BIOEN 326 2013 LECTURE 14: BEAM OR COLUMN BUCKLING

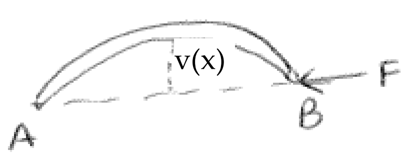
Also read Gere chapter 9

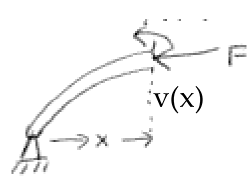
Now we return to ask about axial compressive force on a bar/rod/beam:

This happens often with columns. We learned before that the compressive stress is calculated by dividing the force by the cross-sectional area: $\sigma_x = -F/A$. However, we noted then that we were assuming that the force could not buckle a column. Buckling is when the beam bends, so that it shortens due to bending, as illustrated below.

We noted that a long slender column was susceptible to buckling, and now we are ready to determine exactly when buckling occurs, now that we have learned how to calculate bending of beams. Specifically, we will ask how much force it takes to buckle a rod, which will depend on L and EI. We will also ask whether this force is less than or more than the force needed to exceed the ultimate compressive stress under an axial load. This will determine whether we need to consider buckling or not.

First we will consider a rod held by pins at both ends, so the deflection v(o) = v(L) = 0, as illustrated in the figures above. Between 0 and L, however, we assume that the rod is deflected by a small amount v(x) due to application of a transient lateral load. We ask whether the beam will maintain a bend or straighten when the transient lateral load is removed.

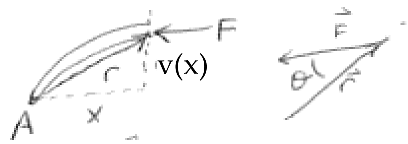




To answer this question, we use a Free Body Diagram on the left half of the beam at position x. Specifically, we will add the moments around the left pin. The pin cannot support a moment, so the internal bending moment at point x must be offset by the moment created by the axial force F applied at all positions x, which is slightly out of alignment with the pin because of the bend. That is, $0 = M(x) + \vec{r} \times \vec{F}$.

M(x) is the internal bending moment at M(x). We learned last lecture that $M(x) = EI\frac{d^2v}{dx^2}$.

To find $\vec{r} \times \vec{F}$, we sketch the geometry.



$\vec{r} \times \vec{F} = rFsin\theta$, and $sin\theta = v(x)/r$, so $\vec{r} \times \vec{F} = v(x)F$

Thus, our moment equation is

$$EI\frac{d^2v}{dx^2} + v(x)F = 0$$

This also has the form:

$$\frac{d^2v}{dx^2} = -\frac{F}{EI}v(x)$$

You may recall that this has the general solution $v(x) = C_1\sin(kx) + C_2\cos(kx)$, with $k = \sqrt{\frac{F}{EI}}$.

This tells us that the shape of the rod is the sum of sinusoidals. To get the specific solution for this problem, we need to remember the boundary conditions. The pins do not restrict the angle, dv/dx, of either end, but do restrict $v(0) = v(L) = 0$.

Thus, $v(0) = 0 = C_1\sin(0) + C_2\cos(0) = 0 + C_2$. This requires that $C_2 = 0$.

In addition, $v(L) = 0 = C_1\sin(kL)$.

One possibility is the trivial solution, which is that $C_1 = 0$, so $v(x) = 0$ for all x. This means that the rod is straight. This trivial solution is possible regardless of k, so regardless of F and EI, one possibility is that the rod is straight. This trivial solution is an unstable equilibrium.

The other possibility is that $\sin(kL) = 0$. This occurs when$kL = 0, \pi, 2\pi, ...$ That is, $kL = n\pi$. If $n = 0$, then $kL = 0$, so $k = 0$, which again is the trivial solution. So we next consider $n = 1$. That is, one solution is that $\sqrt{\frac{F}{EI}}L = \pi$, or

$$F = \frac{\pi^2 EI}{L^2}$$

We refer to this as the Euler buckling force. For this specific force, we have a nontrivial solution. We exchange $k = \sqrt{\frac{F}{EI}} = \sqrt{\frac{\pi^2 EI}{L^2 EI}} = \frac{\pi}{L}$. Thus, the nontrivial solution at the Euler force is

$$v(x) = C_1 \sin(kx) = C_1 \sin\left(\frac{x}{L}\pi\right)$$

This is a half-sinusoid of arbitrary amplitude:

To understand what this means, imagine you steadily increase the axial force on a rod. Below the Euler force, the only solution to the FBD is the trivial solution, so the only equilibrium solution is a straight rod, and if you deflect the beam slightly, it will return to zero deflection. This means that below the buckling force, the beam does not buckle.

At the Euler force, the beam has an infinite number of equilibrium solutions, with arbitrary amplitudes. Thus, when this force is reached, a deflection will be maintained. Indeed, since the beam shortens slightly during deflection, which is energetically favorable as force is applied, the deflection will increase. This means that the beam will buckle at the Euler force if there is any transient deflection.

However, even at the Euler force, the trivial solution is still valid, so a column may survive at or above the Euler force if there is no transient deflection. However, because it is in an unstable equilibrium, any deflection will cause instant buckling.

Now we note the significance of the assumptions of our FBD above.

First, we assumed small deflections. This means that any arbitrary but small amplitude satisfies the equilibrium equation. As the amplitude gets larger, the approximation $M(x) = EI\frac{d^2v}{dx^2}$ no longer applies. While we do not do the calculations, a larger force is necessary to satisfy the equation for larger deflection. Thus, while the sinusoidal solution is only valid for the Euler force, the column buckles at any force equal to or above the Euler force. The buckling will continue until the deflection is enough for the force applied, or until the longitudinal stress due to bending exceeds the ultimate compressive or tensile stress.

We thus don't need to consider the higher modes, with $n = 2, n = 3, etc,$ unless the rod is restricted to prevent the first mode, as in the figure below.

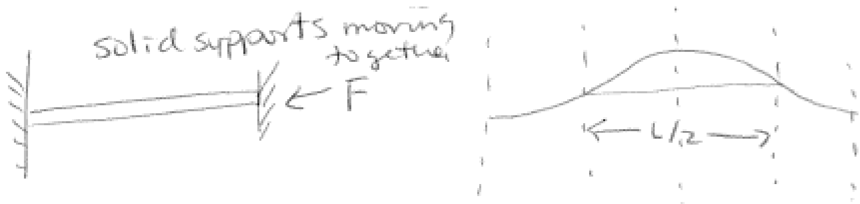
Our second assumption involved the support. If we have different supports at the ends of the rod, we need to start over with the boundary conditions. However, we will take a simpler approach.

If we have a single solid support, we realize that this is like half of the rod we studied above:

Thus, the buckling force is the same as what we calculated above, but for 2L:

$$F = \frac{EI\pi^2}{(2L)^2} = \frac{EI\pi^2}{4L^2}$$

If we have two solid supports that restrict lateral motion but not length, then we have $v(0) = v(L) = 0$, and $\frac{dv}{dx}(0) = \frac{dv}{dx}(L) = 0$.



In this situation, the portion of the sinusoid measured before occurs over the middle L/2, so

$$F = \frac{EI\pi^2}{(L/2)^2} = \frac{4EI\pi^2}{L^2}$$

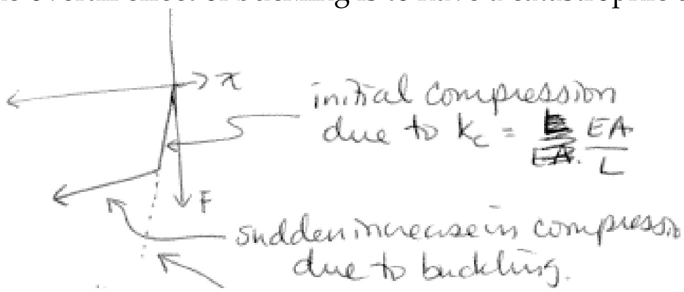
If we have a solid support and a hinge, then deflection and angle are fixed at one end, and just position at the other.

The sinusoid measured before occurs over 0.7L, so

$$F = \frac{EI\pi^2}{(0.7L)^2} \sim \frac{2EI\pi^2}{L^2}$$

Notes:

1. hollow tubes have a higher moment of inertia than a solid tube with the same cross sectional area, by allowing a larger outer radius. Thus, in rods or columns that need to avoid buckling but are constrained in material costs or by weight, a hollow tube is ideal.
2. The overall effect of buckling is to have a catastrophic effect on the force-length curve.



   Buckling has the same effect as yielding, but on compressive rather than tension, and occurs within the linear elastic range of material deformation.

3. If an asymmetric beam is subjected to an axial load, it will buckle in the direction of lowest moment of inertia. Thus, while (for example) a 2X8 on its side is effective for a beam to prevent bending, square posts are used to prevent buckling.
4. We define the **radius of gyration to be $r = \sqrt{\frac{I}{A}}$**, which has units of meters and is another property of cross-sectional geometry. We then define the '**slenderness ratio' to be L/r,** which is unitless. If the rod or beam is asymmetric, we use the smallest value of r for the various orientations that can be used to calculate I for the reason of point 3.
5. Recalling that a column fails due to ultimate compressive stress when $UCS = \sigma_x = F/A$, it can be useful to ask whether the compressive stress at the buckling force, which we will call σ_C is greater than or less than the ultimate compressive stress. That is, will the column already fail due to compression before it buckles, or will it buckle first?

$$\sigma_c = \frac{F_c}{A} = \frac{\pi^2 EI}{L^2 A} = \frac{\pi^2 E}{\left(\frac{L}{r}\right)^2}$$

   This depends on the young's modulus and slenderness ratio. If $\sigma_C < UCS$ then the column will buckle, while if $UCS < \sigma_C$, the column will compress to failure. Thus, the columns 'fails' due to buckling when σ_C is small, so when the rod is soft (small E) or slender (large L/r).

Buckling is part of the normal behavior of many subcellular polymers, like microtubules, actin filaments, and bacterial adhesive organelles called fimbriae. Often, buckling is not associated with failure; that is, when the buckling force is removed, the rods retake their original shape (like the ruler). This may occur because the rod is so slender that it can buckle and curl in circles without exceeding the allowable stresses or strains, or because the rod is held together by reversible bonds that can break and reform.