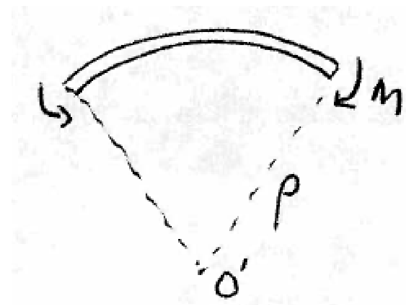
BIOEN 326 2014 LECTURE 9: NORMAL STRESSES ON BEAMS

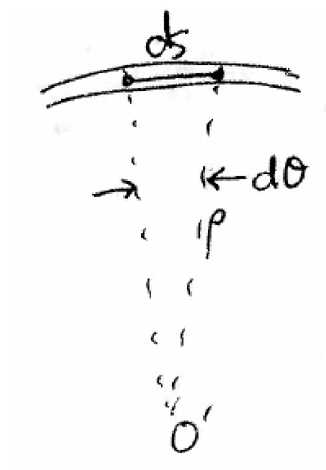
Also read Gere chapter 5.1-5.5

Here we calculate the stresses and strains due to bending moment, M(x) at a position x along the beam.



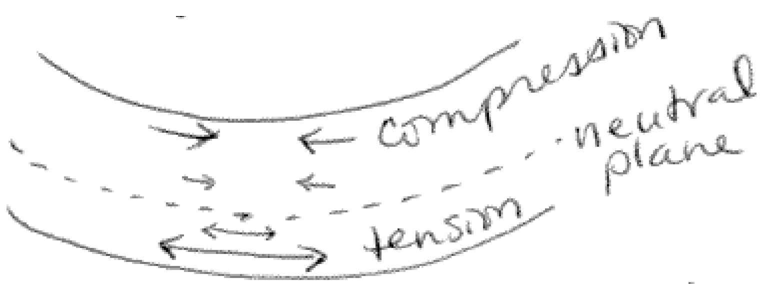
First, we consider pure bending, where V(x) = 0, so M is the same along the beam. Like for pure torsion, we will start by calculating the deformations (strains) expected, then the stresses and finally integrate these to relate this back to the magnitude of the moment.

We define the **radius of curvature, $\boldsymbol{\rho}$**, to be the distance from the beam to where all cross-sectional planes meet, as in figure.

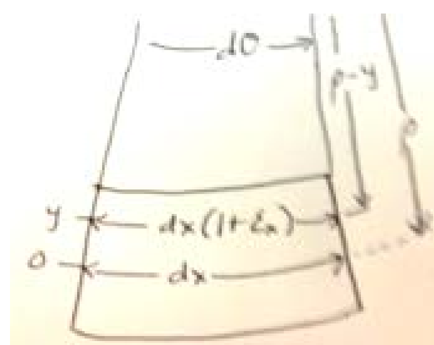


We define the **curvature, $\boldsymbol{\kappa = 1/\rho}$**. Thus a small curvature is a slow bend with a large radius of curvature, etc.

Now we consider an infinitesimally small segment, ds, of this curve. This moves an angle $d\theta$ along the curve. From the definition of circumference, $ds = \rho d\theta$. Thus, $\rho = \frac{dS}{d\theta}$, so $\kappa = \frac{d\theta}{ds}$. The sign convention for κ is the same as for M; smile is positive and frown is negative.



Next we consider the longitudinal strains from this bending. If the beam bends upward (M>0), then the upper region is under compression and the lower region under tension. We define the neutral plane as the surface within the beam that is not under compression or tension. We will learn later how to find this place, but for now, we will call the neutral plane y = 0.



Now consider a segment of length dx. The segment maintains this length on the neutral axis. However, at a height y, it has a length that depends on the longitudinal strain at that height: $dx\big(1+\epsilon_x(y)\big)$. We want to relate this to the curvature, thus the angle $d\theta$, so we note that as before,

$$dx = ds = \rho d\theta$$

but we can also calculate at the new height

$$dx * (1 + \epsilon_x(y)) = (\rho - y) * d\theta$$

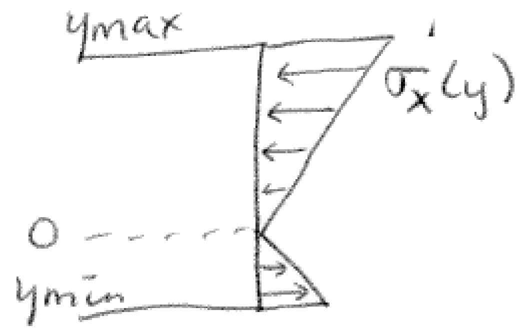
We then subtract these two to obtain:

$$dx * (\epsilon_x(y)) = -y * d\theta$$

And simplify to get:

$$\epsilon_x(y) = -y\frac{d\theta}{dx} = -y\kappa$$

Thus, strain is proportional to the curvature, and varies linearly with the height from the neutral axis.



If we assume the material is linear and isotropic, we also know that $\sigma_x(y) = -yE\kappa$.

Now we will integrate these stresses across the cross sectional area to get the total longitudinal force, which should be zero, since we applied only lateral forces or moments. $\int_A \sigma_x(y)dA = 0$.

When we substitute in the equation we derived above for the stress, we realize that E is a constant and that κ is the same over the entire cross-sectional area, so we take these out of the integral: $E\kappa \int_A ydA = 0$. Thus, we require that $\int_A ydA = 0$. Recall that we calculate the y-position of the centroid with: $y_c = \frac{\int ydA}{A}$, so this simply states that $y_c = 0$, which means that the neutral plane is at the y-position of the centroid. If the beam is symmetric in the y-direction, then the neutral plane is the half-way point.

Next we calculate the contribution of the stress on each element to the bending moment around the neutral axis: $dM = -y\sigma_x(y)dA$. The negative sign must be included because a positive stress applied at a positive y contributes a negative moment. (The stress in the picture above is compressive (negative) for positive y and is contributing a positive moment.) We can substitute in $\sigma_x(y) = -yE\kappa$ to get $dM = +E\kappa y^2 dA$.

The integral of the moment is then: $M = \int_A dM = E\kappa \int_A y^2 dA$

We define $\boldsymbol{I = \int_A y^2 dA}$ **as the moment of inertia** of the area A with respect to the neutral axis. This would be the relevant inertia if you were to spin the cross-sectional area around the neutral axis. This out-of-plane spinning is different from if you spin the area in-plane around the centroid. For that, the relevant inertia is the polar moment of inertia, Ip, which we encountered when we considered torsional moments. Each depends on the square of the distance from the central axis or plane. Both have units of distance to the fourth power.

We can thus rewrite our equation as $M = E\kappa I$, or 'Mike'

$$M = I\kappa E$$

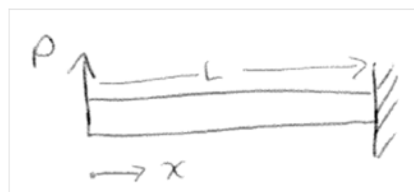
This is the moment-curvature equation.

EI is called the 'flexural rigidity' which determines how easy it is to bend a bar, just as axial and torsional rigidity determined the ease of compressing or twisting a bar.

This argument held for an infinitesimally small segment, so did not actually require that the beam have uniform bending. Thus, we actually have $M(x) = EI\kappa(x)$ or $\kappa(x) = \frac{M(x)}{EI}$.

You may note that the bending moment created no shear stresses within the beam. In the next lecture we will calculate the shear stresses, and will find that these result from the shear forces. Conversely, the shear forces create no longitudinal normal stresses. Thus, we will see that the moment is the only contributor to longitudinal stresses, so the equations derived here are complete.

Example 1.



Consider a simple rectangular cantilever of length L, height H and width W, subjected to an upward load P at the left end and held by a solid support at the right.

a) What is the longitudinal compressive and tensile stress at any location (x,y)?
b) Where do the maximum for these occur, and what are they?

In the previous lecture we determined for this cantilever that $M(x) = xP$.

By symmetry, we know that the neutral axis is the center, so the beam goes from $y = -H/2$ to $y = H/2$.

The moment of inertia is $I = \int_A y^2 dA = W\int_{-\frac{H}{2}}^{\frac{H}{2}} y^2 dy = \frac{Wy^3}{3}\Big|_{-\frac{H}{2}}^{\frac{H}{2}} = \frac{WH^3}{3*8} - \left(-\frac{WH^3}{3*8}\right) = \frac{WH^3}{12}$

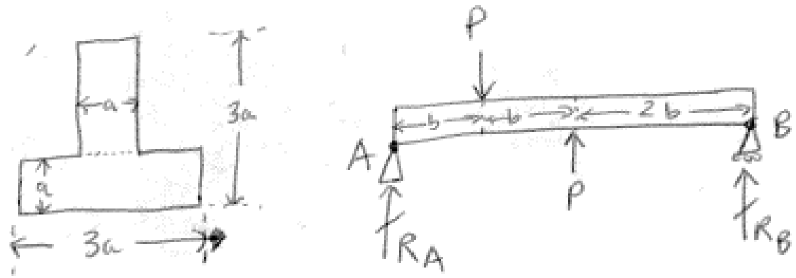
I use that to calculate stress: $\sigma_x(x,y) = -\frac{M(x)}{I}y = -\frac{12P}{WH^3}xy$.

The maximum longitudinal compression is at the base (the solid support), where $x = L$, and at the upper edge, where $y = H/2$. The maximum longitudinal tensile stress is at the bottom of the base, where $y = -H/2$. These values are $\min(\sigma_x) = -\frac{6PL}{WH^2}$ and $\max(\sigma_x) = \frac{6PL}{WH^2}$.

Note that if we wanted to calculate the maximum compressive or tensile stress, without restricting this to 'longitudinal' stress, we would need to apply stress analysis. Before we can that, however, we would need to calculate the shear stress at each location, $\tau(x,y)$, due to the shear force, V(x), since right now we don't know what stresses that applies.

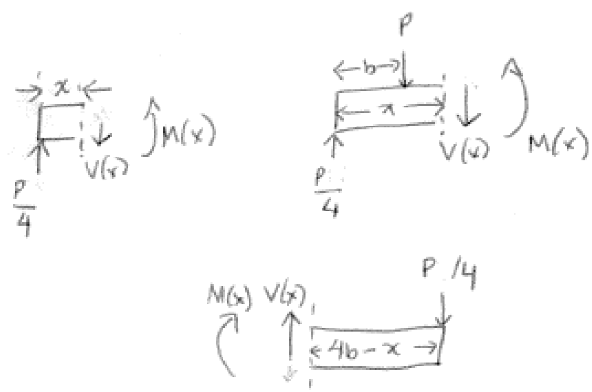
Example 2. Here we ask the same questions for a more complicated situation. The steps are the same, but we need to consider more positions on the beam to figure out the extremes.

A beam with cross-section as shown in panel A is exposed to the forces as shown in the panel B. What is the maximum longitudinal tensile and compressive stress and where are these found?



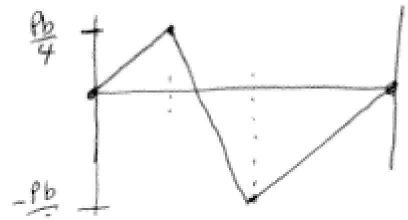
To solve this, we first need to find the maximum and minimum values of M(x).

- For this, we need to solve for the support reactions R_A and R_B. From $\sum F_y = 0$, we have $R_A = -R_B$. We will write the equilibrium equation for moment around point A, the left hand side: $-Pb + 2Pb + 4R_B b = 0$. Thus, $R_B = -\frac{P}{4}$, and from that, $R_A = \frac{P}{4}$.
- Next we solve for M(x) within each segment. I chose the FBD method, with three segments, as shown in the figure. We already had to solve for the support reactions, so I keep the segment with the fewer external forces after the cut. I also note that I don't need to solve for V(x) if I calculate my moments around the cut point.
  - In the left-hand segment, $M(x) - \frac{xP}{4} = 0$; $M(x) = \frac{Px}{4}$.
  - In the central segment, $M(x) + (x-b)P - \frac{Px}{4} = 0$; $M(x) = Pb - \frac{3Px}{4}$.
  - In the right-hand segment, $-M(x) - \frac{(4b-x)P}{4} = 0$; $M(x) = -\frac{(4b-x)P}{4}$.



- To find the min and max of M(x), it is useful to draw the bending moment diagram. Each segment is linear in x, so we can just calculate $M(0), M(b), M(2b), M(4b)$, which are the points between segments. For the internal points, we have two ways to calculate M(x). Since there is no couple , the two values should be the same. (If there were a couple, the two values should differ by M_0, the moment applied by the couple.) Thus,
  - $M(0) = 0$
  - $M(b) = Pb/4$ or $Pb - 3Pb/4 = Pb/4$.
  - $M(2b) = Pb - \frac{3P2b}{4} = -Pb/2$, or $-\frac{(4b-2b)P}{4} = -Pb/2$
  - $M(4b) = -\frac{(4b-4b)P}{4} = 0$

This is shown in the bending moment diagram here. For a sanity check, we note that the beam bends upward on the left part and downward on the right, which makes sense given the forces applied.



- Next, we need to calculate the minimum and maximum value of sigma at these x-positions, which means we need to find the neutral axis, which is the centroid of the cross-section. The easiest way to find the centroid of this cross-section is to consider two separate rectangles. The centroid of the lower rectangle is a height a/2 from the bottom of the beam, while the centroid of the upper rectangle is a height 2a from the bottom. The area of the upper is $2a^2$, and the area of the lower is $3a^2$. In lecture 1, we learned that $y_{com} = 1/M \sum m_i y_i$ where y_i is the center of mass of the i^th subunit. Thus, $y_{com} = \frac{1}{5a^2}\left(3a^2 * \frac{a}{2} + 2a^2 * 2a\right) = \frac{1}{5a^2}\left(\frac{3a^3}{2} + 4a^3\right) = \frac{11a}{10} = 1.1a$. Thus ,the neutral axis is 1.1a from the bottom of the beam, so we refer to this as y = 0. The minimum and maximum values of y are $y_{MIN} = -1.1a, y_{MAX} = 1.9a$.
- We recall from our previous example that $\sigma_x(x,y) = -\frac{M(x)}{I}y$, and the places of interest are $(b, 1.9a), (b, -1.1a), (2b, 1.9a), and\ (2b, -1.1a)$, which are the top and bottom of the beam at the positions of min and max curvature. These evaluate to
  - $\sigma_x = -\frac{1.9}{4}\frac{Pba}{I} = -0.475\ Pba/I$ at $(b, 1.9a)$
  - $\sigma_x = \frac{1.1}{4}\frac{Pba}{I} = \ 0.275\ Pba/I$ at $(b, -1.1a)$
  - $\sigma_x = \frac{1.9}{2}\frac{Pba}{I} = 0.95\ Pba/I$ at $(2b, 1.9a)$
  - $\sigma_x = -\frac{1.1}{2}\frac{Pba}{I} = -0.55\ Pba/I$ at $(2b, -1.1a)$
  - The maximum tensile is thus $0.95\ Pba/I$ at $(2b, 1.9a)$, while the maximum compressive is $-0.55\ Pba/I$ at $(2b, -1.1a)$. In this case, the difference in magnitude of the curvature turned out to be more important than the difference in magnitude of the minimum versus maximum values of y but that will not always be the case for an asymmetric cross-section.
- We have not quite finished however, because we were not given I. Instead, we must calculate it from the given variables, a, b, and P. Note that we could have skipped this step if we only needed to know the locations of the extreme longitudinal stresses. We need $I = \int_A y^2 dA = \int_{A_1} y^2 dA + \int_{A_2} y^2 dA$, for the two rectangles, since integration is a linear operator. Thus, $I = a\int_{-0.1}^{1.9} y^2 dy + 3a\int_{-1.1a}^{-0.1a} y^2 dy = \frac{ay^3}{3}\Big|_{-0.1a}^{1.9a} + \frac{3ay^3}{3}\Big|_{-1.1a}^{-0.1a}$, or $I = a^4\left(\frac{1.9^3}{3} + \frac{0.1^3}{3} - 0.1^3 + 1.1^3\right) = 2.730a^4$. Thus our max and min are:
  - $0.95\frac{Pba}{2.73a^4} = 0.348Pb/a^3$
  - $-0.55\frac{Pba}{2.73a^4} = -0.201\ Pb/a^3$

<u>To summarize:</u>

- We calculate the neutral axis by symmetry or we calculate the y-position of the centroid from the bottom of the beam, $y_c = \frac{\int y dA}{A}$, and use this value as the new y = 0 position.
- Using that new definition for y, we calculate the moment of inertia: $I = \int_A y^2 dA$. The calculus is left as an exercise for the reader:
    - for a rectangle with height H in the y-dimension and width W: $I = WH^3/12$
    - for a circle of radius r or diameter d: $I = \frac{\pi r^4}{4} = \frac{\pi d^4}{64}$
    - for a triangle with height H in the y-dimension and width W: $I = WH^3/36$
    - For a hollow cylinder or rectangle, then the moment of inertia is the difference between the outer geometry and the inner (missing) geometry, because integration is a linear operator. This is the same logic you would use to calculate the area of the cross-sectional geometry for a hollow beam.
- We usually know either M(x) or $\kappa(x)$ for all positions on the bar. Then, we can calculate the other from 'Mike':

$$M(x) = EI\kappa(x)$$
$$\kappa(x) = \frac{M(x)}{EI}$$

- Now we can calculate stress and strain:

$$\sigma_x(x,y) = -\frac{M(x)}{I}y = -E\kappa(x)y$$
$$\epsilon_x(x,y) = -\frac{M(x)}{EI}y = -\kappa(x)y$$