

15.1 Viscoelasticity

The material response discussed in the previous chapters was limited to the response of elastic materials, in particular to linearly elastic materials. Most metals, for example, exhibit linearly elastic behavior when they are subjected to relatively low stresses at room temperature. They undergo plastic deformations at high stress levels. For an elastic material, the relationship between stress and strain can be expressed in the following general form:

$$\sigma = \sigma(\varepsilon). \quad (15.1)$$

Equation (15.1) states that the normal stress σ is a function of normal strain ε only. The relationship between the shear stress τ and shear strain γ can be expressed in a similar manner. For a linearly elastic material, stress is linearly proportional to strain, and in the case of normal stress and strain, the constant of proportionality is the elastic modulus E of the material (Fig. 15.1):

$$\sigma = E\varepsilon. \quad (15.2)$$

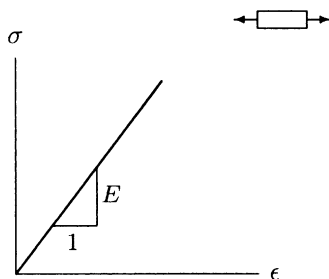


Fig. 15.1 Linearly elastic material behavior

While investigating the response of an elastic material, the concept of time does not enter into the discussions. Elastic materials show time-independent material behavior. Elastic materials deform instantaneously when they are subjected to externally applied loads. They resume their original (unstressed) shapes almost instantly when the applied loads are removed.

There is a different group of materials—such as polymer plastics, almost all biological materials, and metals at high temperatures—that exhibits gradual deformation and recovery when they are subjected to loading and unloading. The response of such materials is dependent upon how quickly the load is applied or removed, the extent of deformation being dependent upon the rate at which the deformation-causing loads are applied. This time-dependent material behavior is called *viscoelasticity*. Viscoelasticity is made up of two words: viscosity and elasticity. *Viscosity* is a fluid property and is a measure of resistance to flow. *Elasticity*, on the other hand, is a solid material property. Therefore, a viscoelastic material is one that possesses both fluid and solid properties.

For viscoelastic materials, the relationship between stress and strain can be expressed as:

$$\sigma = \sigma(\varepsilon, \dot{\varepsilon}). \quad (15.3)$$

Equation (15.3) states that stress, σ , is not only a function of strain, ε , but is also a function of the *strain rate*, $\dot{\varepsilon} = d\varepsilon/dt$, where t is time. A more general form of Eq. (15.3) can be obtained by including higher order time derivatives of strain. Equation (15.3) indicates that the stress–strain diagram of a viscoelastic material is not unique but is dependent upon the rate at which the strain is developed in the material (Fig. 15.2).

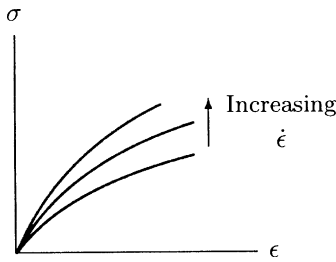


Fig. 15.2 Strain rate ($\dot{\epsilon}$) dependent viscoelastic behavior

15.2 Analogies Based on Springs and Dashpots

In Sect. 13.8, while covering Hooke's Law, an analogy was made between linearly elastic materials and linear springs. An elastic material deforms, stores potential energy, and recovers deformations in a manner similar to that of a spring. The elastic modulus E for a linearly elastic material relates stresses and strains, whereas the constant k for a linear spring relates applied forces and corresponding deformations (Fig. 15.3). Both E and k are measures of stiffness. The similarities between elastic materials and springs suggest that springs can be used to represent elastic material behavior. Since these similarities were first noted by Robert Hooke, elastic materials are also known as *Hookean solids*.

When subjected to external loads, fluids deform as well. Fluids deform continuously, or *flow*. For fluids, stresses are not dependent upon the strains but on the strain rates. If the stresses and strain rates in a fluid are linearly proportional, then the fluid is called a *linearly viscous fluid* or a *Newtonian fluid*. Examples of linearly viscous fluids include water and blood plasma. For a linearly viscous fluid,

$$\sigma = \eta(\dot{\epsilon}). \quad (15.4)$$

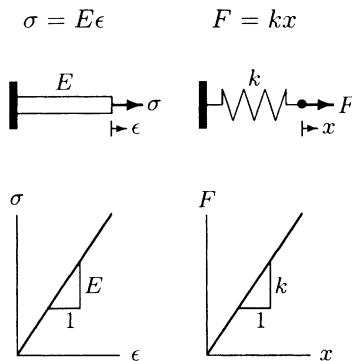


Fig. 15.3 Analogy between a linear spring and an elastic solid

In Eq. (15.4), η (eta) is the constant of proportionality between the stress σ and the strain rate $\dot{\epsilon}$, and is called the *coefficient of viscosity* of the fluid. As illustrated in Fig. 15.4, the coefficient of viscosity is the slope of the $\sigma - \dot{\epsilon}$ graph of a Newtonian fluid. The physical significance of this coefficient is similar to that of the coefficient of friction between the contact surfaces of solid bodies. The higher the coefficient of viscosity, the “thicker” the fluid and the more difficult it is to deform. The coefficient of viscosity for water is about 1 centipoise at room temperature, while it is about 1.2 centipoise for blood plasma.

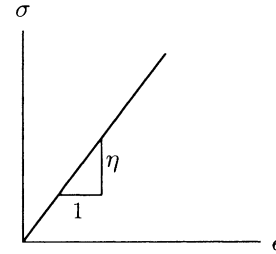


Fig. 15.4 Stress-strain rate diagram for a linearly viscous fluid

The spring is one of the two basic mechanical elements used to simulate the mechanical behavior of materials. The second basic mechanical element is called the *dashpot*, which is used to simulate fluid behavior. As illustrated in Fig. 15.5, a dashpot is a simple piston-cylinder or a syringe type of arrangement. A force applied on the piston will advance the piston in the direction of the applied force. The speed of the piston is dependent upon the magnitude of the applied force and the friction occurring between the contact surfaces of the piston and cylinder. For a linear dashpot, the applied force and speed (rate of displacement) are linearly proportional, the *coefficient of friction* μ (mu) being the constant of proportionality. If the applied force and the displacement are both in the x direction, then,

$$F = \mu \dot{x}. \quad (15.5)$$

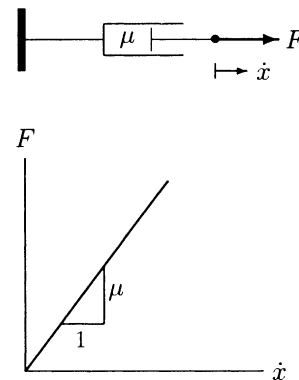


Fig. 15.5 A linear dashpot and its force-displacement rate diagram

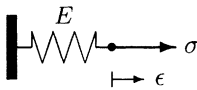
In Eq. (15.5), $\dot{x} = dx/dt$ is the time rate of change of displacement or the speed.

By comparing Eqs. (15.4) and (15.5), an analogy can be made between linearly viscous fluids and linear dashpots. The stress and the strain rate for a linearly viscous fluid are, respectively, analogous to the force and the displacement rate for a dashpot; and the coefficient of viscosity is analogous to the coefficient of viscous friction for a dashpot. These analogies suggest that dashpots can be used to represent fluid behavior.

15.3 Empirical Models of Viscoelasticity

Springs and dashpots constitute the building blocks of model analyses in viscoelasticity. Springs and dashpots connected to one another in various forms are used to construct empirical viscoelastic models. Springs are used to account for the elastic solid behavior and dashpots are used to describe the viscous fluid behavior (Fig. 15.6). It is assumed that a constantly applied force (stress) produces a constant deformation (strain) in a spring and a constant rate of deformation (strain rate) in a dashpot. The deformation in a spring is completely recoverable upon release of applied forces, whereas the deformation that the dashpot undergoes is permanent.

SPRING: ELASTIC SOLID

$$\sigma = E \epsilon$$


DASHPOT: VISCOUS FLUID

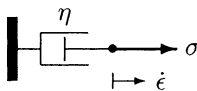
$$\sigma = \eta \dot{\epsilon}$$


Fig. 15.6 Spring represents elastic and dashpot represents viscous material behaviors

15.3.1 Kelvin–Voight Model

The simplest forms of empirical models are obtained by connecting a spring and a dashpot together in parallel and in series configurations. As illustrated in Fig. 15.7, the *Kelvin–Voight model* is a system consisting of a spring and a dashpot connected in a parallel arrangement. If subscripts “s” and “d” denote the spring and dashpot, respectively, then a stress σ applied to the entire system will produce stresses σ_s and σ_d in the spring and the dashpot. The total stress

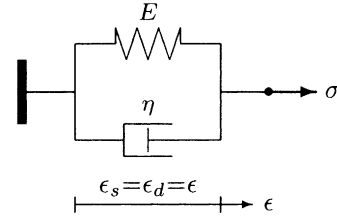


Fig. 15.7 Kelvin–Voight model

applied to the system will be shared by the spring and the dashpot such that:

$$\sigma = \sigma_s + \sigma_d. \quad (15.6)$$

As the stress σ is applied, the spring and dashpot will deform by an equal amount because of their parallel arrangement. Therefore, the strain ϵ of the system will be equal to the strains ϵ_s and ϵ_d occurring in the spring and the dashpot:

$$\epsilon = \epsilon_s = \epsilon_d. \quad (15.7)$$

The stress–strain relationship for the spring and the stress–strain rate relationship for the dashpot are:

$$\sigma_s = E \epsilon_s, \quad (15.8)$$

$$\sigma_d = \eta \dot{\epsilon}_d. \quad (15.9)$$

Substituting Eqs. (15.8) and (15.9) into Eq. (15.6) will yield:

$$\sigma = E \epsilon_s + \eta \dot{\epsilon}_d. \quad (15.10)$$

From (15.7), $\epsilon_s = \epsilon_d = \epsilon$. Therefore,

$$\sigma = E \epsilon + \eta \dot{\epsilon}. \quad (15.11)$$

Note that the strain rate $\dot{\epsilon}$ can alternatively be written as $d\epsilon/dt$. Consequently,

$$\sigma = E \epsilon + \eta \frac{d\epsilon}{dt}. \quad (15.12)$$

Equation (15.12) relates stress to strain and the strain rate for the Kelvin–Voight model, which is a two-parameter (E and η) viscoelastic model. Equation (15.12) is an *ordinary differential equation*. More specifically, it is a first order, linear ordinary differential equation. For a given stress σ , Eq. (15.12) can be solved for the corresponding strain ϵ . For prescribed strain ϵ , it can be solved for stress σ .

Note that the review of how to handle ordinary differential equations is beyond the scope of this text. The interested

reader is encouraged to review textbooks in “differential equations.”

15.3.2 Maxwell Model

As shown in Fig. 15.8, the *Maxwell model* is constructed by connecting a spring and a dashpot in a series. In this case, a stress σ applied to the entire system is applied equally on the spring and the dashpot ($\sigma = \sigma_s = \sigma_d$), and the resulting strain ϵ is the sum of the strains in the spring and the dashpot ($\epsilon = \epsilon_s + \epsilon_d$). Through stress–strain analyses similar to those carried out for the Kelvin–Voight model, a differential equation relating stresses and strains for the Maxwell model can be derived in the following form:

$$\eta \dot{\sigma} + E\sigma = E\eta \dot{\epsilon}. \quad (15.13)$$

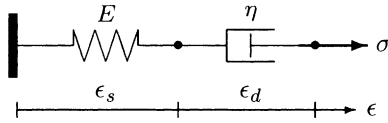


Fig. 15.8 Maxwell model

This is also a first order, linear ordinary differential equation representing a two-parameter (E and η) viscoelastic behavior. For a given stress (or strain), Eq. (15.13) can be solved for the corresponding strain (or stress).

Notice that springs are used to represent the elastic solid behavior, and there is a limit to how much a spring can deform. On the other hand, dashpots are used to represent fluid behavior and are assumed to deform continuously (flow) as long as there is a force to deform them. For example, in the case of a Maxwell model, a force applied will cause both the spring and the dashpot to deform. The deformation of the spring will be finite. The dashpot will keep deforming as long as the force is maintained. Therefore, the overall behavior of the Maxwell model is more like a fluid than a solid, and is known to be a *viscoelastic fluid* model. The deformation of a dashpot connected in parallel to a spring, as in the Kelvin–Voight model, is restricted by the response of the spring to the applied loads. The dashpot in the Kelvin–Voight model cannot undergo continuous deformations. Therefore, the Kelvin–Voight model represents a *viscoelastic solid* behavior.

15.3.3 Standard Solid Model

The Kelvin–Voight solid and Maxwell fluid are the basic viscoelastic models constructed by connecting a spring and a dashpot together. They do not represent any known real material. However, in addition to springs and dashpots,

they can be used to construct more complex viscoelastic models, such as the standard solid model. As illustrated in Fig. 15.9, the *standard solid model* is composed of a spring and a Kelvin–Voight solid connected in a series. The standard solid model is a three-parameter (E_1 , E_2 , and η) model and is used to describe the viscoelastic behavior of a number of biological materials such as the cartilage and the white blood cell membrane. The material function relating the stress, strain, and their rates for this model is:

$$(E_1 + E_2)\sigma + \eta \dot{\sigma} = (E_1 E_2 \epsilon + E_1 \eta \dot{\epsilon}). \quad (15.14)$$

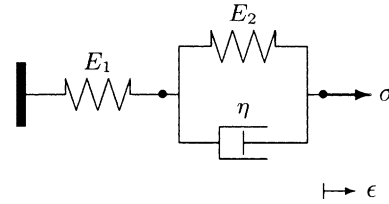


Fig. 15.9 Standard solid model

In Eq. (15.14), $\dot{\sigma} = d\sigma/dt$ is the stress rate and $\dot{\epsilon} = d\epsilon/dt$ is the strain rate. This equation can be derived as follows. As illustrated in Fig. 15.10, the model can be represented by two units, A and B, connected in a series such that unit A is an elastic solid and unit B is a Kelvin–Voight solid. If σ_A and ϵ_A represent stress and strain in unit A, and σ_B and ϵ_B are stress and strain in unit B, then,

$$\sigma_A = E_1 \epsilon_A, \quad (i)$$

$$\sigma_B = E_2 \epsilon_B + \eta \frac{d\epsilon_B}{dt} = \left(E_2 + \eta \frac{d}{dt} \right) \epsilon_B. \quad (ii)$$

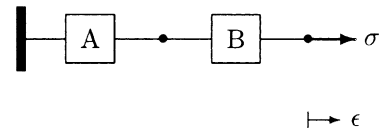


Fig. 15.10 Standard solid model is represented by units A and B

Since units A and B are connected in a series:

$$\epsilon_A + \epsilon_B = \epsilon, \quad (iii)$$

$$\sigma_A = \sigma_B = \sigma. \quad (iv)$$

Substitute Eq. (iv) into Eqs. (i) and (ii) and express them in terms of strains ϵ_A and ϵ_B :

$$\epsilon_A = \frac{\sigma}{E_1}, \quad (v)$$

$$\varepsilon_B = \frac{\sigma}{E_2 + \eta(d/dt)}. \quad (vi)$$

Substitute Eqs. (v) and (vi) into Eq. (iii):

$$\frac{\sigma}{E_1} + \frac{\sigma}{E_2 + \eta(d/dt)} = \varepsilon.$$

Employ cross multiplication and rearrange the order of terms to obtain

$$(E_1 + E_2)\sigma + \eta \frac{d\sigma}{dt} = E_1 E_2 \varepsilon + E_1 \eta \frac{d\varepsilon}{dt}$$

15.4 Time-Dependent Material Response

An empirical model for a given viscoelastic material can be established through a series of experiments. There are several experimental techniques designed to analyze the time-dependent aspects of material behavior. As illustrated in Fig. 15.11a, a *creep and recovery (recoil)* test is conducted by applying a load (stress σ_0) on the material at time t_0 , maintaining the load at a constant level until time t_1 , suddenly removing the load at t_1 , and observing the material response. As illustrated in Fig. 15.11b, the *stress relaxation* experiment is done by straining the material to a level ε_0 and maintaining the constant strain while observing the stress response of the material. In an *oscillatory response* test, a harmonic stress is applied and the strain response of the material is measured (Fig. 15.11c).

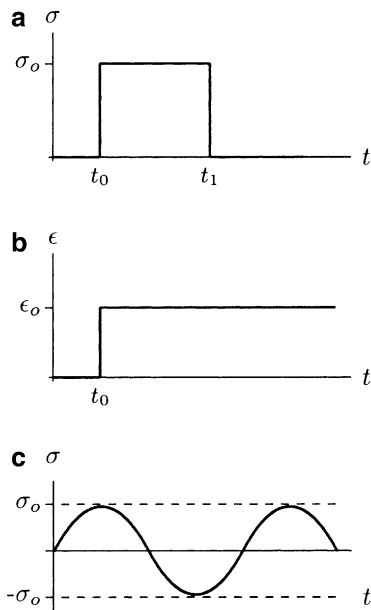


Fig. 15.11 (a) Creep and recovery, (b) stress relaxation, and (c) oscillatory response tests

Consider a viscoelastic material. Assume that the material is subjected to a creep test. The results of the creep test can be represented by plotting the measured strain as a function of time. An empirical viscoelastic model for the material behavior can be established through a series of trials. For this purpose, an empirical model is constructed by connecting a number of springs and dashpots together. A differential equation relating stress, strain, and their rates is derived through the procedure outlined in Sect. 15.3 for the Kelvin–Voight model. The imposed condition in a creep test is $\sigma = \sigma_0$. This condition of constant stress is substituted into the differential equation, which is then solved (integrated) for strain ε . The result obtained is another equation relating strain to stress constant σ_0 , the elastic moduli and coefficients of viscosity of the empirical model, and time. For a given σ_0 and assigned elastic and viscous moduli, this equation is reduced to a function relating strain to time. This function is then used to plot a strain versus time graph and is compared to the experimentally obtained graph. If the general characteristics of the two (experimental and analytical) curves match, the analyses are furthered to establish the elastic and viscous moduli (material constants) of the material. This is achieved by varying the values of the elastic and viscous moduli in the empirical model until the analytical curve matches the experimental curve as closely as possible. In general, this procedure is called *curve fitting*. If there is no general match between the two curves, the model is abandoned and a new model is constructed and checked.

The result of these mathematical model analyses is an empirical model and a differential equation relating stresses and strains. The stress–strain relationship for the material can be used in conjunction with the fundamental laws of mechanics to analyze the response of the material to different loading conditions.

Note that the deformation processes occurring in viscoelastic materials are quite complex, and it is sometimes necessary to use an array of empirical models to describe the response of a viscoelastic material to different loading conditions. For example, the shear response of a viscoelastic material may be explained with one model and a different model may be needed to explain its response to normal loading. Different models may also be needed to describe the response of a viscoelastic material at low and high strain rates.

15.5 Comparison of Elasticity and Viscoelasticity

There are various criteria with which the elastic and viscoelastic behavior of materials can be compared. Some of these criteria are discussed in this section.

An elastic material has a unique stress–strain relationship that is independent of the time or strain rate. For elastic

materials, normal and shear stresses can be expressed as functions of normal and shear strains:

$$\sigma = \sigma(\epsilon) \quad \text{and} \quad \tau = \tau(\gamma).$$

For example, the stress–strain relationships for a linearly elastic solid are $\sigma = E\epsilon$ and $\tau = G\gamma$, where E and G are constant elastic moduli of the material. As illustrated in Fig. 15.12, a linearly elastic material has a unique normal stress–strain diagram and a unique shear stress–strain diagram.

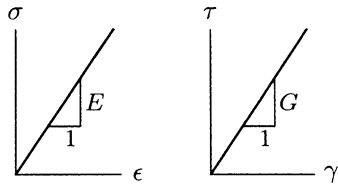


Fig. 15.12 An elastic material has unique normal and shear stress–strain diagrams

Viscoelastic materials exhibit time-dependent material behavior. The response of a viscoelastic material to an applied stress not only depends upon the magnitude of the stress but also on how fast the stress is applied to or removed from the material. Therefore, the stress–strain relationship for a viscoelastic material is not unique but is a function of the time or the rate at which the stresses and strains are developed in the material:

$$\sigma = \sigma(\epsilon, \dot{\epsilon}, \dots, t) \quad \text{and} \quad \tau = \tau(\gamma, \dot{\gamma}, \dots, t).$$

Consequently, as illustrated in Fig. 15.13, a viscoelastic material does not have a unique stress–strain diagram.

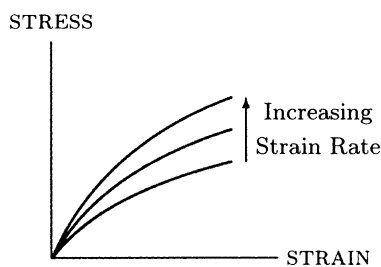


Fig. 15.13 Stress–strain diagram for a viscoelastic material may not be unique

For an elastic body, the energy supplied to deform the body (strain energy) is stored in the body as potential energy. This energy is available to return the body to its original (unstressed) size and shape once the applied stress is removed. As illustrated in Fig. 15.14, the loading and unloading paths for an elastic material coincide. This indicates that there is no loss of energy during loading and unloading.

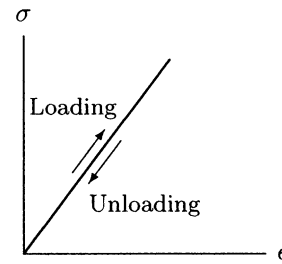


Fig. 15.14 For an elastic material, loading and unloading paths coincide

For a viscoelastic body, some of the strain energy is stored in the body as potential energy and some of it is dissipated as heat. For example, consider the Maxwell model. The energy provided to stretch the spring is stored in the spring while the energy supplied to deform the dashpot is dissipated as heat due to the friction between the moving parts of the dashpot. Once the applied load is removed, the potential energy stored in the spring is available to recover the deformation of the spring, but there is no energy available in the dashpot to regain its original configuration.

Consider the three-parameter standard solid model shown in Fig. 15.9. A typical loading and unloading diagram for this model is shown in Fig. 15.15. The area enclosed by the loading and unloading paths is called the *hysteresis loop*, which represents the energy dissipated as heat during the deformation and recovery phases. This area, and consequently the amount of energy dissipated as heat, is dependent upon the rate of strain employed to deform the body. The presence of the hysteresis loop in the stress–strain diagram for a viscoelastic material indicates that continuous loading and unloading would result in an increase in the temperature of the material.

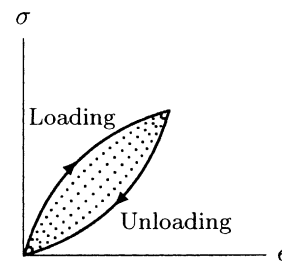


Fig. 15.15 Hysteresis loop

Note here that most of the elastic materials exhibit plastic behavior at stress levels beyond the yield point. For elastic–plastic materials, some of the strain energy is dissipated as heat during plastic deformations. This is indicated with the presence of a hysteresis loop in their loading and unloading diagrams (Fig. 15.16). For such

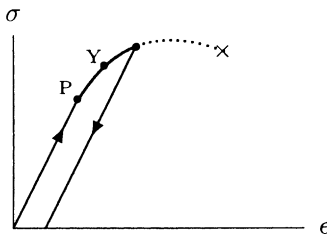


Fig. 15.16 Hysteresis loop for an elastic-plastic material

materials, energy is dissipated as heat only if the plastic region is entered. Viscoelastic materials dissipate energy regardless of whether the strains or stresses are small or large.

Since viscoelastic materials exhibit time-dependent material behavior, the differences between elastic and viscoelastic material responses are most evident under time-dependent loading conditions, such as during the creep and stress relaxation experiments.

As discussed earlier, a creep and recovery test is conducted by observing the response of a material to a constant stress σ_0 applied at time t_0 and removed at a later time t_1 (Fig. 15.17a). As illustrated in Fig. 15.17b, such a load will cause a strain $\epsilon_0 = \frac{\sigma_0}{E}$ in a linearly elastic material instantly at time t_0 . This constant strain will remain in the material until time t_1 . At time t_1 , the material will instantly and completely recover the deformation. To the same constant loading condition, a viscoelastic material will respond with a strain gradually increasing between times

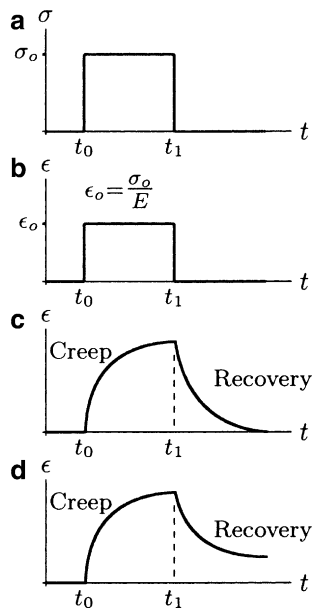


Fig. 15.17 Creep and recovery

t_0 and t_1 . At time t_1 , gradual recovery will start. For a viscoelastic solid material, the recovery will eventually be complete (Fig. 15.17c). For a viscoelastic fluid, complete recovery will never be achieved and there will be a residue of deformation left in the material (Fig. 15.17d).

As illustrated in Fig. 15.18a, the stress relaxation test is performed by straining a material instantaneously, maintaining the constant strain level ϵ_0 in the material, and observing the response of the material. A linearly elastic material response is illustrated in Fig. 15.18b. The constant stress $\sigma_0 = E\epsilon_0$ developed in the material will remain as long as the strain ϵ_0 is maintained. In other words, an elastic material will not exhibit a stress relaxation behavior. A viscoelastic material, on the other hand, will respond with an initial high stress that will decrease over time. If the material is a viscoelastic solid, the stress level will never reduce to zero (Fig. 15.18c). As illustrated in Fig. 15.18d, the stress will eventually reduce to zero for a viscoelastic fluid.

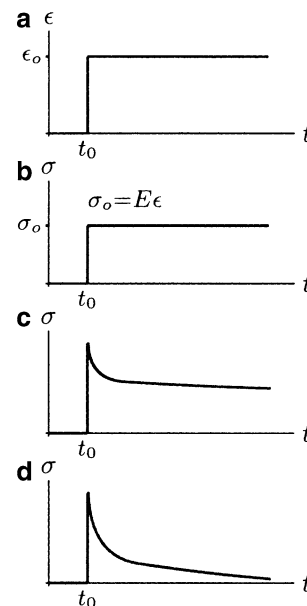


Fig. 15.18 Stress relaxation

Because of their time-dependent material behavior, viscoelastic materials are said to have a “memory.” In other words, viscoelastic materials remember the history of deformations they undergo and react accordingly.

Almost all biological materials exhibit viscoelastic properties, and the remainder of this chapter is devoted to the discussion and review of the mechanical properties of biological tissues including bone, tendons, ligaments, muscles, and articular cartilage.

15.6 Common Characteristics of Biological Tissues

One of the objectives of studies in the field of biomechanics is to establish the mechanical properties of biological tissues so as to develop mathematical models that help us describe and further investigate their behavior under various loading conditions. While conducting studies in biomechanics, it has been a common practice to utilize engineering methods and principles, and at the same time to treat biological tissues like engineering materials. However, living tissues have characteristics that are very different than engineering materials. For example, living tissues can be self-adapting and self-repairing. That is, they can adapt to changing mechanical demand by altering their mechanical properties, and they can repair themselves. The mechanical properties of living tissues tend to change with age. Most biological tissues are composite materials (consisting of materials with different properties) with nonhomogeneous and anisotropic properties. In other words, the mechanical properties of living tissues may vary from point to point within the tissue, and their response to forces applied in different directions may be different. For example, values for strength and stiffness of bone may vary between different bones and at different points within the same bone. Furthermore, almost all biological tissues are viscoelastic in nature. Therefore, the strain or loading rate at which a specific test is conducted must also be provided while reporting the results of the strength measurements. These considerations require that most of the mechanical properties reported for living tissues are only approximations and a mathematical model aimed to describe the behavior of a living tissue is usually limited to describing its response under a specific loading configuration.

From a mechanical point of view, all tissues are composite materials. Among the common components of biological tissues, collagen and elastin fibers have the most important mechanical properties affecting the overall mechanical behavior of the tissues in which they appear. *Collagen* is a protein made of crimped fibrils that aggregate into fibers. The mechanical properties of collagen fibrils are such that each fibril can be considered a mechanical spring and each fiber as an assemblage of springs. The primary mechanical function of collagen fibers is to withstand axial tension. Because of their high length-to-diameter ratios (aspect ratio), collagen fibers are not effective under compressive loads. Whenever a fiber is pulled, its crimp straightens, and its length increases. Like a mechanical spring, the energy supplied to stretch the fiber is stored and it is the release of this energy that returns the fiber to its unstretched configuration when the applied load is removed. The individual fibrils of the collagen

fibers are surrounded by a gel-like *ground substance* that consists largely of water. Collagen fibers possess a two-phase, solid–fluid, or viscoelastic material behavior with a relatively high tensile strength and poor resistance to compression.

The geometric configuration of collagen fibers and their interaction with the noncollagenous tissue components form the basis of the mechanical properties of biological tissues. Among the noncollagenous tissue components, *elastin* is another fibrous protein with material properties that resemble the properties of rubber. Elastin and microfibrils form elastic fibers that are highly extensible, and their extension is reversible even at high strains. Elastin fibers behave elastically with low stiffness up to about 200% elongation followed by a short region where the stiffness increases sharply until failure (Fig. 15.19). The elastin fibers do not exhibit considerable plastic deformation before failure, and their loading and unloading paths do not show significant hysteresis. In summary, elastin fibers possess a low-modulus elastic material property, while collagen fibers show a higher modulus viscoelastic material behavior.

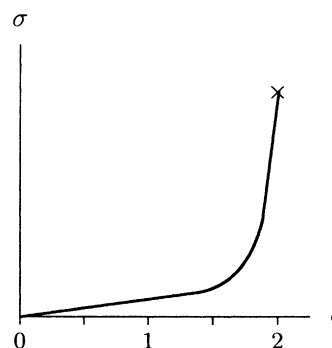


Fig. 15.19 Stress–strain diagram for elastin

15.7 Biomechanics of Bone

Bone is the primary structural element of the human body. Bones form the building blocks of the skeletal system that protects the internal organs, provides kinematic links, provides muscle attachment sites, and facilitates muscle actions and body movements. Bone has unique structural and mechanical properties that allow it to carry out these functions. As compared to other structural materials, bone is also unique in that it is self-repairing. Bone can also alter its shape, mechanical behavior, and mechanical properties to adapt to the changes in mechanical demand. The major factors that influence the mechanical behavior of bone