma(R) = R\frac{d\phi}{dx}$

The same logic would have applied for any r, so $\gamma(r) = r\frac{d\phi}{dx}$. That is, the shear strain is zero along the central axis, and increases linearly with radius, so is largest at the outside. It is also proportional to the rate of twist.

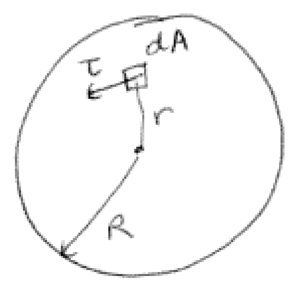
By integrating dx and $d\phi$ along the length of the bar, we also see that $\frac{\phi(L)}{L} = \frac{d\phi}{dx}$, so $\gamma(r) = r\frac{\phi(L)}{L}$.

<u>Stresses in Torsion</u>

Now we consider the stress on the bar. To apply this, we assume that the total stress and strain remain within the proportional limit, so linear materials properties apply and there is a shear modulus, G.

To calculate the stresses, we take a cross-section, and consider an element dA.

- assume a distance r from the center.
- assume a shear stress τ, on the cross section at this element.
- τ must act tangent to the surface, to create the shear strain that moved b to b' in the previous diagram.
- After drawing the shear stress in that direction in this diagram, we can calculate the moment due to the shear stress: $dM = \vec{r} \times \vec{F} = r\tau dA$.
- Because we are within the proportional limit, $\tau = G\gamma = Gr\frac{\phi(L)}{L}$.
- Thus, $dM = Gr^2\frac{\phi(L)}{L}dA$



We can integrate this over the cross-sectional area to get the total moment, which will equal the moment calculated by the applied forces. We call the overall moment the torque, T:

- $T = \int_A dM = \int_A Gr^2\frac{\phi(L)}{L}dA$
- Note that $G\frac{\phi(L)}{L}$ is constant with respect to dA, so can be taken outside of integration
- This leaves $\int_A r^2 dA = I_p$, which we define as the **polar moment of inertia**.
- To calculate I_p, we realize that the integration over area is a double integral with respect to radius and angle, and that $dA = dr \times rd\theta$. Thus, $I_p = \int_A r^2 dA = \int_0^R dr \int_0^{2\pi} rd\theta\, r^2$, or $I_p = \int_A r^2 dA = \int_0^R 2\pi r^3 = \frac{2\pi R^4}{4}$. Thus, $I_p = \frac{\pi}{2}R^4$, for a circle.
- For a hollow cylinder, we subtract the missing polar moment of inertia of the hollow portion from the outer circle, so if the inner and outer radii are Ri and Ro respectively, then $I_p = \frac{\pi}{2}(R_o^4 - R_i^4)$.
- Thus, $T = G\frac{\phi(L)}{L}I_p$, or $\phi(L) = \frac{TL}{GI_p}$.

- We define the **torsional stiffness, or torsional spring constant** of the bar to be $k_T = \frac{GI_p}{L}$. This is a property of the bar that determines how much it rotates in response to a torque: $T = k_T\phi(L)$, analogous to F = kx.
- We define the **torsional rigidity** of the bar to be GI_p. This is a property of the cross-section of the bar.

We can now exchange this value for $\phi(L)$ back into the equation for τ: $\tau(r) = Gr\frac{\phi(L)}{L} = \frac{GrTL}{LGI_p}$, to calculate the stress on any element from the torque. That is: $\tau(r) = r\frac{T}{I_p}$.

And, we can substitute this into the equation for strain: $\gamma(r) = \frac{\tau}{G} = r\frac{T}{GI_p}$.

Analogies to Axial loads:

Note that we just studied responses to axial forces and now to torsion. Let's compare the two.

step	Pure torsion	Pure Axial Force
calculate internal force or moment	$T = M = \sum \vec{r_i} \times \vec{F_i}$	$N = \sum P_i$
calculate stress	$\tau(r) = r\frac{T}{I_p}, \max(\tau) = R\frac{T}{I_p}$	$\sigma = \frac{N}{A}$
calculate strain	$\gamma(r) = r\frac{T}{GI_p}$	$\epsilon = \frac{N}{EA}$
calculate deformation	$\phi(L) = \frac{T}{k_T}, k_T = \frac{GI_p}{L}$	$\delta = \frac{N}{k_a}, k_a = \frac{EA}{L}$

Note that torque has units Nm, and Ip has units m^4, while force has units N and area units m^2. The inclusion of the r in the stress and strain calculations for pure torsion makes up for the difference.

Until now, we considered prismatic bars. If the bar is nonprismatic, we use the same techniques as for axial loads to consider each region separately for segmented bars, or integration for continually changing bars.

Failure and Stress Analysis

For pure torsion, the stress varies with the location of the element, with the highest stress at the surface. For any such situation, we need to consider two maximums when we perform a failure analysis.

1. We need to consider the maximum level of stress (of each kind) over all the elements (positions), which I refer to as $\max(\tau)$ and $\max(\sigma)$ and $\max(-\sigma)$.
2. We need to consider the maximum shear and normal stresses over all orientations at each of those elements, which we call the principle stresses and τ_{MAX}.

Now we apply this to pure torsion.

1. At every element, we have $\sigma_x = \sigma_\phi = \sigma_r = \tau_{xr} = \tau_{r\phi} = 0$, and one nonzero $\tau_{x\phi}$, which we called $\tau(r) = r\frac{T}{I_p}$. Thus, the maximum is: $\max(\tau(r)) = R\frac{T}{I_p}$, which occurs at every point on the surface.
2. At any element, we are already at the orientation providing maximum shear stress, so $\tau_{MAX} = \tau(r)$ and $\max(\tau_{MAX}) = R\frac{T}{I_p}$. The principle stresses at this location are calculated from $\sigma_{1,2} = \frac{\sigma_x+\sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x-\sigma_y}{2}\right)^2 + \tau^2} = 0 \pm \tau = \pm\tau$.

To test for failure, we have to compare these values to UTS, USS, and UCS as in previous situations. We may also compare to **allowable** stresses, which are even lower, by some required safety factor, which addresses acceptable risk, lifetime of use, and other such policies.

<u>Combined Loading</u>

We have now learned how to analyze the effect of either an axial load or a pure torsion on a bar. However, in most real situations, a bar is subjected to multiple types of loads at the same time, which we call **combined loadings**. (This is covered in chapter 7.4 of Gere, but touched on earlier.) To understand the stress and strain of elements under combined loadings, we can apply **superposition**, meaning we simply add together the stresses or strains at each element that result from each of the types of loads, <u>if the following assumptions are met</u>:

a. small deformations (so one load doesn't change the geometry used to calculate a second)
b. linearity (within proportional limit so stress and strain are linear functions of load)
c. no interaction between the various loads; stresses and strains due to one load are not affected by another load

Since we already are assuming small deformations and linearity for most problems where we calculate stress and strain, and since the condition (c) is usually met, we can usually apply superposition.

The approach for combined loadings is

1. select an element in the object of interest
2. Calculate the internal forces and moments that result from each type of load (e.g the internal axial force, the twisting moment or torque, and (after next lecture) the bending moment and shear force).
3. Calculate the normal and shear stresses due to each these forces and moments.
4. Add these individual stresses to obtain the combined stresses.
5. Determine the principal stresses and maximum shear stress using stress analysis. (if needed, e.g. to compare to ultimate stresses).
6. Determine the strains using Hooke's Law in 3D (if needed, eg for deformations).
7. Depending on the question or purpose of the analysis, you may need to repeat for multiple elements of interest, to catch the true maxima or to integrate all deformations.