Biology 427 Biomechanics
Lecture 9. Shape and stress: architecture in biology.

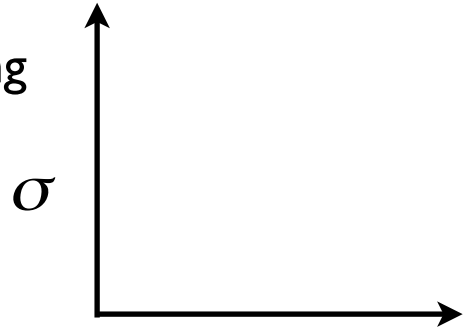
- Brief recap of fracture and viscoelasticity.
- Failure and safety factors: many ways to break up
- The flexural stiffness of a structure (EI): an introduction to structural mechanics.
- The deflection of a simple beam -- both stiffness and shape determine deformation.
- A joint effort – analyses of femoral mechanics

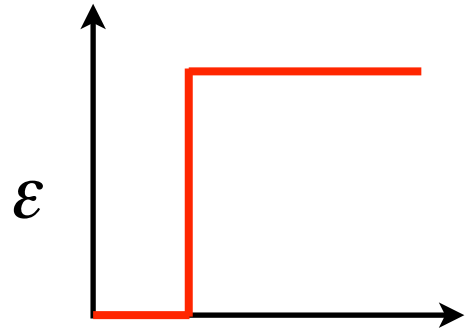
A stress-relaxation test: apply a constant strain, measure stress

$\sigma = E\varepsilon$

Solid

Spring

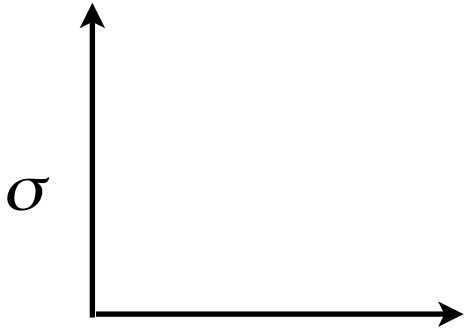


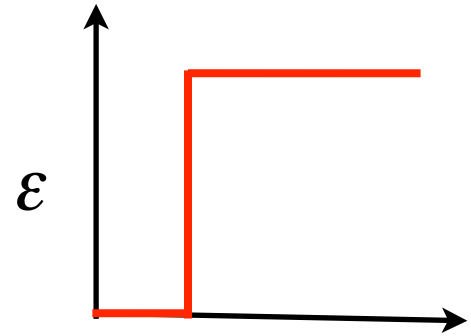


$\sigma = \mu d\varepsilon / dt$

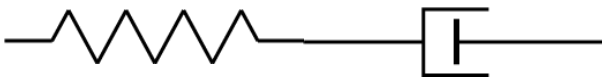
Fluid

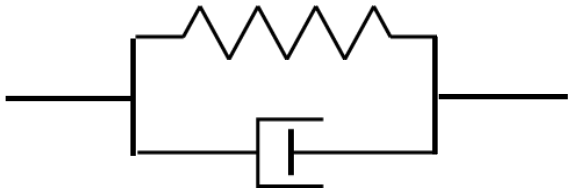
Dashpot



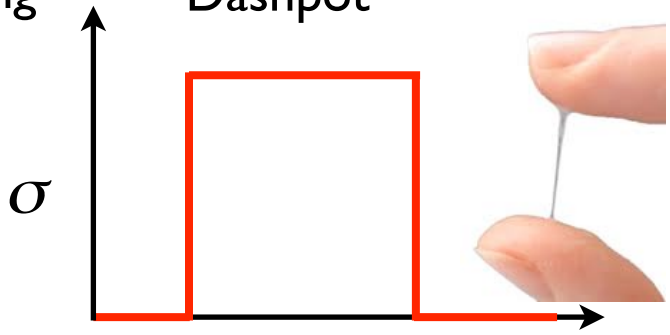


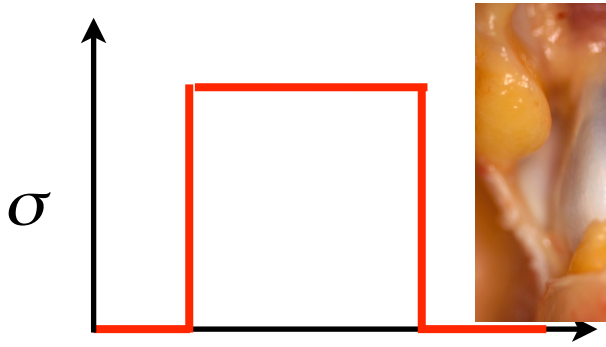
Creep characteristics of mucus and cartilage

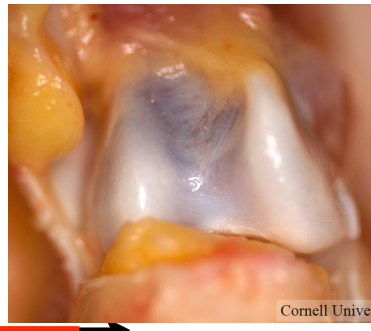


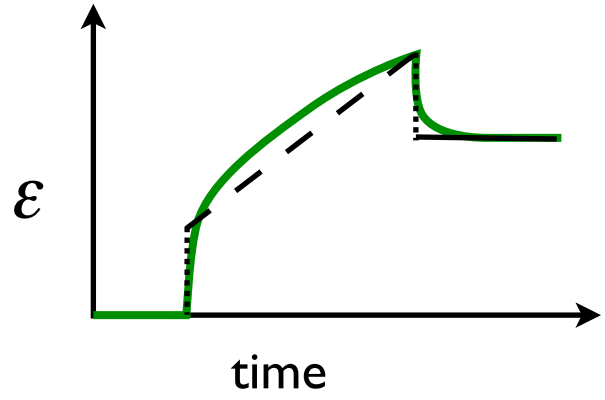


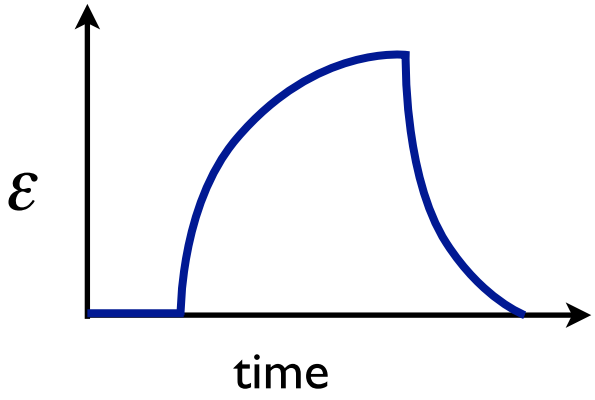
Spring Dashpot



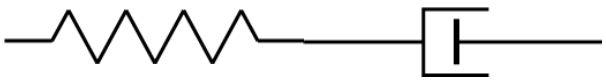




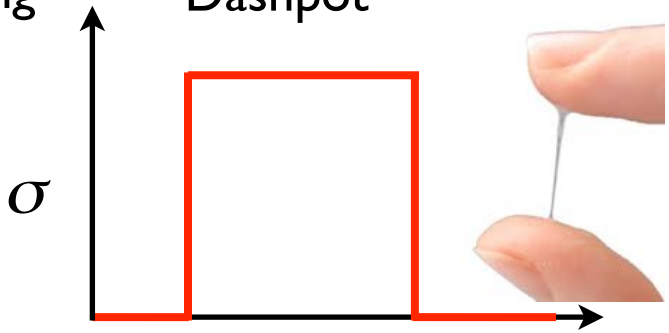


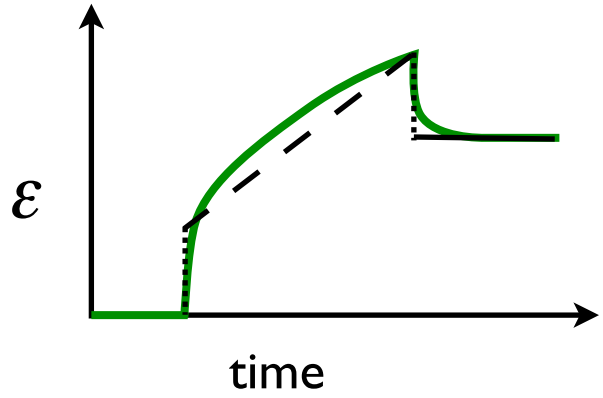


Creep characteristics of mucus



Spring Dashpot





$$\sigma_s = \sigma_d = \sigma_{total}$$

$$\varepsilon_s + \varepsilon_d = \varepsilon_{total}$$

$$\mu d\varepsilon_d / dt = \sigma$$

$$\varepsilon_d = \sigma t / \mu$$

$$\sigma / E + \sigma t / \mu = \varepsilon_{total}$$

So far: stress, strain, stiffness, strength, toughness, resilience.
Linear vs. non-linear, isotropic vs. anisotropic, time-dependent (vs not).

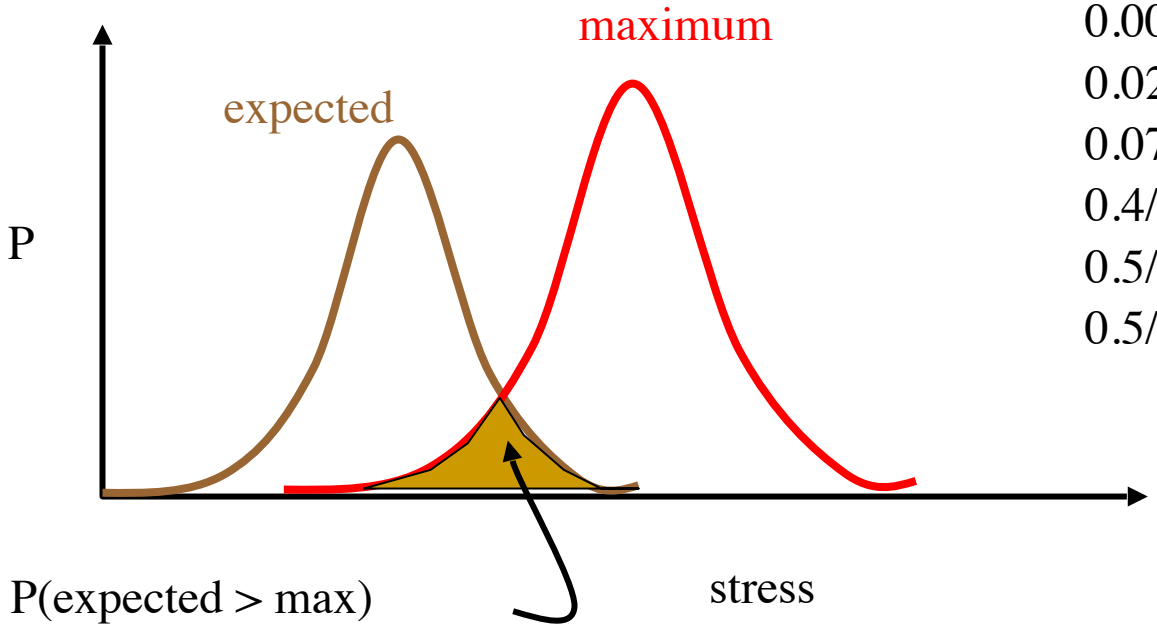
Performance: to what extent are biological materials and structures designed to withstand forces?

Safety factor *: $n = \sigma_{br}/\sigma_{ex}$

	Bones		Tendon
	Tensile	Compressive	
Strength (MPa)	172	284	84
Dog jumping	68-80(2-3)	100(2.8)	84(1.0)
Kangaroo hopping	60(3)	90(3.2)	40-80(1-2)
Elephant running	45-69(2.5-4)	57-85(3.3-5)	
Man weight lifting		(1 - 1.7)	
Goose flying		50(6)	

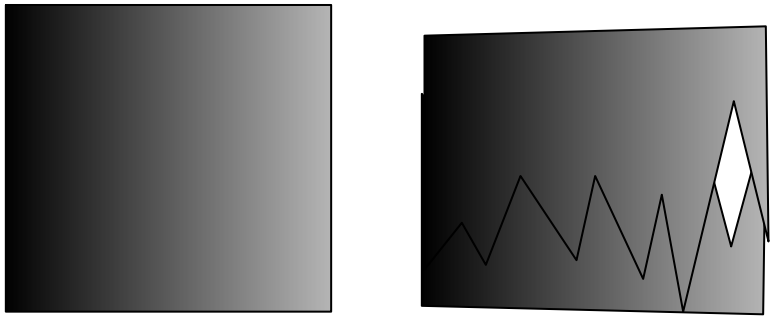
*Alexander, R.McN. 1981. Sci. Prog. Oxf. 67:109-120

But … there is variation in the materials and loads may be unpredictable (they too may vary). The geometric arrangement matters as well!



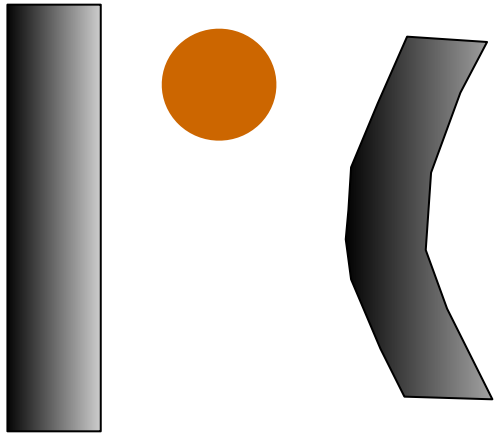
0.0004/year human humerus
0.0006/year human femur
0.02/life either
0.07/life viverids
0.4/life gibbons
0.5/life red deer antlers
0.5/life spider webs

Size and shape matter too



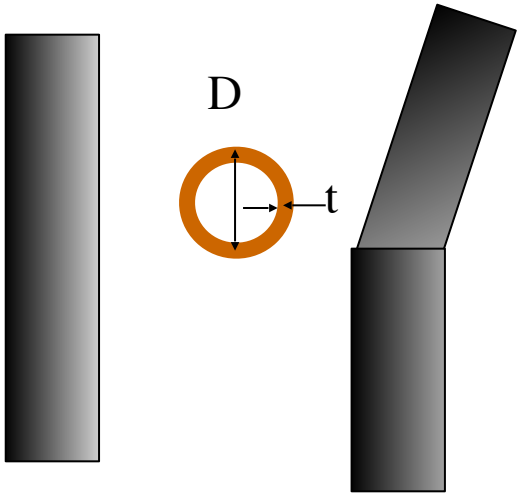
Compressive failure

$\sigma > \sigma_{max}$



Euler buckling

$F_E = n\,\pi^2\,E\,I/L^2$



Local buckling

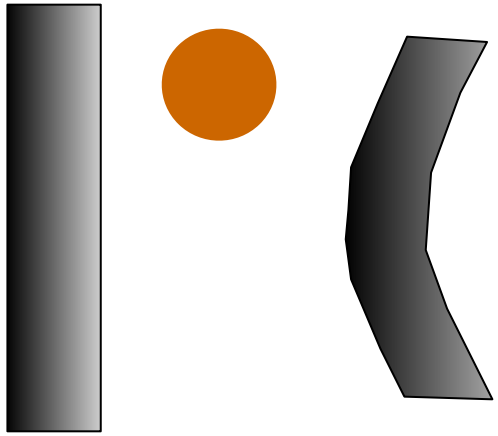
$\sigma_L = k\,E\;\;t/\,D\;\;\;(\mathrm{k} \sim 0.7)$

Size and shape matter too

$$I = \int y^2 dA$$

EI : Flexural stiffness of a structure

I = second moment of area:
a measure of how shape affects stress distribution in a structure



Euler buckling

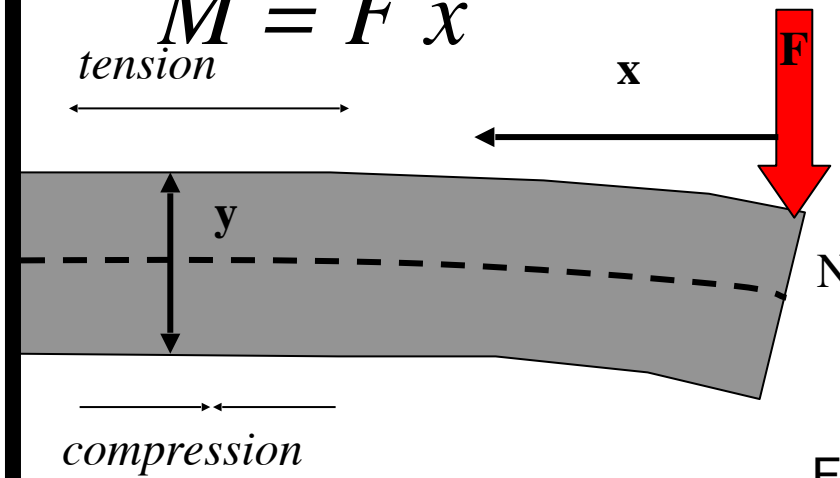
$$F_E = n\,\pi^2\,E\,I/L^2$$

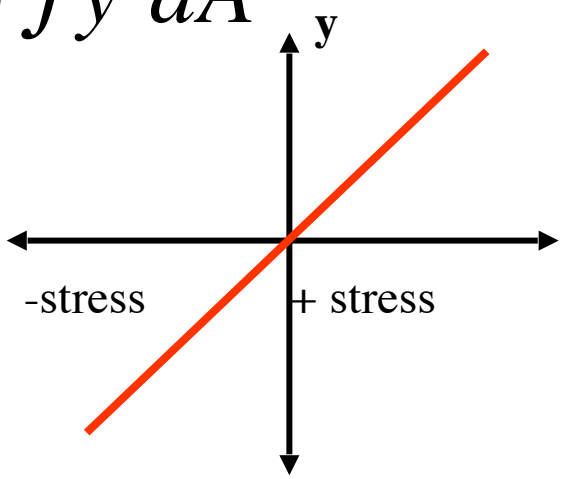
E I :

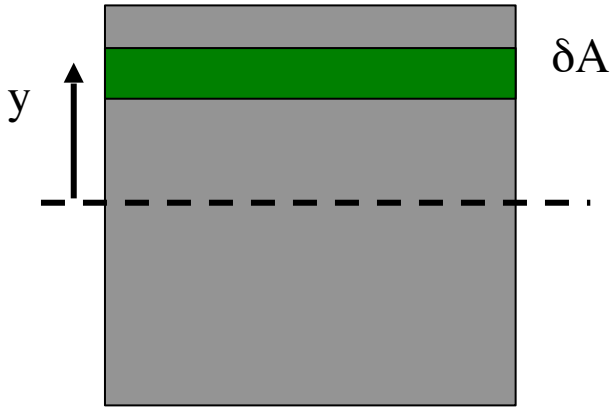
Flexural stiffness of a structure

$$M = (\sigma/y) \int y^2 dA$$

$$M = F\,x$$







Force on dA $F = \sigma_y\, dA$

Moment about neutral axis

$dM = F\,y = (\sigma_y\, dA)\, y$

The total moment about axis

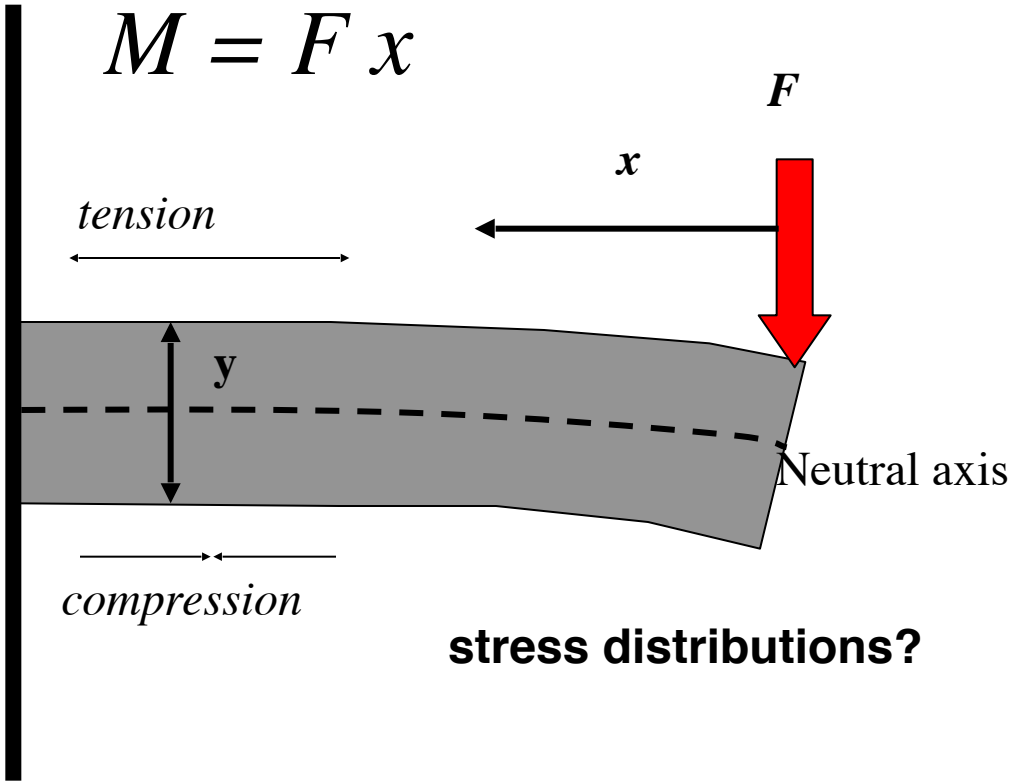
$$M = \int dM = \int \sigma\, y\, dA = \int (k\, y)\, y\, dA$$

E I :

Flexural stiffness of a structure

$$M = (\sigma/y) \int y^2 dA$$

$$M = F\,x$$



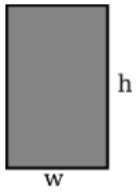
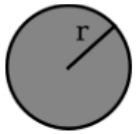
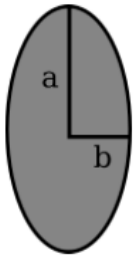
stress distributions?

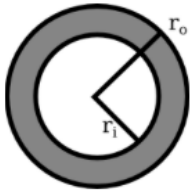
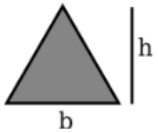
$$M = (\sigma/y)\, I$$

$$F\,x = (\sigma/y)\, I$$

$$\sigma = F\,x\,y/\,I$$

$$\varepsilon = F\,x\,y/\,E\,I$$

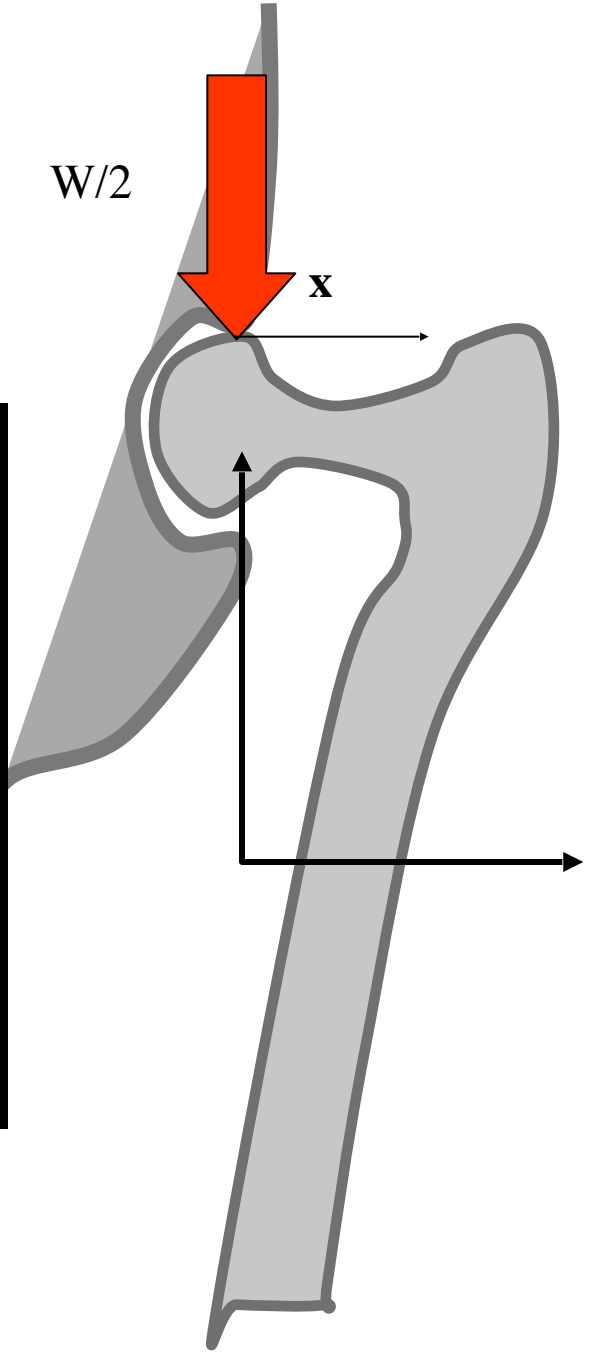
Shape	Area	I
	wh	$\frac{wh^3}{12}$
	πr^2	$\frac{\pi r^4}{4}$
	πab	$\frac{\pi a^3 b}{4}$

Shape	Area	I
	$\pi(r_o^2 - r_i^2)$	$\frac{\pi(r_o^4 - r_i^4)}{4}$
	$\frac{bh}{2}$	$\frac{bh^3}{36}$

https://en.wikipedia.org/wiki/List_of_area_moments_of_inertia

Where do you think the tensile stress is greatest?

Where is the most likely zone for failure?

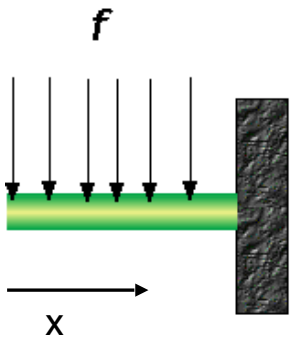


$$\sigma = F\ x\ y/I$$

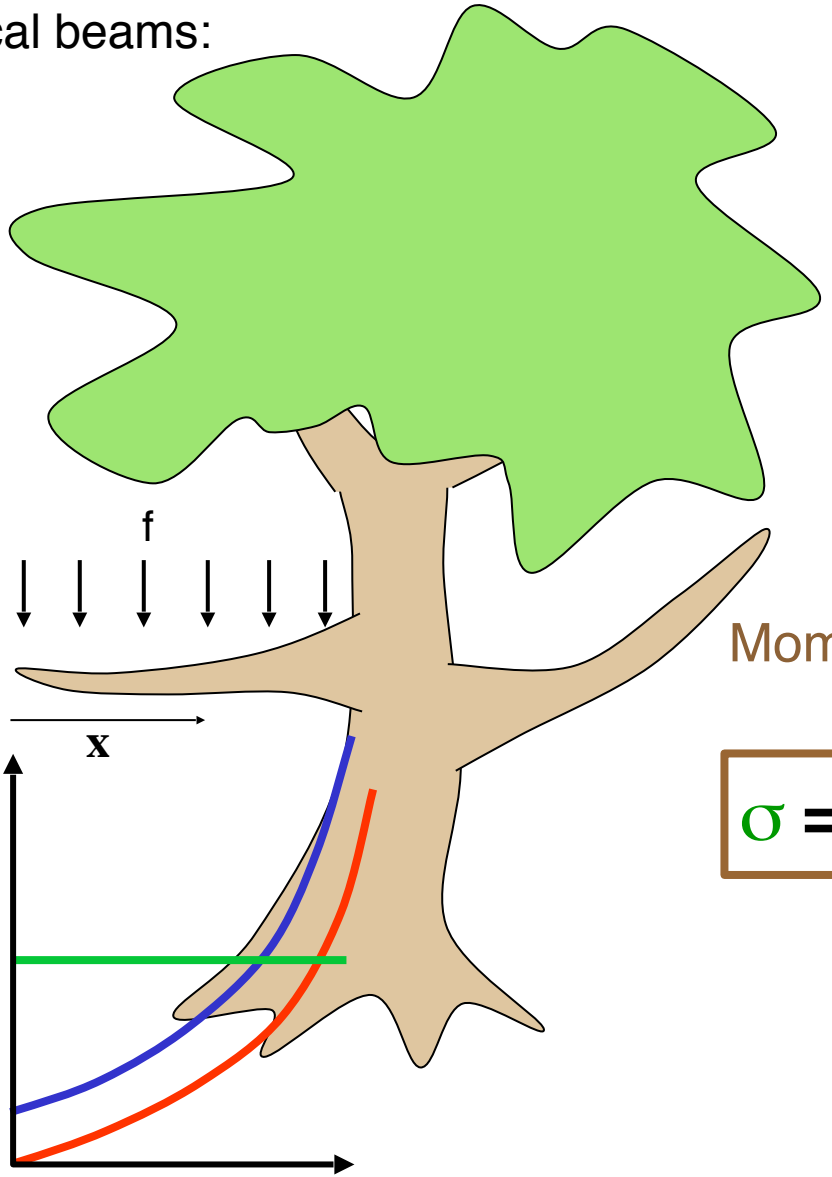
$$I = \pi R^4/4$$

Stress distributions in biological beams: tree branches

$\sigma = M\,y/I$



Tree branches support their own weight,
plus the weight of leaves, fruit, etc.



Moment $M = Fx^2/3L$

$$\sigma = F\,x^2\,y/\,3\,L\,I$$

How does the design of branches affect the distribution of bending stresses?

$I = \pi R^4/4$

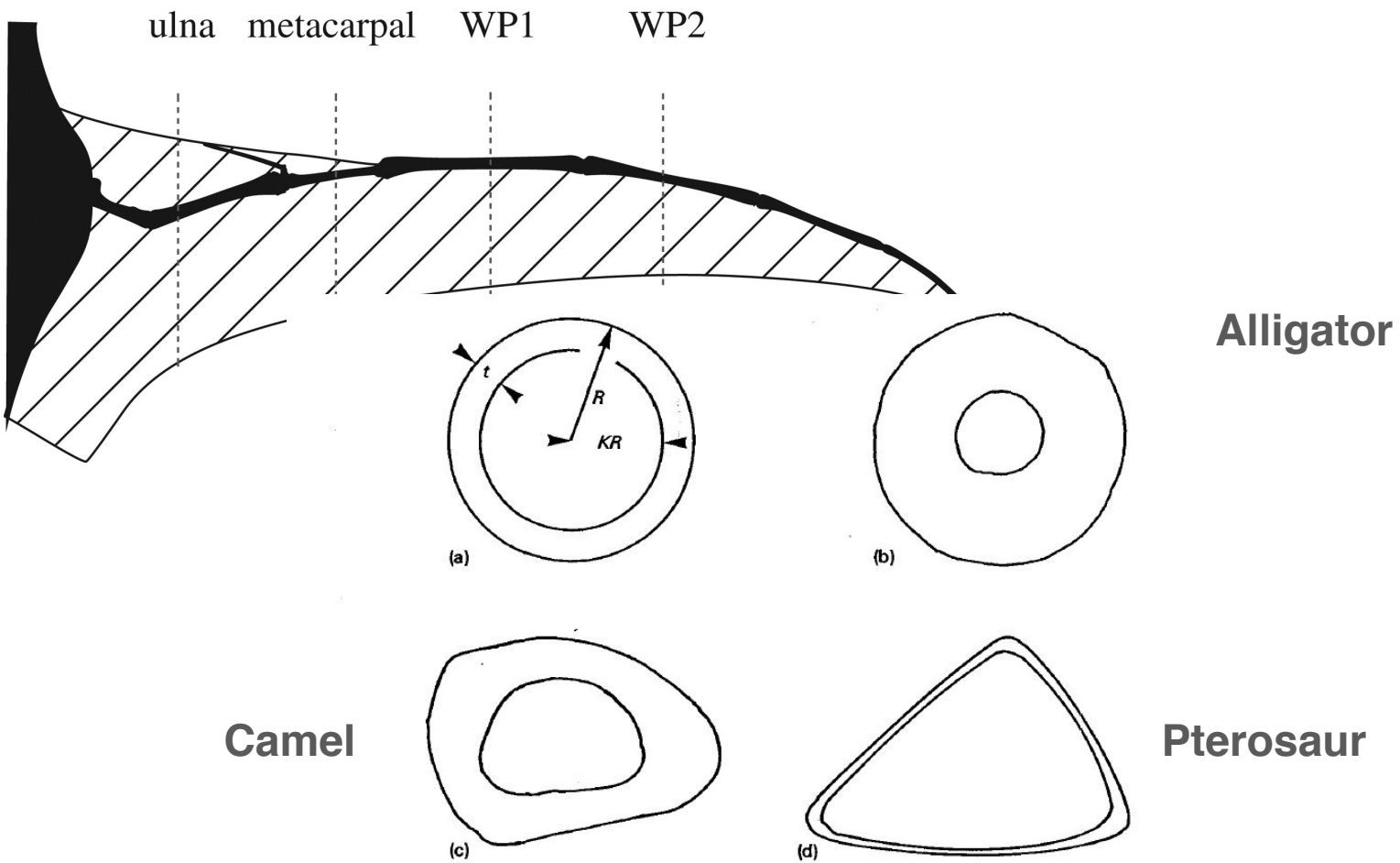
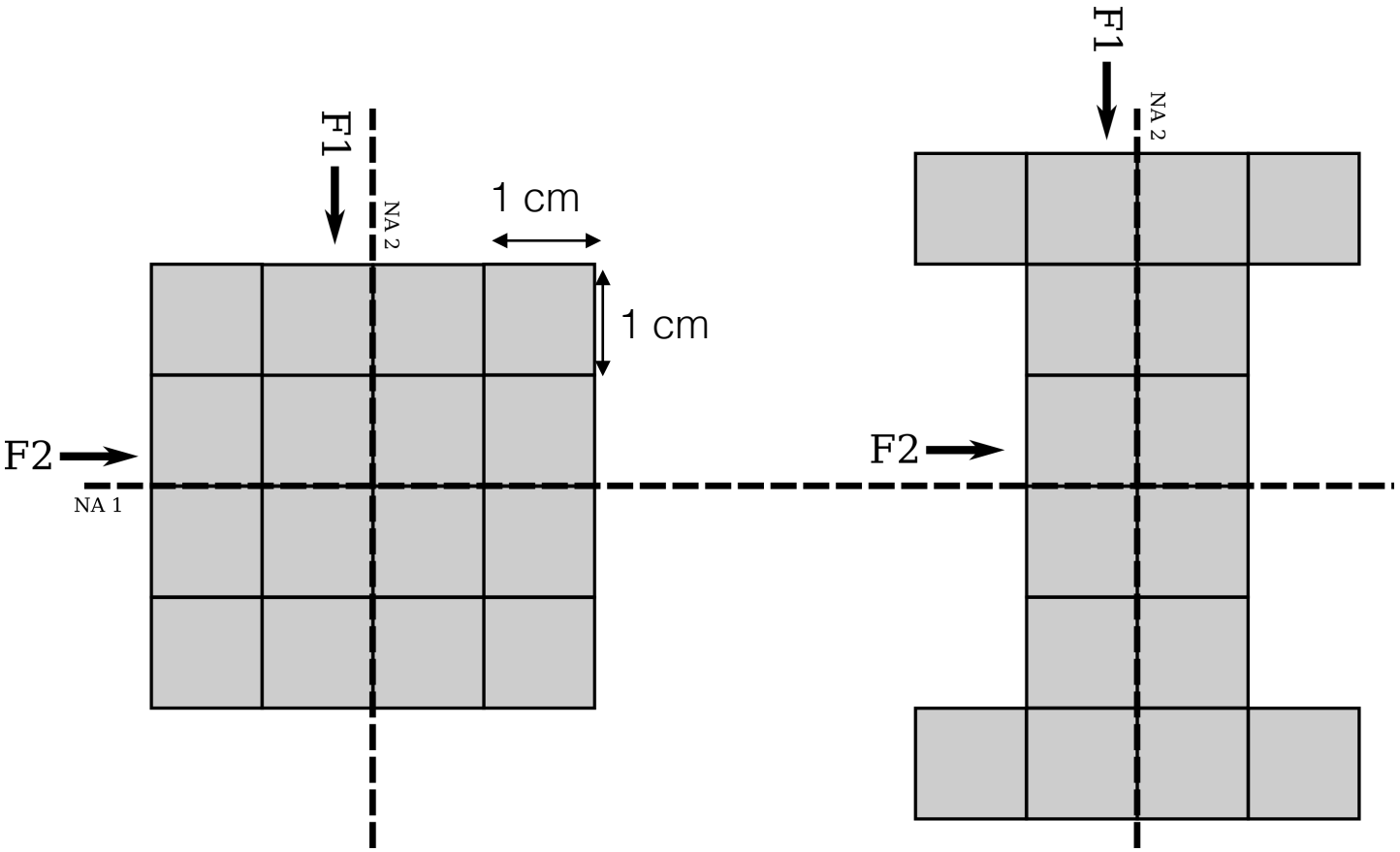
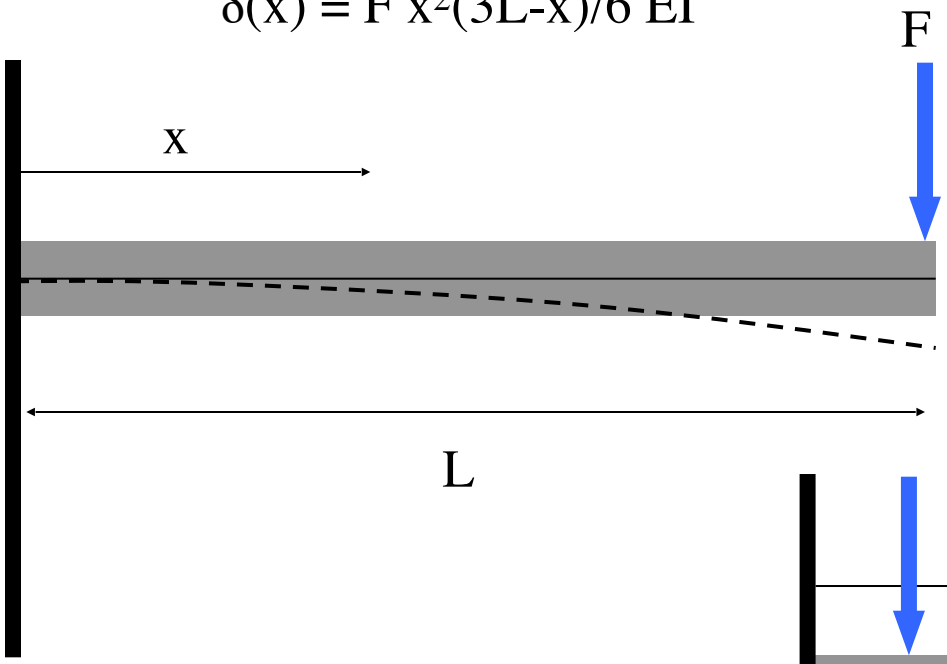


FIG. 1. (a) Conventions for R, t and K used in text. (b), (c), (d) Sketches of sections of bones of different values of K and R/t. (b) $K = 0{\cdot}35$, $R/t = 1{\cdot}5$; alligator femur. (c) $K = 0{\cdot}57$, $R/t = 2{\cdot}4$; camel tibia. (d) $K = 0{\cdot}91$, $R/t = 11$; *Pteranodon* first phalanx.

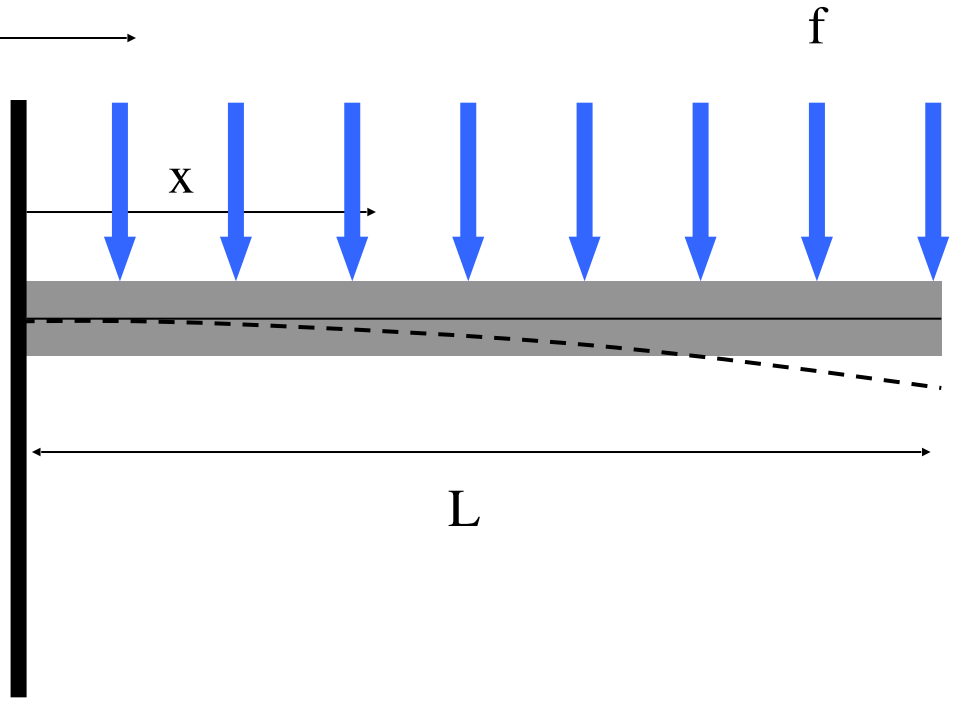
Area?
Second Moment of Area?



$$\delta(x) = F\, x^2(3L-x)/6\, EI$$



$$\delta(x) = f\, x^2(x^2 + 6L^2 - 4Lx)/24\, EI$$



http://www.aps.anl.gov/asd/me/Calculators/ElasticBeam2.html