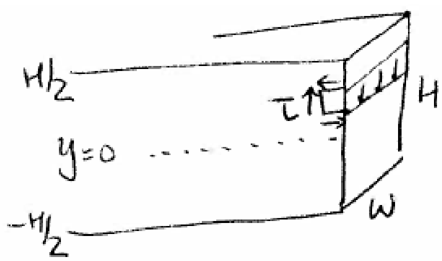
BIOEN 326 2014 LECTURE 10: SHEAR STRESSES ON BEAMS

Also read Gere chapter 5.7-5.8

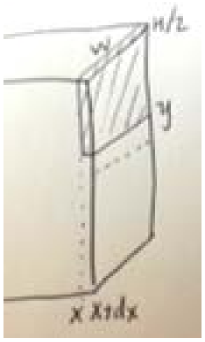
Here we calculate the shear stresses and strains due to internal shear force, V(x) along beam. We know from equilibrium equations that the shear stresses must integrate to the internal shear force at that point: $V = \int_A \tau(y)dA$. However, we don't know how the shear stress is distributed over the height. This is far from trivial, and we will only derive this for rectangular cross-sections. We will also ignore the sign of the shear stress as for torsion, since it doesn't really matter.

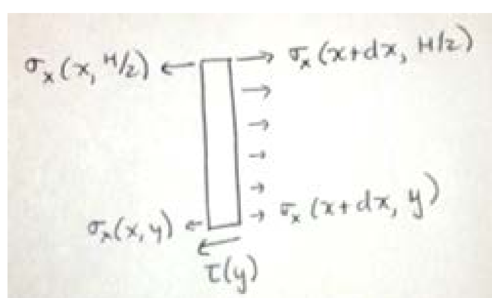


Since there are not forces or differences in geometry in the z-direction (the width), we will always consider a small element as infinitesimally small in x and y, but having the entire width W. Recall that $\tau_{xy} = \tau_{yx}$ for this element.

If this element is on the upper or lower surface of the beam, there is nothing to apply a force on the outer face (the y-face), so the $\tau_{xy}\left(\frac{H}{2}\right) = \tau_{xy}\left(-\frac{H}{2}\right) = 0$. This argument did not arise when we considered the shear stress due to a shear force in lecture 3, because we assumed then that the object was sufficiently large and flat that we could ignore the edge effects along the thin edges. In contrast, a shear stress on the side of the beam can be nonzero, because it is the z-face and the shear stress we consider here is in the x-y plane.

Now consider a big chunk of the beam from x to dx and from an arbitrary y to H/2, as shown in the illustration to the left below. The internal shear stress and bending moments are $M(x)$ and $V(x)$ at the left edge and $M(x + dx)$ and $V(x + dx)$ at the right edge. We apply the equilibrium equation for the forces in the x-direction on this element, which is shown in the sketch here. The forces will have contributions from both the longitudinal normal stress $\sigma_x(x, y)$, and the shear stress $\tau_{xy}(x, y)$, as shown in the illustration to the right below.





Since we need to sum forces, not stresses, we have to integrate each stress over the surface area, which means over width W and for all y' between y and H/2 for the side walls, or over Wdx for the bottom wall. We integrate $\sigma_x(x, y)$ over the x-faces at x to get $-W\int_y^{\frac{H}{2}} \sigma_x(x, y')dy'$ and $x + dx$ to get $W\int_y^{\frac{H}{2}} \sigma_x(x + dx, y')dy'$. We multiply $\tau(x, y)$ at the lower y-face by its area to get $-Wdx\tau(x, y)$.

We thus have:

$$-Wdx\tau(x,y) - W\int_{y}^{\frac{H}{2}}\sigma_x(x,y')dy' + W\int_{y}^{\frac{H}{2}}\sigma_x(x+dx,y')dy' = 0$$

As long as we know M(x), we already know how to calculate these longitudinal stresses; recall that $\sigma_x(x,y) = -\frac{M(x)}{I}y$. Thus:

$$-Wdx\tau(x,y) - W\int_{y}^{\frac{H}{2}} -\frac{M(x)}{I}y'dy' + W\int_{y}^{\frac{H}{2}} -\frac{M(x+dx)}{I}y'dy' = 0$$

Since M(x) doesn't change over the height, this can be removed from the integral:

$$-Wdx\tau(x,y) + W\frac{M(x)}{I}\int_{y}^{\frac{H}{2}} y'dy' - W\frac{M(x+dx)}{I}\int_{y}^{\frac{H}{2}} y'dy' = 0$$

Now note that $\int_{y}^{\frac{H}{2}} y'dy'$ appears in two terms, and is equal to $\frac{y'^2}{2}$ evaluated from y to H/2, or $\frac{H^2}{8} - \frac{y^2}{2}$. We thus combine terms, and substitute in this value:

$$0 = -Wdx\,\tau(x,y) + W\frac{M(x) - M(x+dx)}{I}\left(\frac{H^2}{8} - \frac{y^2}{2}\right)$$

Then we bring the term containing shear stress to one side of the equation and divide by Wdx:

$$\tau(x,y) = \frac{1}{I}\frac{M(x) - M(x+dx)}{dx}\left(\frac{H^2}{8} - \frac{y^2}{2}\right)$$

Finally, we realize that $\frac{M(x+dx)-M(x)}{dx}$ is the definition of the derivative $\frac{dM}{dx}$ as $dx \to 0$. Thus:

$$\tau(x,y) = -\frac{1}{I}\frac{dM}{dx}\left(\frac{H^2}{8} - \frac{y^2}{2}\right)$$

We then remember that $V(x) = dM/dx$, so $\tau(x,y) = -\frac{V(x)}{I}\left(\frac{H^2}{8} - \frac{y^2}{2}\right)$. Finally, we argue that the sign of the shear stress is not important to this analysis, since both signs lead equally to failure (unlike normal stress). Thus, we conclude that the shear stress is:

$$\tau_{xy}(x,y) = \frac{|V(x)|}{2I}\left(\frac{H^2}{4} - y^2\right)$$

At any given cross section (at any x), the maximum τ_{xy} occurs at the height where $\frac{H^2}{4} - y^2$ is maximum, which occurs at y = 0. That is, at any given cross-section, the maximum value of the shear stress τ_{xy}, which we call $\max(\tau_{xy})$, occurs at the neutral plane.

Note the distinction between maximum shear stress, τ_{max}, which is the maximum shear stress in an element calculated from the state of stress in that element using stress analysis, and the maximum *value* of shear stress, $\max(\tau)$, which is the maximum value of shear stress in a specified orientation (here, τ_{xy}, which we are simply calling τ), after comparing all locations in an object. That is, according to the conventions we use in this class, τ_{Max} is the maximum when considering all orientations of a specific element, and $\max(\tau)$ is the maximum when considering all locations but only one orientation.

We can further simplify this by replacing $I = \frac{H^3W}{12}$ and $HW = A$ for a rectangle, so $\frac{H^2}{8I} = \frac{12H^2}{8H^3W} = \frac{3}{2HW} = \frac{3}{2A}$. Thus, for a given cross section with shear force V,

$$\max\left(\tau_{xy}(y)\right) = \frac{3V}{2A}$$

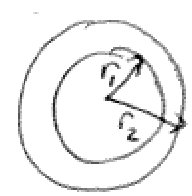
If we want the maximum value, $\max(\tau_{xy}(x, y))$ over the entire beam, we also need to find where $|V(x)|$ is maximum. (We use the absolute value since we do not care about the sign of the shear stress, just the magnitude). You can easily find the extremes of $V(x)$ from the shear force diagram, so this identifies the x-position for the maximum load.

In summary, for a beam with rectangular cross section,

$$\max(\tau_{xy}) = \frac{3\max(|V(x)|)}{2A}$$

Which occurs on the neutral axis where shear force is extreme.

For a beam with a circular cross-section, we skip the derivation. In this case, $\tau_{xy}(x, y, z)$ depends on the z-position as well, since the beam is not uniform in the z-dimension. However, there is still no shear stress on the upper and lower surface, and the maximum value for τ_{xy} still occurs at the neutral axis, with



$$\max(\tau_{xy}) = \frac{4\max(|V(x)|)}{3A}$$

Finally, for a hollow cylinder with outer radius r2 and inner radius r1,

$$\max(\tau_{xy}) = \frac{4\max(|V(x)|)}{3A}\left(\frac{r_2^2 + r_1r_2 + r_1^2}{r_2^2 + r_1^2}\right)$$

<u>Testing for failure</u>.

We are now ready to determine whether the beam is expected to fail. To do this, we …

1. Find the extremes of M(x) and V(x) to identify the locations of interest.
2. comparing these locations, find the extremes of compressive, tensile and shear stress: $\max(\sigma_x)$, $\min(\sigma_x)$ and $\max(\tau_{xy})$.
3. Apply stress analysis at each location of interest to identify the principal stresses σ_1 and σ_2 and maximum shear stress τ_{max} at each of those locations.
4. Compare the values of these at each location of interest to identify the largest tensile stress $\max(\sigma_1)$, largest compressive stress $-\min(\sigma_2)$, and largest shear stress $\max(\tau_{max})$
5. Compare these to the ultimate or allowable stresses of the same type.

Additional comments:

- You may note that shear stress is maximum at the neutral axis, where longitudinal stress is zero, while longitudinal stress is extreme at the upper and lower surfaces, where shear stress is zero. Thus, we only need to identify three spots: one with pure shear, one with uniaxial compression, and one with uniaxial tension, and doing stress analysis on each is easy, since each only has one nonzero stress. Recall that uniaxial stress is already aligned along the principle axes, and that $\tau_{max} = \sigma_x/2$, while pure shear is rotated 45 degrees from the principle axes and has $\sigma_{1,2} = 0 \pm R = \pm\tau_{xy}$. Thus, we have nonzero shear stress where longitudinal stress is extreme, and nonzero compressive and tensile stress where τ_{xy} is highest.
- We will see in the example below that the magnitude of longitudinal stress from the bending moment is usually much larger than the magnitude of shear stress from the internal shear force, for long thin beams. For this reason, failure will usually occur at the locations that provide $\max(\sigma_x)$ or $\min(\sigma_x)$, even if failure occurs due to $\tau_{max} = \sigma_x/2$ at this location. However, loads may be applied in a way that causes stresses due to V(x) to be greater, so you do need to check this by completing the complete or at least a convincing preliminary calculation.
- In a situation of combined loads, then the steps are the same, but you will need to consider the combination of loads to calculate $\max(\sigma_x)$, $\min(\sigma_x)$ and $\max(\tau_{xy})$. Note that axial load will only affect σ_x and torsional load will only affect $\tau_{x\phi}$. At the top and bottom of the beam, $\tau_{x\phi} = \tau_{xz}$, while at the side of the beam, $\tau_{x\phi} = \tau_{xy}$. In each situation, you will need to consider whether to look at additional positions, and you will need to include all the relevant stresses at each of these locations in your stress analysis.

Example 1:

Consider a simple rectangular cantilever of height H=2W, width W, and length L=100W, subjected to an upward load P at the left end and held by a solid support at the right. Find the maximum compressive, tensile and shear stresses that you would use to test whether the beam is within allowable limits, and indicate where these occur on the beam.

1. Find V and M. From examples in the previous lectures, we learned that V(x) = P, is the same for all x, and M(x) = xP, has an extreme at x = L; M = PL = 100 PW
2. We use these locations to:
   a. find max $(\sigma_x) = \frac{6PL}{WH^2}$ (from previous lecture). We substitute in the values for L and H: $\max(\sigma_x) = \frac{600PW}{W4W^2} = 150\ P/W^2$. This occurs at the bottom of the beam at the solid support, $x = L, y = -W$.
   b. Find $\min(\sigma_x) = -\frac{600PW}{W4W^2} = -150\ P/W^2$. This occurs at the top of the beam at the solid support, so at $x = L, y = W$.
   c. Find $\max(\tau_{xy}) = \frac{3V}{2A} = \frac{3P}{3W^2} = 0.75\ P/W^2$. This occurs at the neutral axis of the beam, all along its length, so at all $(x, 0)$.
3. Apply stress analysis at these positions.
   a. At $(L, -W)$, $\sigma_1 = 150P/W^2$, $\sigma_2 = 0$, and $\tau_{max} = \max(\sigma_x)/2 = 75\ P/W^2$.
   b. At (L, W), $\sigma_1 = 0$, $\sigma_2 = -150P/W^2$ and $\tau_{max} = 75\ P/W^2$.
   c. At $(x, 0)$, $\sigma_{1,2} = \pm\max(\tau_{xy}) = \pm 0.75\ P/W^2$, and $\tau_{max} = \tau_{xy} = 0.75P/W^2$
4. Thus,
   a. $\max(\sigma_1) = 150\ P/W^2$ (bottom of beam at support)
   b. $-\min(\sigma_2) = 150\ P/W^2$ (top of beam at support)
   c. $\max(\tau_{max}) = 75\ P/W^2$ (top and bottom of beam at support)
5. We would compare those to UTS, UCS, and USS respectively.

Note that the bending moment causes 100-fold larger shear stress than does the internal shear force, for this example where the length is 100-fold larger than the half-height and width. This is because the equation for maximum longitudinal stress has the parameter L, which arose because the shear force load P was multiplied by the length to determine the moment. This is generally the case for slender beams. However, if you do not know that one is much larger, one should rigorously check both $\max(\sigma_x)$ and $\max(\tau_{xy})$ when testing for failure.

To summarize the three beam lectures:

1. External shearing loads, P, induce shear forces V(x) and bending moments M(x) within the beams. External couples produce bending moments. You can calculate V(x) and M(x) from these external loads and couples using free body diagrams or the integration method.
2. The bending moment M(x) determines the longitudinal stress, $\sigma_x(x,y) = -\frac{M(x)}{I}y$. You calculate $I = \int_A y^2 dA$, where y is the height from the neutral plane (the y-position of the centroid of the cross-section).
3. The shear force V(x) determines the shear stresses
   a. The distribution and maximum values depend on geometry, not just I or A:
   b. For rectangular cross section, $\tau(x,y) = \frac{V(x)}{2I}\left(\frac{H^2}{4} - y^2\right)$, so shear stress varies parabolically with height. max(τ(x)) = $\frac{V(x)}{2I}\left(\frac{H^2}{4}\right) = \frac{3V(x)}{2A}$ occurs at y = 0, the neutral axis.
   c. For circular cross –section, max($\tau(x)$) = $\frac{4V(x)}{3A}$ occurs at y = 0 also.
4. The beam usually fails at the x-position where the bending moment is maximum, and the y-position of the upper or lower surface. The failure here could be due to compressive stress (inside the bend), tensile stress (outside the bend) or shear stress (both locations). To be rigorous, you should also check the principle stresses and maximum shear stress at the neutral plane where the shear force has the highest magnitude, but these are usually lower for slender beams, except for certain loading conditions.

Deflection due to shear forces and bending moments .

Soon we will determine how beams deflect under these kinds of loads. This will require us to integrate the accumulated strains over the length of the beam. Slender beams can undergo large deflections while remaining within the proportional limit.