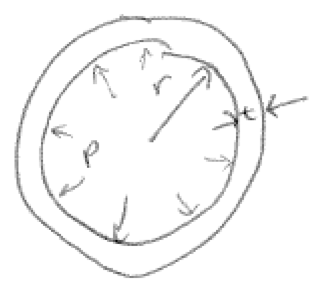
BIOEN 326 2014 LECTURE 16: PRESSURE VESSELS

Also read Gere chapter 7.1-7.7

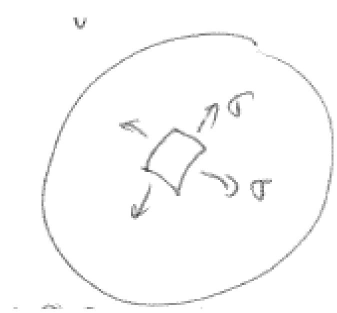
Arteries, balloons, compressed air cylinders, and similar objects are called **pressure vessels** because they hold fluid under pressure. A common question with pressure vessels is how much pressure the vessel can tolerate before failing.

Certain genetic or acquired diseases can weaken blood vessel walls, so that the pressurized blood can cause a balloon-like bulge to appear in artery walls. This is called an **aneurism**. In addition to the conditions that weaken the vessel, hypertension is a risk factor, since higher pressure can cause more damage. The aneurism is itself a pressure vessel, and at risk for rupturing, again particularly with high blood pressure. Even if aneurisms don't rupture, the irregular patterns of blood flow in and around the aneurism cause the endothelial cells lining the vessel to become inflamed, which increases the risk of thrombosis. The thrombus can break free and lodge downstream, causing heart attacks, strokes, or other problems. Thus, there is significant medical interest in understanding the properties of arteries as pressure vessels, and in understanding how they fail.

Compressed air tanks are made of rigid material, and expand little under pressure, while arteries and balloons are made of soft extensible materials that can expand large amounts within their regular working conditions. We will start with small deformations, as explained in Gere, but then extend our analysis to consider softer materials in order to apply our knowledge to aneurisms.

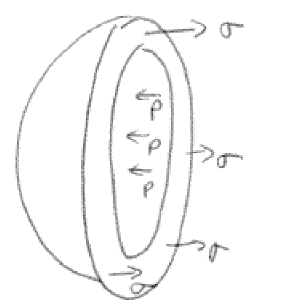


We thus start by asking how much stress and strain is in a pressure vessel under pressure P. We consider atmospheric pressure to be zero, since materials are considered to be under no stress and strain at atmospheric pressure. Thus, the pressure P is the increase over atmospheric. We also assume radius r and thickness t, with $r \gg t$, meaning the radius is perhaps at least 5 times as large as the material is thick. Therefore, we refer to these as **thin-walled pressure vessels**.



To calculate stress and strain, we first note the following. By symmetry, the in-plane stress at any point in the vessel must be the same in all locations and in both in-plane axes. That is, the element is under uniform biaxial stress, $\sigma_x = \sigma_y$, which we will simply call σ. So we already know that $\tau_{xy} = 0$ and we only have one stress to find. Now we draw a free body diagram for a spherical pressure vessel. Regardless of how we cut it, the cross-section looks the same.

We now want to add up all forces applied by one half of the sphere to the other.

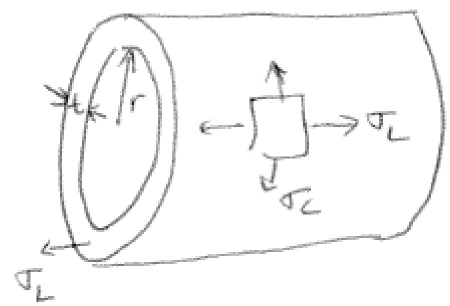


First we need to integrate the normal stress over the area of the wall cross-section, which is the perimeter times the thickness, or $A_{wall} = 2\pi rt$, so $F_\sigma = \sigma 2\pi rt$. This force is pulling, since the pressurized vessel is stretched so under tension.

Second we need to integrate the pressure over the area of the fluid in cross-section, which is πr^2 , so $F_p = P\pi r^2$. This force is pushing. To be at equilibrium, these must balance, so $\sigma 2\pi rt = P\pi r^2$. We simplify and rearrange to get

$$\sigma = \frac{pr}{2t}$$

Thus the stress in the pressure vessel wall is proportional to pressure, but also to the radius of the vessel and inversely proportional to the thickness. This is called **Laplace's Law for Pressure Vessels**.

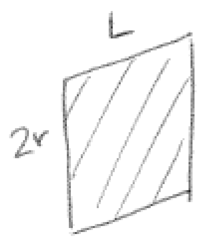


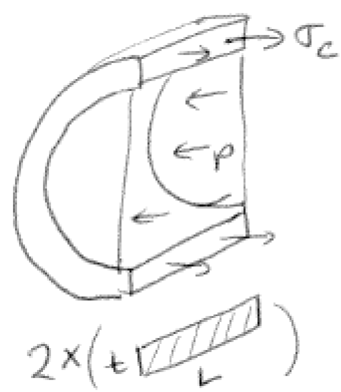
Now we turn to cylindrical pressure vessels. We still have r and t defined as before, with the same requirement for thin walls. However, we now have two stresses to consider.

The **longitudinal stress $\boldsymbol{\sigma_L}$** parallel to long axis and **circumpherential stress $\boldsymbol{\sigma_C}$** perpendicular to long axis.

If we cut a cross-section, we get the exact same conditions on the cross-section as before, so we don't need to repeat the derivation to know:

$$\sigma_L = \frac{pr}{2t}$$





If we cut the other way, for a section of arbitrary length L, we get two pieces of material with length L and thickness t, for an area of $2Lt$, over which σ_C acts.

We also get a fluid surface of area $2rL$, over with P acts. Thus, our equilibrium equation is $\sigma_c 2Lt = P2rL$, which simplifies to

$$\sigma_c = \frac{pr}{t} = 2\sigma_L$$

Thus, cylindrical pressure vessels are under asymmetric biaxial stress, with the highest stress in the circumpherential direction, and two-fold higher than that of a sphere of same r, t, and P.

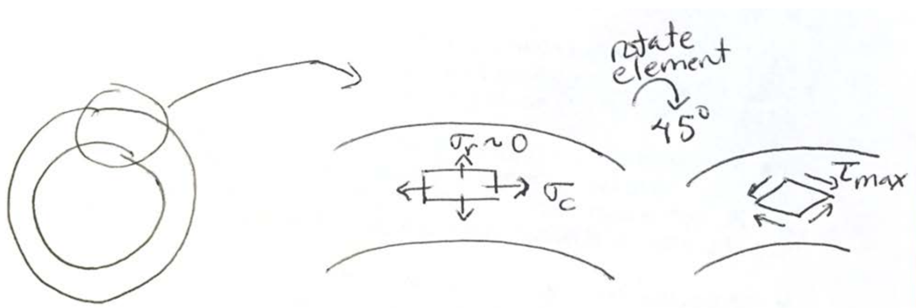
Before we apply stress analysis to calculate the maximum shear stress, we need to consider the out-of-plane normal stress. This is the stress perpendicular to the plane of the vessel wall. We will call this σ_r, since it points in the radial direction, that is, along a line from the center of the sphere or cylinder through the plane of the vessel wall, as illustrated in the central panel below. For both shapes, inside the vessel, $\sigma_r = -p$ since there is compression due to the fluid pressure. Outside the vessel, $\sigma_r = 0$ since we have normalized to atmospheric pressure. The material in the cylindrical pressure vessel is thus under triaxial stress with $\sigma_C = \frac{pr}{t}, \sigma_L = \frac{pr}{2t}, 0 < \sigma_r < p$. However, since $r \gg t$, $\frac{pr}{t} \gg p$, and we can neglect the nonzero stress $-p$ inside the vessel when we consider the overall state of state of stress. We also note that there is no shear stress in this coordinate system ($\tau_{CL} = \tau_{Cr} = \tau_{Lr} = 0$). Therefore:

For a sphere

$$\sigma_x = \sigma_y = \frac{pr}{2t}, \sigma_r \sim 0, \tau_{CL} = \tau_{Cr} = \tau_{Lr} = 0$$

and for a cylinder,

$$\sigma_c = \frac{pr}{t}, \sigma_L = \frac{pr}{2t}, \sigma_r \sim 0, \tau_{xy} = \tau_{xr} = \tau_{yr} = 0$$



The maximum in plane shear stress is $R = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}{}^2} = \sqrt{(0)^2 + 0^2} = 0$ for a sphere and $R = \sqrt{\left(\frac{\frac{pr}{2t} - \frac{pr}{t}}{2}\right)^2 + 0^2} = \sqrt{\left(\frac{pr}{4t}\right)^2} = \frac{pr}{4t}$ for a cylinder. However, we also need to consider the maximum out-of-plane shear stress, which is caused by the difference in maximum in-plane normal stress and σ_r. That is, if we consider a planar element that is within the cut plane of the vessel as shown above, we can draw the imaginary element rotated by 45 degrees, and see that it is sheared, with the shear stress being τ_{max} as we defined it before, but for $\sigma\ and\ \sigma_r$ or $\sigma_c\ and\ \sigma_r$. The fact that the stress perpendicular to this planar element ($eg\ \sigma_L$) is nonzero does not affect this calculation; we can still use the stress analysis equations for rotation within any given plane.

Thus, for a sphere, $\tau_{max} = R = \sqrt{\left(\frac{\sigma_x - \sigma_z}{2}\right)^2 + \tau^2} = \sqrt{\left(\frac{pr/2t}{2}\right)^2 + 0^2} = \frac{pr}{4t}$, and for a cylinder, $\tau_{max} = \sqrt{\left(\frac{pr/t}{2}\right)^2 + 0^2} = \frac{pr}{2t}$.

In summary, for pressure vessels, assuming thin walls ($r \gg t$), and that r and t are known:

For a spherical pressure vessel, the walls are under approximately biaxial symmetric plane stress, with $\sigma = \frac{pr}{2t}$, which is also the maximum normal stress. There is no compressive stress. The maximum shear stress is out-of-plane, so at a diagonal angle from inside to outside the vessel, at $\tau_{Max} = \frac{pr}{4t}$.

For a cylindrical pressure vessel, the walls are under approximately biaxial asymmetric plane stress, with the maximum stress around the circumpherence, $\sigma_C = \frac{pr}{t}$, and a lower stress along the axis, $\sigma_L = \frac{pr}{2t}$. There is no compressive stress. The maximum shear stress is out-of-plane, at a diagonal from inside to outside and in the circumpherential direction, with $\tau_{Max} = \frac{pr}{2t}$.

Large deformations in pressure vessels.

We should note that in these calculations, we used r and t in the current condition of stress. This means either that we have measured the dimensions (r and t) under stress, or that deformations are very small ($\epsilon \ll 1$) and we measured r_0 and t_0, the dimensions without stress, so that we can assume $r \sim r_0$ and $t \sim t_0$. However, if we only know r_0 and t_0, and deformations are large, we need to develop new theory and equations to address this.

Many biological pressure vessels, such as arteries, can stretch to several fold their native size. Indeed, they need to do so in order to regulate blood flow to the downstream tissues. Thus, if we use the equations provided above and we only know the unstressed dimensions, r_0 and t_0, we should check afterwards if $\epsilon = \frac{\sigma}{E} = \frac{pr}{Et} \ll 1$, so $p \ll \frac{Et}{r}$. If the strain is not small (say, less than 0.05), then we have violated this assumption.

Now consider what happens if strains are significant, but the material is still linear. The derivation in Gere is true for the actual radius and thickness of the vessel wall, which are related to the nominal (no force) radius and thickness by the strains. If ϵ_C is the strain in the circumferential direction, ϵ_T is the strain in the thickness, C_0 is the circumference and t_0 the thickness of the unstressed vessel, then $C = C_0(1 + \epsilon_C)$ and $t = t_0(1 + \epsilon_T)$. Of course, the radius of a vessel is proportional to the circumference ($C = 2\pi r$), so $r = r_0(1 + \epsilon_C)$, where r_0 is the unstressed radius.

If the material has linear elasticity, we can use Hooke's law in 3D for triaxial stress ($\epsilon_x = \frac{1}{E}\sigma_x - \frac{v}{E}\sigma_y - \frac{v}{E}\sigma_z$), considering $\sigma_x = \sigma_C$, $\sigma_y = \sigma_L = \sigma_C/2$, and $\sigma_z = \sigma_T = 0$. Thus:

$$\epsilon_C = \frac{1}{E}\sigma_C - \frac{v}{E}\sigma_L = \frac{(1 - v/2)}{E}\sigma_C$$

$$\epsilon_T = -\frac{v}{E}\sigma_C - \frac{v}{E}\sigma_L = -\frac{3v}{2E}\sigma_C$$

Thus for the cylinder:

$$\sigma_C = \frac{pr}{t} = \frac{pr_0(1+\epsilon_C)}{t_0(1+\epsilon_T)} = \frac{pr_0\left(1 + \frac{(1-v/2)}{E}\sigma_C\right)}{t_0\left(1 - \frac{3v}{2E}\sigma_C\right)}$$

Finding a solution to this will take more time than we have today, so we consider the situation where $v = 0$ as an illustration of the issue at hand. That is, we consider what happens as the radius increases with pressure, without adding to the problem by also decreasing the thickness.

$$\sigma_C = \frac{pr_0\left(1 + \frac{(1)}{E}\sigma_C\right)}{t_0(1)} = \frac{pr_0 + \frac{pr_0}{E}\sigma_C}{t_0}$$

$$\sigma_C t_0 = pr_0 + \frac{pr_0}{E}\sigma_C$$

$$\sigma_C t_0 - \frac{pr_0}{E}\sigma_C = pr_0$$

$$\sigma_C\left(t_0 - \frac{pr_0}{E}\right) = pr_0$$

$$\sigma_C = \frac{pr_0}{\left(t_0 - \frac{pr_0}{E}\right)}$$

I find it clarifying to rewrite this as follows:

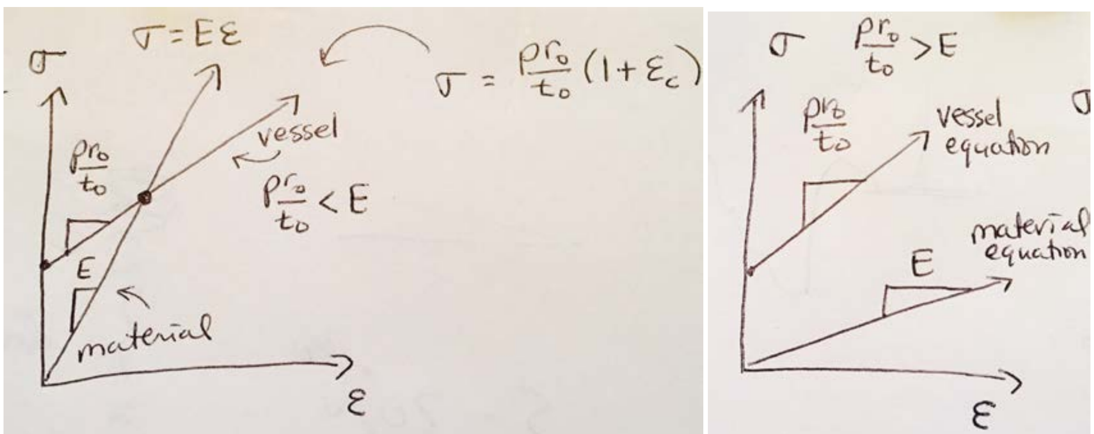
$$\sigma_C = \frac{\frac{pr_0}{t_0}}{\left(1 - \frac{pr_0}{Et_0}\right)}$$

Since $\epsilon_C = \frac{\sigma_C}{E}$, we can also write $\epsilon_C = \frac{\frac{pr_0}{Et_0}}{\left(1-\frac{pr_0}{Et_0}\right)}$.

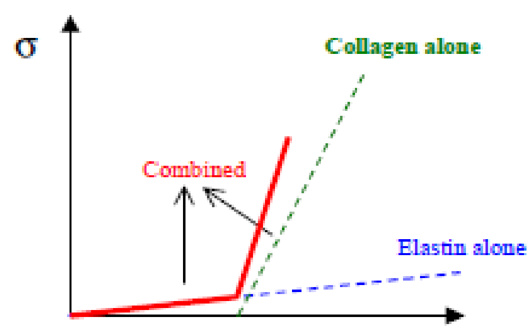
This equation tells us that when $\frac{pr_0}{Et_0} \ll 1$, then $\epsilon_C \ll 1$. In this case, the denominator in the equation for stress above approaches 1, and we get $\sigma_C = \frac{pr_0}{t_0}$, which is the same as when we neglected the effect of strain on the geometry ($r \sim r_0$) which is another way of saying $\epsilon_C \ll 1$.

However, consider what happens at a critical pressure $\frac{p_{crit} r_0}{Et_0} = 1$. Then, $\sigma_C = \infty$. That is, the stress (and strain) increase assymptotically with pressure and the vessel explodes, regardless of the ultimate stresses the material can withstand. Actually, the material will reach the ultimate stress sometime before this pressure, but we can see that there is no way to subject the vessel to a higher pressure than this regardless of the specific value of ultimate stress.

We can also illustrate this graphically, by plotting the equation for the pressure vessel and the equation for the material property on the same stress-strain plot. The solution must satisfy both equations, so is the intersection of the two lines. When $\frac{pr_0}{t_0} < E$, which is the same as $\frac{pr_0}{Et_0} < 1$, the two lines intersect (left panel below), so there is a solution, and the vessel will not explode as long as that solution occurs prior to failure of the material. In contrast, when $\frac{pr_0}{t_0} > E$, which is the same as $\frac{pr_0}{Et_0} > 1$, the two lines could never intersect (right panel below), so there is no solution, and the vessel explodes.



Nonlinear elasticity in pressure vessels.



Fortunately, our arteries do not have linear elasticity. Arterial walls are made up of elastin fibers in addition to wavy collagen fibers. At low strains, the material is soft because only the elastin is stretched, but at high strains, the buckled collagen fibers straighten and are engaged, so the material is stiff. This leads to a two-phase elastic behavior, shown here. In general, materials for which the slope of the stress-strain curve increases with strain are called **strain-hardening**. This is one type of nonlinear material behavior, which we will address later this quarter.

Another material with nonlinear elastic behavior is rubber. Consider what happens when you fill a cylindrical balloon with air. When the balloon is partially full, part of the balloon is small, while the other part is distended. Clearly, the strain and stress in the two parts of the balloon are different. Yet, the pressure inside must be the same as it is all in one compartment. Thus, there must be two solutions for strain that satisfy both Laplace's law $\left(\sigma_C = \frac{pr}{t} = \frac{pr_0(1+\epsilon_C)}{t_0(1+\epsilon_T)}\right)$ and the nonlinear equation for the stress strain curve that will determine ϵ_C and ϵ_T from σ_C and σ_L.