# One-dimensional heat conduction in cylindrical coordinates

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In lecture you saw that the 1-D heat transfer equation in a flat plate or wall is

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2},$$

where $T$ is temperature, $t$ is time, $x$ is position, and $\alpha$ is the thermal diffusivity [$m^2/s$].

In a cylinder, the equation for 1-D radial heat transfer is

$$\frac{\partial T}{\partial t} = \frac{\alpha}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right), \text{ i.e. } \frac{\partial T}{\partial t} = \alpha\left(\frac{1}{r}\frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial r^2}\right).$$

The solution can be obtained by assuming that $T(r,t) = X(r)*\Theta(t)$. Substituting $X*\Theta$ into the partial differential equation lets us break it into two ordinary differential equations:

$$\frac{d\Theta}{dt} + \lambda^2\alpha\Theta = 0 \quad \text{and} \quad \frac{d^2X}{dr^2} + \frac{1}{r}\frac{dX}{dr} + \lambda^2 X = 0.$$

The first-order equation is easy to solve once we know $\lambda$, and it gives an exponential factor. The second-order equation is called "Bessel's equation of order zero." It can be solved using power series, producing a solution of the form

$$X(r) = \sum_{n=0}^{\infty} a_n r^n \Rightarrow \sum_{m=0}^{\infty} \frac{(-1)^m \lambda^{2m}}{2^{2m}(m!)^2} r^{2m}.$$

This power series is defined as a Bessel function, represented by $J(x)$. The eigenvalue $\lambda$ depends on the boundary conditions. Our boundary condition is that the temperature around the circumference, $T(R,t)$, is 0° for $t>0$. There is an infinite set of specific $\lambda$ values that satisfy this boundary condition. For any one value of $\lambda$, we can solve the differential equation for $\Theta(t)$ to get an exponential decay function. The product of $\Theta(t)_n$ and $X(r)_n$ gives one complete solution, $T(r,t)_n$, to the heat transfer equation given our boundary condition.

We next apply the initial condition, which assumes that the temperature is uniform throughout the cylinder for $t<0$. With difficulty, all of the infinite set of solutions $T(r,t)_n$ can be summed, with appropriate weighting factors (coefficients) to produce almost any mathematical function. The summation that produces a uniform temperature $T(r,0)=1$ is:[1]

$$1 = 2\sum_{n=0}^{\infty} \frac{J_0\left(\beta_n \frac{r}{R}\right)}{\beta_n J_1(\beta_n)}$$

when we multiply each term in the summation by the exponential decay function, we get

$$T* = 2\sum_{n=0}^{\infty} e^{-\beta_n^2 \frac{\alpha t}{R^2}} \frac{J_0\left(\beta_n \frac{r}{R}\right)}{\beta_n J_1(\beta_n)}$$

where $R$ is the radius of the can, $J_0$ and $J_1$ are Bessel functions of the first kind with order 0 and 1, and $\beta_n$ is the $n$th root of $J_0(x)=0$. The Bessel functions can be implemented in Excel with the functions BESSELJ(X,0) and BESSELJ(X,1) for $J_0$ and $J_1$, respectively.

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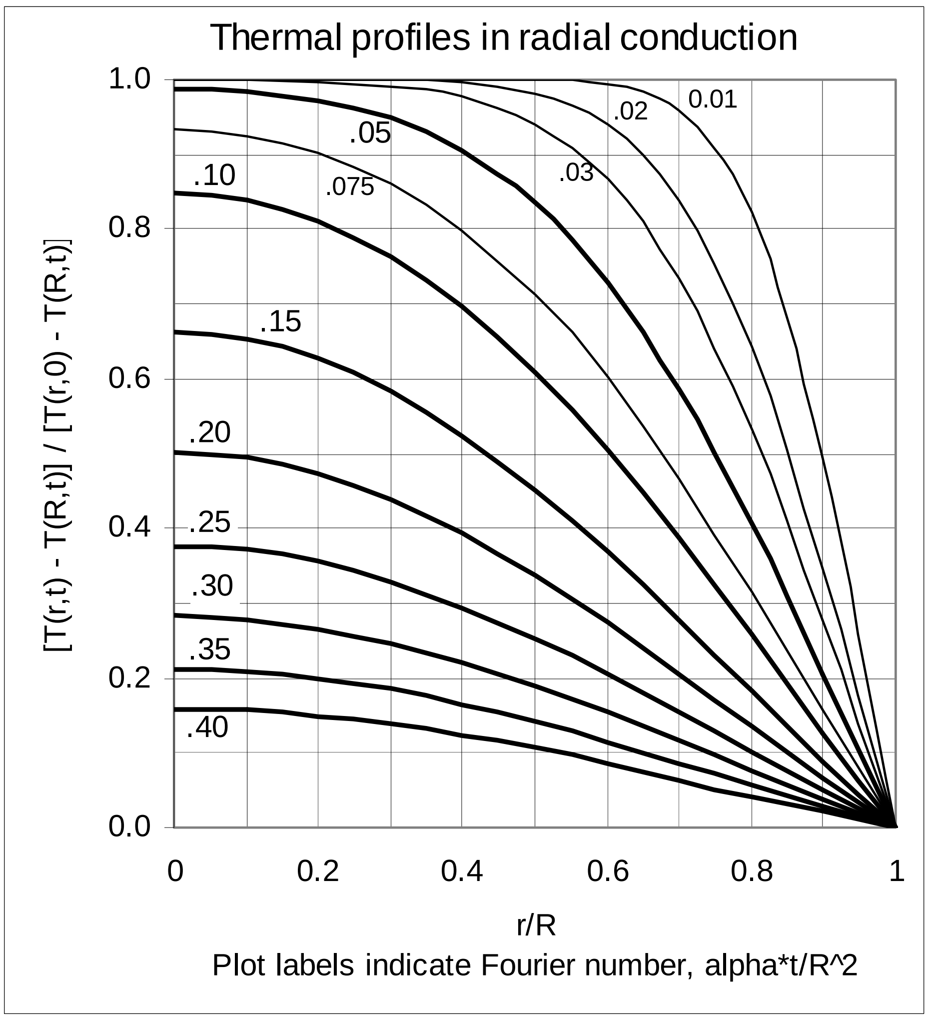
[1] Adapted from Carslaw & Jaeger, *Conduction of Heat in Solids*, 2nd Ed., pp. 198-200.

Of course, this equation assumes that the initial temperature is 1 degree. To make the problem applicable to any initial temperature, we define $T^*$ as a dimensionless temperature. In it, $T(r,0)$ is the initial temperature, and $T(R,t)$ is the boundary condition, i.e. the temperature of around the circumference for time $t>0$.

$$T^* = \frac{T(r,t) - T(R,t)}{T(r,0) - T(R,t)} = 2\sum_{n=0}^{\infty} e^{-\beta_n^2 \frac{\alpha t}{R^2}} \frac{J_0\left(\beta_n \frac{r}{R}\right)}{\beta_n J_1(\beta_n)}$$

Don't panic! This theoretical result has been plotted using Excel, in terms of dimensionless position (r/R) and dimensionless temperature. Each curve shows one instant in time, represented by a dimensionless time variable called the Fourier number, $Fo = \alpha t / R^2$. The fourier number is used instead of seconds because the rate of heat conduction depends on the thermal diffusivity, $\alpha$.

$$\frac{T(r,t) - T(R,t)}{T(r,0) - T(R,t)}$$




Note that $T(r,0)$ is the initial temperature and $T(R,t)$ is the temperature of the ice water.