b = actual dimension of small side of column;
C_f = depth factor;
d = actual dimension of large side of column;
e = eccentricity of longitudinal load;
E = modulus of elasticity of wood parallel to grain;
F_b = allowable unit stress in bending;
F'_c = allowable unit stress in compression parallel to grain;
K = ratio of effective to unsupported length of column;
L = unsupported length of column;
M = bending moment;
P = longitudinal load;
S = section modulus;
$\bar{\alpha}$ = duration of load factor;
β, γ = amplification coefficients; and
ν = input constant used by computer program to determine maximum cross-sectional area desired.

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RELIABILITY ANALYSIS OF FRAME STRUCTURES

By John Stevenson[1] and Fred Moses,[2] A. M. ASCE

INTRODUCTION

There are several relatively new circumstances which have brought the inadequacy of the factor of safety deterministic design concept to the fore. Included among these is the appearance of a vast array of new structural materials from the aerospace industry which not only exhibit in the mean improved strength and stiffness or weight characteristics but also show an increased variation as compared to conventional construction materials. This greater variation necessitates such high factors of safety that the improved properties of the material are largely left unrealized.

Another consideration influencing design safety is the increase in the extremes of the structural load environment. Nuclear power reactors and weapons effects, undersea and space exploration present the designer with a load spectrum many times broader in terms of structural survival and function than here-to-fore encountered. The luxury of picking the worst possible load condition and designing for that load is no longer economically feasible under broad spectrum load conditions. Some considerations must be given to the statistics of extremes for the rational design of such structures.

A third factor is concerned with economic costs, and in the world of finite capital supply the imprecise and sometimes erroneous factor of safety concept design must give way to acceptable risk design with all factors kept in their proper economic perspective. The concept of risk must be quantified before acceptable risk levels can be determined. At the very least, reliability

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analysis can be a useful tool in determining the factor of safety level used in structural codes by adding an element of objective evaluation to the largely intuitive safety factor decision process.

The particular analysis developed herein is that of determining the overall probability of failure of structures whose failure mechanisms is expressed as a linear combination of several structural resistances and loadings. The particular class of structure studied herein is limited to frame structures suitable for plastic or mechanism analysis. It should be understood, however, that these reliability techniques developed herein are applicable to a wide range of structural and other engineering problems, as, for example, electrical networks. Nor is there any requirement that the basic phenomenon that is being described statistically need be linear in nature. Thus a greater range of structural problems including trusses, plates and shells are amenable to reliability analysis.

The failure of indeterminate frame structures subject to plastic hinge formation is characterized by the linear combination of moment resistances or plastic moment capacities and load effects. The moment resistance and load variables in reliability analysis are described by three quantities, mean value, standard deviation and statistical distribution function. This description contrasts with deterministic analysis where only mean values are considered explicitly and safety or load factors are used to compensate for undefined standard deviations and distribution functions.

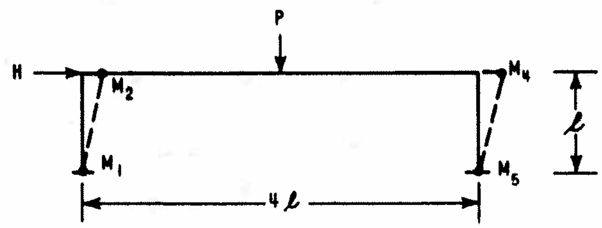
The statistical variables are the plastic moment capacity or resistance of structural members at hinge points and the applied moments at hinges resulting from load. Member moment resistances and load induced moments seem to be the largest or most composite statistical phenomenon readily identifiable in plastic frame analysis that can be measured and a sample population developed without undue expense. In the reliability analysis, all the statistical variables are assumed to be independent, continuous, random variables with known mean, standard deviation and probability density function. Reliability based design will reflect, however the degree of uncertainty or confidence that is had in the values chosen for the mean standard deviation and distribution function.

The theoretical application of mathematical statistics and probability theory to structural safety and design has been explored in depth particularly in the past 10 yr (7,13,2,16). The purpose herein is to develop reliability computational schemes employing limit analysis and apply them to the redundant, indeterminate frame structures including load and strength variation (2). The correlation between failure modes is considered by an approximate method using a Gaussian distribution for all the conditional probability distributions (16). Approximate, generalized methods of reliability analysis are developed for computing the failure probability of a single failure mode. This utilizes Pearson family of distributions and avoids the n-fold multiple integrals generally required for this type of problem. The Pearson distribution uses the first four statistical moments and thus may be as appropriate in matching limited available data as other commonly used frequency distributions. A comparison of the Pearson distribution with exact probability analysis shows relatively small errors. The ultimate purpose of the approximate methods presented herein is to be able to include reliability analysis in an iterative design and optimization program. Only the reliability against plastic hinge collapse has been considered. No attempt is made to evaluate reliability

associated with column and frame buckling or to include large deflection effects.

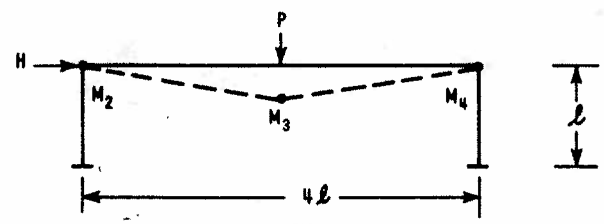
THEORY

There are two basic problems requiring solution in determining accurate overall structural reliability of a framed structure. The first problem is associated with evaluating the failure probability of the individual failure modes or mechanisms which describe the structure. In this instance the major difficulty encountered is that, in general, even though the distribution describing



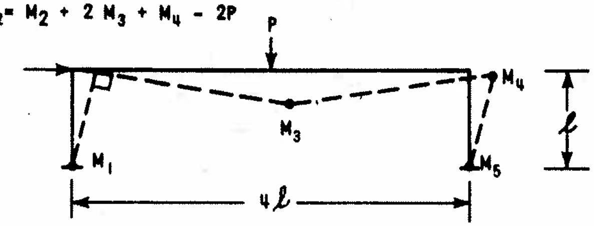
FRAME FAILURE MECHANISM Z_1:

$Z_1 = M_1 + M_2 + M_4 + M_5 - H\ell$



BEAM FAILURE MECHANISM Z_2:

$Z_2 = M_2 + 2\,M_3 + M_4 - 2P$



COMBINED FAILURE MECHANISM Z_3:

$Z_3 = M_1 + 2\,M_3 + 2\,M_4 + M_5 - 2P\ell - H\ell$

FIG. 1.—FAILURE MECHANISMS FOR SINGLE BENT FRAME

the components of the mechanism or failure mode is known, the statistical distribution of the total mechanism is not. Notable exceptions to this difficulty are found when the components are normally distributed or when the number of components is large enough for the central limit theorem to be applicable. In these cases the distribution of the failure modes can also be considered normal. In cases where normality does not apply, two techniques are available to determine failure mode reliability: Recursive function integration (17), and the Pearson distribution method developed herein.

The second major problem encountered is that of incorporating correlation between failure modes into the calculation of overall reliability. Only upper and lower bounds on overall reliability can be determined without explicitly considering the degree of correlation between failure modes. As an example consider the single bent frame of Fig. 1. The simple beam and combined failure mechanisms for this example take the form

$$Z_2 = M_2 + 2M_3 + M_4 - 2Pl \quad (1)$$

$$Z_3 = M_1 + 2M_3 + 2M_4 + M_5 - 2Pl - Hl \quad (2)$$

in which M_i = plastic moment capacity of the ith joint; H = horizontal load on joint 2; P = vertical load on joint 3; Z_i = reserve strength of the ith failure mode; and l = unit length of member. Eqs. 1 and 2 are correlated in moment resistance and load as M_3, M_4 and P are common to both equations. The degree of correlation depends on the number and relative magnitude of common terms.

All individual failure modes for a structure are assumed to have the same basic form. For the kth mode

$$Z_k = \sum_{i=1}^{n} A_{ki} M_i - \sum_{j=1}^{m} B_{kj} S_j \quad (3)$$

in which M_i = independent continuous random variable denoting the structural resistance of a structural member at the ith point in the structure; S_j = independent continuous random variable denoting the effect of the jth load on the structure; A_{ki} = resistance coefficient determined by the position and condition of the ith point related to the kth failure mode; B_{kj} = load coefficient determined by the position and magnitude of the jth load on the structure related to kth failure mode; and Z_k = continuous random variable denoting the reserve strength of the kth failure mode of the structure. It is assumed that the probability density functions, the mean values and the standard deviations or frequency table estimates of same are known for M_i and S_j.

It should be noted that while this paper deals only with a simple form of the variables M and R the methods developed in this section are applicable to systems of the form

$$Z_k = \sum_{i=1}^{n} M_{ki}(x_i) - \sum_{j=1}^{m} S_{kj}(x_j) \quad (4)$$

in which $M(x_i)$ and $S(x_j)$ can be nonlinear functions of a statistical variable.

The probability of failure of the kth mode described in Eq. 3 is expressed as

$$\Pr(Z_k < 0) = \int_{-\infty}^{0} \Pr(Z_k = t)\,dt \quad (5)$$

The probability density function $\Pr(Z_k = t) = p(Z_k)$ is generally unknown unless Z_k is made up of normally distributed variables in which case then Z_k is also normally distributed.

Defining the kth failure mode in more compact form:

$$Z_k = \sum_{i=1}^{N} C_{ki} x_i \quad (6)$$

in which the C_{ki} replaces A_{ki} and B_{kj} and x_i replaces M_i and R_j. A set of functions ϕ_j are defined to carry out the recursive integrations.

$$\left.\begin{aligned} \phi_1(X) &= \Pr(C_{k1}x_1 < X) = \int_{-\infty}^{X} \Pr(C_{k1}x_1 = t)\,dt \\ \phi_2(X) &= \Pr(C_{k1}x_1 + C_{k2}x_2 < X) \\ &= \int_{-\infty}^{X} \phi_1(X - t)\ \Pr(A_{k2}x_2 = t)\,dt \end{aligned}\right\} \quad (7a)$$

Finally,

$$\left.\begin{aligned} \phi_N(X) &= \Pr(C_{k1}x_1 + C_{k2} \ldots C_{kN-1}x_{N-1} + C_{kN}x_N < X) \\ &= \int_{-\infty}^{X} \phi_{N-1}(X - t)\ \Pr(C_{kN}x_N = t)\,dt \end{aligned}\right\} \quad (7b)$$

It can be seen by inspection of Nth term in Eq. 7b and the left side of Eq. 6 that they are the same when X is taken equal to zero.

$$\Pr(Z_k < 0) = \phi_N(X)|_{X=0} \quad (8)$$

The function $\phi_1(X)$ is determined by integration of the density function of x_1 which is known. For any value X the function $\phi_N(X)$ is calculated recursively. Similar results could be obtained by using characteristic functions of the X_i and inverting to find the distribution of Z_k or by using n-fold multiple integrals.

It is apparent that recursive integration will involve a large amount of computation. As a result, another method for determining the probability of failure was developed which utilizes a Pearson distribution (5). This method has the advantage of rapid evaluation when compared to the recursive function integration scheme. The number of operations required to evaluate the recursive integration failure probability increases exponentially with the number of terms in the failure mode expression. The Pearson distribution method on the other hand is largely independent of the complexity or the number of terms in the failure modes and as a practical matter can be evaluated more quickly by at least one or two orders of magnitude.

The first four statistical central moments are required to develop the Pearson distributions. These moments for the individual $C_{ki}X_i$ can be calculated from the known distribution of the $C_{ki}x_i$. The first four moments of the independent $C_{ki}x_i$ are combined to form the first four moments of Z_k using well known relationships for the sum of independent random variables (5,15). It should be noted that the even moments always take positive values while odd numbered moments are positive or negative depending on the relative magnitude of $C_{kj}x_i$. Once the first four statistical moments describing the Z_k distribution are known, the parameters necessary to develop the distribution of Z_k in Pearson form (15) can be determined. The one major disadvantage of the Pearson method is the large number of possible Pearson distributions which must be considered over the full range of the parameters and the discontinuities at or near the interface of two Pearson distributions.

Failure Mode Correlation.—With the assumption that all failure modes are statistically independent, the overall probability of failure of the structure containing N failure modes is determined in terms of the individual mode failure probability given in Eq. 5.

$$\Pr(\text{failure}) = 1 - \{[1 - \Pr(Z_1 < 0)]\,[1 - \Pr(Z_2 < 0)] \ldots [1 - \Pr(Z_N < 0)]\} \qquad (9)$$

If complete statistical or condition dependence is assumed, the probability of failure of the structure is

$$\Pr(\text{failure}) = \text{Max}\,\{\Pr(Z_k < 0)\} \qquad (10)$$

When some dependence exists between failure modes the assumption of independence usually yields an upper bound and dependence, a lower bound on the exact value of P_f. Proof of this statement is related to the signs of the B_{kj} in Eq. 3. To determine the exact probability of overall failure, correlation between failure modes must be utilized. If Z_i and Z_j are independent, random variables the probability of their equal to specific value Z_i^*, Z_j^* is

$$\Pr(Z_i = Z_i^*, Z_j = Z_j^*) = \Pr(Z_i = Z_i^*)\ \Pr(Z_j = Z_j^*) \qquad (11)$$

If there is dependence between Z_i and Z_j, the conditional distribution of Z_i given Z_j or (Z_i/Z_j) is required:

$$\Pr(Z_i = Z_i^*, Z_j = Z_j^*) = \Pr\left(Z_i = \frac{Z_i^*}{Z_j} = Z_j^*\right)\ \Pr(Z_j = Z_j^*) \qquad (12)$$

The parameters of the conditional distribution of (Z_i/Z_j) are developed from the parameters describing Z_i and Z_j failure modes and the correlation coefficient. A measure of the degree of dependence is expressed by the simple correlation coefficient $\rho = 0$, while complete dependence requires $\rho = 1$. Negative correlation can also exist but is of no interest in this study. To determine the explicit degree of correlation between the Z, use is made of the basic assumption that there is zero correlation between individual X. An intermediate step in evaluating the correlation between the Z requires the evaluation of their covariance.

$$\mu^2_{Z_j}, Z_k = \sum_{i=1}^{n} C_{ji} C_{ki}\, \mu^2_{X_i} \qquad (13)$$

in which C_{ji} = constant coefficient of the ith statistical variable of the jth equation; $\mu^2_{X_i}$ = the variance or second central moment of X_i; and $\mu^2_{Z_j}, Z_k$ = the covariance of Z_j, Z_k. The simple correlation coefficient between the Z is expressed as

$$\mu_{Z_j}, Z_k = \frac{\mu^2_{Z_j Z_k}}{\mu_{Z_j}\ \mu_{Z_k}} = \rho_{jk} \qquad (14)$$

The correlation coefficients thus determined in general form can then be used to find the first and second moments of the conditional distribution of Z_j given $Z_k \ldots Z_n$ (3,4).

Reliability of Overall Structure.—The probability of failure of the structure as a whole requires an evaluation of a sequence of the Z_k probability of failure. The probability of the first failure mode forming a collapse mechanism is expressed in Eq. 5 by setting $k = 1$. The probability of failure of the structure due to the formation of a collapse mechanism in the second mode assuming it has survived the first mode loading is expressed as:

$$\Pr(Z_1 > 0, Z_2 < 0) = \Pr\left(\frac{Z_1 > 0}{Z_2 < 0}\right)\ \Pr(Z_2 < 0) \qquad (15)$$

$$= \int_{-\infty}^{0} \Pr\left(\frac{Z_1 > 0}{Z_2 = t}\right)\ \Pr(Z_2 = t)\,dt \qquad (16)$$

$$= \int_{t_2=-\infty}^{t_2=0} \int_{t_1=0}^{t_1=\infty} \Pr\left(\frac{Z_1 = t_1}{Z_2 = t_2}\right)\ \Pr(Z_2 = t_2)\,dt_1\,dt_2 \qquad (17)$$

The sequence is extended to $n - 1$ modes having survived and the nth mode failure:

$$\Pr(Z_1 > 0, Z_2 > 0 \ldots Z_{n-1} > 0, Z_n < 0)$$
$$= \Pr(A), \Pr(B) \ldots \Pr(N-1)\ \Pr(N) \qquad (18)$$

in which $\Pr(A) = \Pr(Z_1 > 0/Z_2 > 0, Z_3 > 0 \ldots Z_{n-1} > 0, Z_n < 0)$ and analogous definitions for $\Pr(B)$, $\Pr(N-1)$ and $\Pr(N)$. The probability of failure of the structure is expressed as:

$$\Pr(\text{failure}) = \Pr(Z_1 < 0) + \Pr(Z_1 > 0, Z_2 < 0) + \ldots \Pr(Z_1 > 0, Z_2 > 0 \ldots Z_n < 0) \qquad (19)$$

The procedure just outlined is quite general and is valid for any distribution of Z_k. However, a number of serious difficulties arise when evaluation of the terms in Eq. 19 is attempted. The conditional and multivariate distribution of Z_k is in general not known even for known distribution of Z_k except in the case of the normal distribution. The evaluation of mean values of the conditional distributions requires the solution of n determinates of size $n - 1$ which is quite time consuming. The series of products in Eq. 18 requires a numerical integration technique over n multiple integrals. It is obvious from the difficulties just expressed that some simplifying modifications of the procedure are necessary if solutions to practical problems are to be realistic goals.

The first modification assumes that the conditional distribution of Z_k can be taken as normal. There is no basis in theory for this assumption, but results obtained agree well with Monte Carlo solutions where Monte Carlo probabilities of failure for non-normal distributions are determined by the simulated testing of a large number of trial structures. The method simply ratios the number of failures to the total number of trials to determine probability of failure.

The second modification is to approximate the multiple integrals encountered by a single integral. Eq. 19 is approximated by a single integral when the n - 1 terms are taken at the mean values of their range of integration and the integration is performed on the n-th term only. The individual failure modes are so arranged as to place them in the order of decreasing probabilities of failure. For the cases studied, this ordering yields the closest conservative results between exact and approximate solutions.

$$\Pr(Z_1 > 0, Z_2 > 0 \ldots Z_{n-1} > 0, Z_n < 0)$$

$$\simeq \int_{-\infty}^{0} \Pr(E) \; \Pr(F) \ldots \Pr(G) \; dt_n \quad \ldots \ldots \quad (20a)$$

in which:

$$\Pr(E) = \Pr\left(\frac{Z_1 > 0}{Z_2 = Z_2^*, Z_3 = Z_3^* \ldots Z_{n-1} = Z_{n-1}^*, Z_n = t_n}\right) \quad (20b)$$

$$\Pr(F) = \Pr\left(\frac{Z_2 > 0}{Z_3 = Z_3^* \ldots Z_{n-1} = Z_{n-1}^*, Z_n = t_n}\right) \quad \ldots \ldots \quad (20c)$$

$$\Pr(G) = \Pr(Z_n = t_n) \quad \ldots \ldots \quad (20d)$$

and Z_k^* = the mean value of Z_k in the positive region. In general the first term of the conditional distribution would also require integration over the range of that term for each value of n. In the case of the normal distribution which is the assumed distribution of all conditional probabilities there exists a polynomial expression for the evaluation of Z_k over the positive range. The final modification is to limit to three the number of terms considered in evaluation of the conditional distribution of the Z_n. The choice of three terms in evaluation of the multivariate probabilities of failure is a device which greatly simplifies generalized computer formulation of the reliability analysis. It also speeds solution by reducing the number and size of determinate evaluations without adversely effecting accuracy in those instances where a high degree of correlation between failure modes has a significant effect on overall probability of failure. The modes which are kept in the evaluation of the conditional failure of Z_k are the modes with the highest correlation as determined by their respective correlation coefficients.

RELIABILITY ANALYSIS TEST EXAMPLES

For the most part the actual solutions of the test examples are programmed for an 1107 UNIVAC computer using an ALGOL 60 compiler. The first example presented is that of the fixed end beam which is used to demonstrate in detail the reliability methods used to develop the single mechanism probability of failure. This example also is used to determine the effect of different statistical distributions on the failure probability and also serves as a means for quantitative comparison between the recursive function integration and the Pearson methods for various statistical distributions.

A three failure mechanism simple frame bent is solved in detail in order to demonstrate the methods for determining the effect of failure mechanism correlation on overall structure reliability. While not shown herein solutions for relatively large two-bay two stcry examples and examples employing high and low correlation have been studied which show the applicability of reliability

analysis to extremes of behavior with regard to dependency between failure mechanism (15).

Single Failure Mode Fixed End Beam.—The single failure mode describing the fixed end beam shown in Fig. 2 is

$$Z = M + 2M + M - Pl \quad \ldots \ldots \quad (21)$$

which if the form of Eq. 1 is used:

$$Z_1 = \sum_{i=1}^{3} A_{1i} M_i - \sum_{j=1}^{1} B_{1j} S_j \quad \ldots \ldots \quad (22)$$

in which $A_{11} = 1$; $A_{12} = 2$; $A_{13} = 1$; $B_{11} = 1$; and $S_1 = Pl$. For this first example it is assumed M_1, M_2, M_3, and S_1 are independent continuous random variables.

Table 1 together with additional fixed end beam solutions provide information for a number of interesting comparisons. These comparisons may be

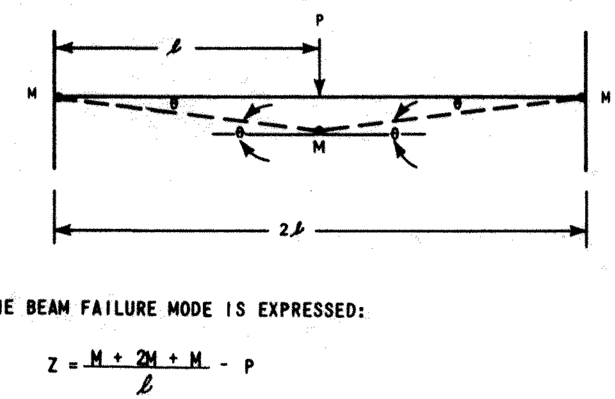


FIG. 2.—FIXED END BEAM WITH CONCENTRATED LOAD FAILURE MODE

summarized as: (1) The effect of the statistical mean on failure probability; (2) the effect of the coefficient of variation on failure probability; and (3) the effect of varying statistical distribution on failure probability.

In Fig. 3 the effects that mean value and overall coefficient of variation have on probability of failure for the fixed end beam are presented. The inconsistency of the factor of safety or ratio of capacity to load means as a measure of reliability is also shown in this figure. As an example compare the probability of failure for a safety factor of 1.5 and coefficients of variation of 0.20 and 0.10 respectively. These variations which are usually not even specified in design differ by a factor of two while their related probabilities of failure differ by a factor of approximately 325. Thus for the same nominal factor of safety the actual reliability can vary greatly as a function of moment resistance and load variability. Conversely designs with widely varying safety factors or means can have the same reliability. As can be seen in Fig. 3 the 0.10 and 0.20 coefficients of variation curves have the same reliability, 0.001 even though their respective safety factors are 1.40 and 1.96.

In Fig. 4 the influence that moment resistant versus load variability have on failure probability is presented. Here can be seen the effect of the coefficient of variation of load on the reliability of a particular structure. Consider the case in Fig. 4 where the coefficient of variation of the moment resistances is 0.10 and the load coefficient of variation varies between 0.20 and 0.05 or by a factor of four. In this instance the failure probability ranges between 0.011 and 0.0000013 or by a factor of 8450. Thus the relative coefficient of variation

TABLE 1.—SOLUTION OF FIXED BEAM EXAMPLES

Moments $M_1 - M_3$, in kip-feet (1)	Load P, in kips (2)	CV[a] moment/load (3)	Normal Solution Number 1[b] (4)	Normal Solution Number 2[c] (5)	Normal Solution Number 3[d] (6)	Log Normal Solution Number 2[c] (7)	Log Normal Solution Number 3[d] (8)	Weibull Solution Number 2[c] (9)	Weibull Solution Number 3[d] (10)
350	120	0.10/0.10	8.75(- 2)	8.75(- 2)	8.74(- 2)	8.94(- 2)	8.94(- 2)	7.81(- 2)	8.43(- 2)
350	120	0.20/0.20	1.70(- 1)	1.70(- 1)	1.70(- 1)	2.37(- 1)	2.38(- 1)	2.53(- 1)	2.55(- 1)
350	120	0.05/0.20	2.06(- 1)	2.06(- 1)	2.06(- 1)	1.93(- 1)	1.94(- 1)	2.12(- 1)	2.17(- 1)
350	120	0.20/0.05	1.35(- 1)	1.35(- 1)	1.35(- 1)	1.30(- 1)	1.30(- 1)	1.37(- 1)	1.38(- 1)
400	120	0.10/0.10	4.91(- 3)	4.91(- 3)	4.91(- 3)	6.40(- 3)	6.43(- 3)	4.01(- 3)	3.08(- 3)
400	120	0.20/0.20	4.09(- 2)	4.09(- 2)	4.09(- 2)	9.82(- 2)	9.83(- 2)	9.54(- 2)	9.66(- 2)
400	120	0.05/0.20	5.12(- 2)	5.12(- 2)	5.12(- 2)	6.27(- 2)	6.28(- 2)	3.82(- 2)	3.80(- 2)
400	120	0.20/0.05	2.55(- 2)	2.55(- 2)	2.55(- 2)	1.55(- 2)	1.55(- 2)	3.08(- 2)	3.06(- 2)
450	120	0.10/0.10	1.22(- 4)	1.20(- 4)	1.22(- 4)	2.21(- 4)	2.36(- 4)	1.47(- 4)	1.22(- 4)
450	120	0.20/0.20	3.28(- 2)	3.28(- 2)	3.28(- 2)	3.59(- 2)	3.59(- 2)	3.07(- 2)	3.28(- 2)
450	120	0.05/0.20	7.41(- 3)	7.41(- 3)	7.41(- 3)	1.69(- 2)	1.70(- 2)	2.20(- 3)	1.04(- 3)
500	120	0.10/0.10	2.26(- 6)	2.15(- 6)	2.24(- 6)	3.23(- 6)	3.82(- 6)	4.12(- 6)	2.26(- 6)
500	120	0.20/0.20	9.82(- 3)	9.82(- 3)	9.82(- 3)	1.19(- 2)	1.21(- 2)	9.24(- 3)	9.82(- 3)
500	120	0.05/0.20	6.27(- 4)	6.27(- 4)	6.27(- 4)	3.92(- 3)	4.08(- 3)	3.73(- 5)	1.42(- 8)
500	120	0.20/0.05	7.64(- 4)	7.64(- 4)	7.64(- 4)	4.98(- 5)	5.12(- 5)	1.49(- 3)	1.53(- 3)

[a] CV—coefficient of variation $[\mu^{(2)1/2}/\mu^{(1)}]$.
[b] Solution Number 1—Normally Distributed Sum of Normals.
[c] Solution Number 2—Recursive Function Integration.
[d] Solution Number 3—Pearson Distribution Method.

of load versus moment resistance has an important effect on the reliability of the structure.

The effects of the statistical distribution used is considered in Fig. 5. The first series of curves 1A, 1B and 1C where the variations or moments is high the log normal curve 1B shows lower failure probabilities because its skew positive nature would tend to develop more high moment variants than a sym-

metrical normal distribution with the same mean and variance. As the Weibull distribution is skew negative for the range of coefficients of variation used in this study, a proportionably higher number of low moment resistances are determined as compared to the normal distribution thus a higher failure prob-

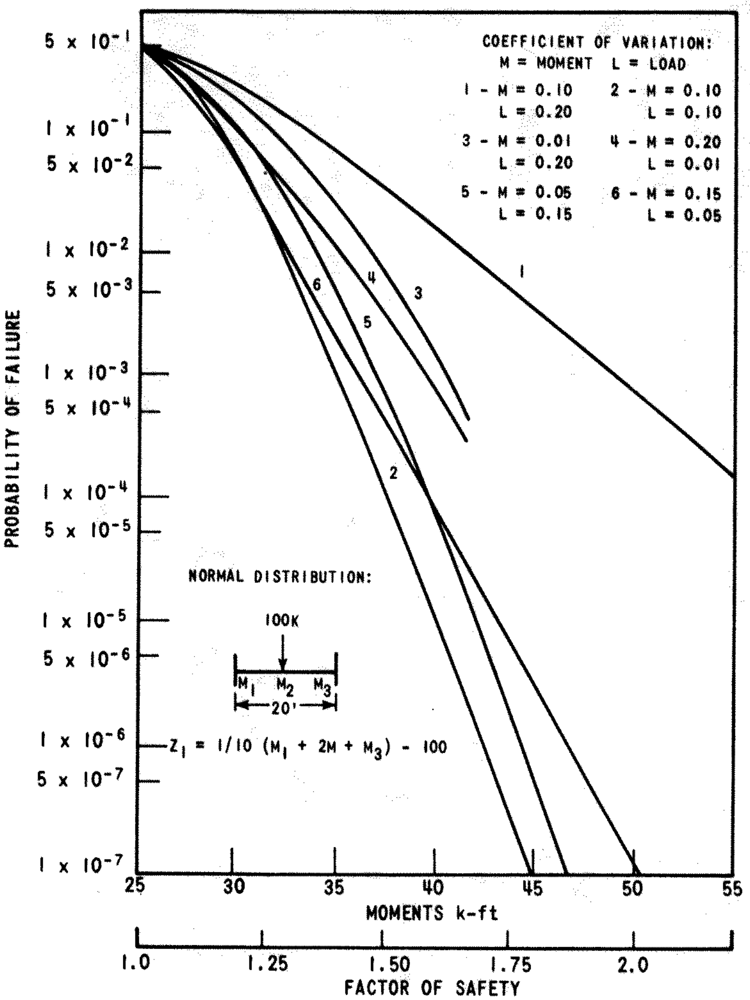


FIG. 3.—EFFECT OF MEAN VALUE AND COEFFICIENTS OF VARIATION ON PROBABILITY OF FAILURE

ability would result. In the 3A, 3B and 3C series of curves the high load variation governs thus the log normal and Weibull distributions switch places with respect to relative high or low failure probabilities. In the 2A-2C set of curves where neither moment nor load variation is obviously dominant the three distribution failure probability curves are closely grouped.

In this last instance three phenomena are active which tend to cause somewhat mixed results. The first factor is that the coefficient of variation of the sum of independent component variations is less than the coefficient of variation of lesser number of larger component variation whose total mean values are equal.

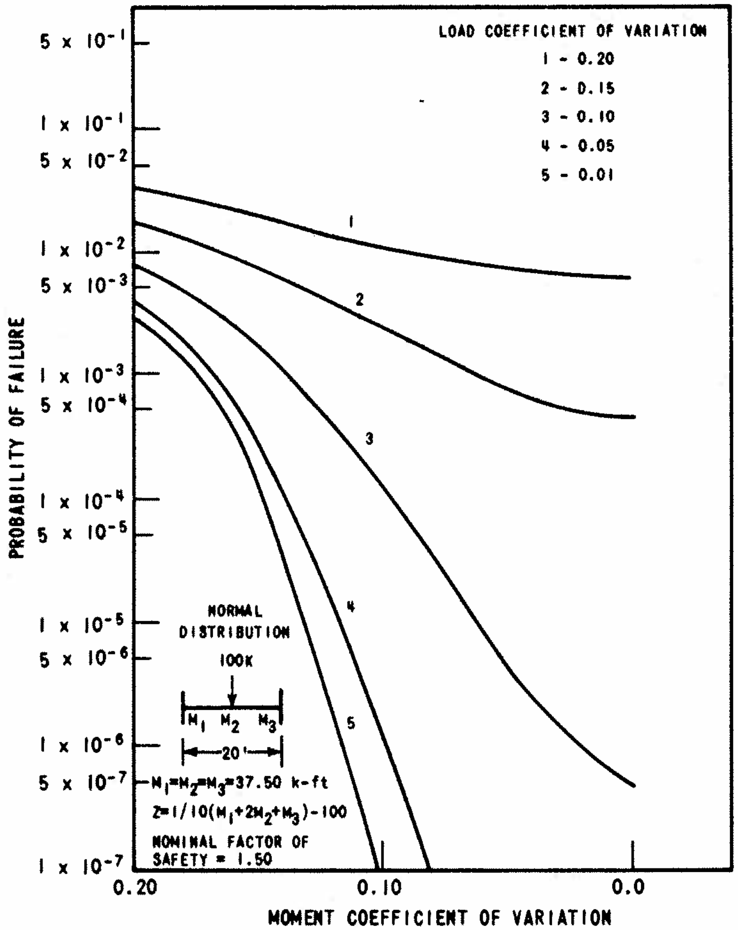


FIG. 4.—EFFECT OF DIFFERENT COEFFICIENTS OF VARIATION FOR MOMENT RESISTANCE AND LOAD ON PROBABILITY OF FAILURE (FS = 1.50)

The second factor effecting failure probability is the statistical distribution of the sum of moment resistance tending toward normality by the central limit theorem. This phenomenon by reducing the positive skewness of the moment resistance distribution tends to emphasize the positive skewness of the load distribution thereby increasing the influence of the load distribution on failure probability as compared to the normal distribution. Finally a counter factor is introduced by mean values of moment resistance being larger than the mean value of load as the factor of safety tends to increase. Because the coefficient of variation is fixed this tends to increase its standard deviation.

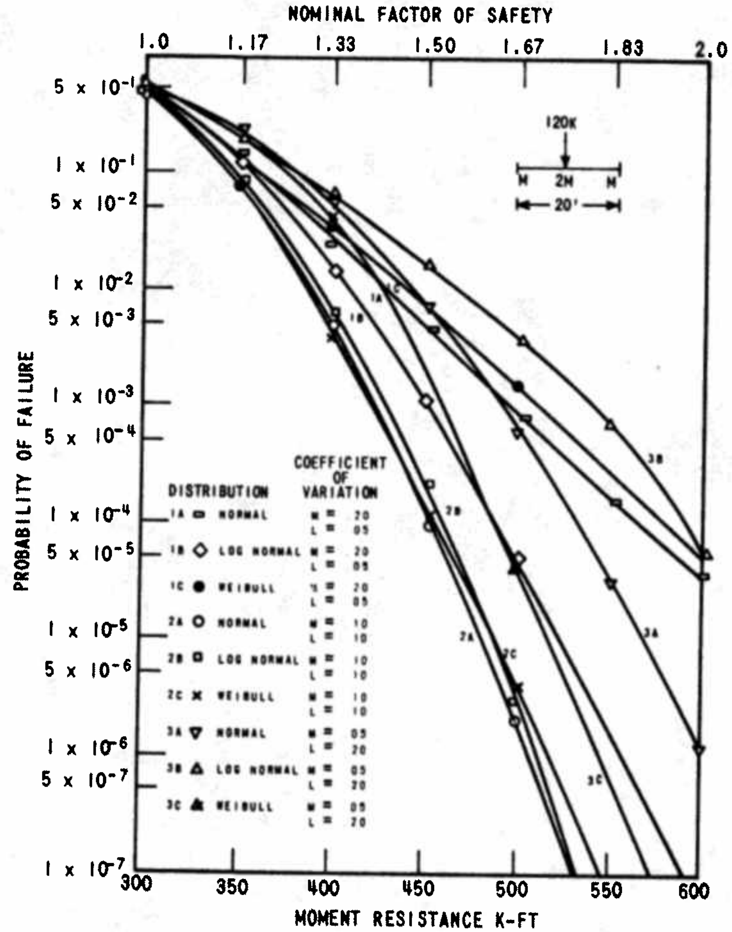


FIG. 5.—PROBABILITY OF FAILURE OF FIXED END BEAM AS FUNCTION OF STATISTICAL DISTRIBUTION

Three Failure Mode Single Bent Frame.—The single bent frame presented in Fig. 1 is described by three modes of failure.

The individual probabilities of failure of Z_1, Z_2, and Z_3 can be determined as shown previously. The overall probability of failure of the structure can be expressed as

$$\Pr(\text{failure}) = \Pr(Z_1 < 0) + \Pr(Z_1 > 0, Z_2 < 0) + \Pr(Z_1 > 0, Z_2 > 0, Z_3 < 0) \quad (23)$$

in which the various probability terms are as defined previously. It can be shown that when the conditional distributions and correlations between the failure mechanisms are expressed exactly, the order in which the failure modes are considered is immaterial.

The exact solution for the third term of Eq. 23 requires the double integration of product terms. The evaluation of the first and second term of Eq. 23 proceed in like manner except no double integration is involved. The evaluation of the numerator of conditional terms $\Pr(Z_1 > 0)$ would normally require an additional integration but in this particular paper use is made of a polynomial expansion equal to the area under the normal density function.

TABLE 2.—SIMPLE BENT FRAME TRIAL EXAMPLES—NORMAL DISTRIBUTION COEFFICIENT OF VARIATION—10 % MOMENTS; 10 % LOADS

Moments, in kip-feet		Loads, in kips		Mode Probability of Failure[a] Normal Solution Number 1			Independent	Exact over-all P_f	Approximate over-all P_f	Approximate error, as a percentage	Monte Carlo[b] P_f
M_C (1)	M_B (2)	H (3)	P (4)	1 (5)	2 (6)	3 (7)	(8)	(9)	(10)	(11)	(12)
						(a) High Range FS—1.0					
50	300	20	30	5.0(- 1)	5.0(- 1)	1.20(- 1)	7.80(- 1)	7.09(- 1)	7.09(- 1)	0.0	7.09(- 1)
100	200	20	30	5.0(- 1)	5.0(- 1)	0.0	7.50(- 1)	5.58(- 1)	5.58(- 1)	0.0	5.87(- 1)
100	200	40	20	5.0(- 1)	5.0(- 1)	3.10(- 4)	7.50(- 1)	6.46(- 1)	6.46(- 1)	0.0	—
60	300	20	30	3.25(- 1)	7.97(- 2)	4.32(- 2)	4.06(- 1)	3.45(- 1)	3.65(- 1)	+ 5.8	—
50	320	20	30	5.00(- 1)	3.30(- 1)	5.58(- 2)	6.84(- 1)	6.29(- 1)	6.29(- 1)	0.0	—
						(b) Middle Range FS—1.5					
75	450	20	30	1.74(- 4)	3.56(- 5)	2.03(- 5)	2.31(- 4)	2.06(- 4)	2.06(- 4)	0.0	—
150	300	20	30	3.09(- 4)	1.42(- 5)	0.0	3.24(- 4)	3.10(- 4)	3.10(- 4)	0.0	—
60	400	20	30	4.32(- 2)	9.91(- 3)	7.23(- 4)	5.33(- 2)	5.18(- 2)	5.18(- 2)	0.0	—
						(c) Low Range FS—1.7					
100	500	20	30	6.18(- 7)	3.28(- 7)	0.0	9.39(- 7)	6.63(- 7)	6.63(- 7)	0.0	—
90	620	20	30	2.24(- 8)	7.46(- 9)	7.45(- 9)	3.75(- 8)	2.41(- 8)	2.39(- 8)	- 0.8	—

[a] Solution based on normally distributed sum of normals.
[b] Monte Carlo solution based on 8000 simulated trial loads.
Failure Modes: l = 10 ft

1) $\frac{1}{l}(M_1 + M_3 + M_4 + M_5) - H$

2) $\frac{1}{l}(M_2 + 2M_3 + M_4) - 2V$

3) $\frac{1}{l}(M_1 + 2M_3 + 2M_4 + M_5) - H - 2V$

Approximate Solution.—In those instances where simplifying modifications assuming the mean value in the positive range of integration are employed, a least upper bound solution results when the failure mechanisms are ordered in terms of decreasing probability of failure. The approximate solution is reduced to a single integral by replacing the integration on Z_2 over its positive range in Eq. 23 by the single average value of Z_2 in the positive range. The numerical results of this and other examples by means of the exact and approximate solutions are shown in Table 2. Also shown in Table 2 is the close and conservative agreement achieved between exact and approximate method of analysis developed in this study over a number of trial examples. The table also shows the error associated with the assumption of independence of failure modes. Error in excess of 30 % is noted in some cases. A comparison between the exact and approximate or simplified methods of analysis shows a maximum loss in accuracy of + 5.8 %.

A further demonstration of the effect of statistical distribution on probability of failure is shown for a optimum design single frame bent which recently appeared in the literature (10). Table 3 shows the range of reliability as a function of statistical distribution for the same design M_C = 7 k-ft and M_B = 5 k-ft as determined for a single frame bent.

TABLE 3.—PROBABILITY OF FAILURE OF OPTIMUM DESIGN SINGLE BENT FRAME FOR VARIOUS DISTRIBUTIONS

Moment resistance distribution (1)	Load distribution (2)	Probability of failure of structure[a] (3)	Remarks (4)
Normal	Normal	4.80(- 5)	
Log Normal	Log Normal	4.87(- 5)	
Weibull	Weibull	4.80(- 5)	
Weibull	Weibull	4.09(- 5)	Start at 0.5 $\mu_x^{(1)'}$
Normal	Log Normal	4.80(- 5)	
Log Normal	Normal	7.13(- 6)	
Log Normal	Weibull	3.31(- 8)	
Weibull	Log Normal	1.84(- 4)	
Normal	Weibull	1.53(- 5)	
Weibull	Normal	5.79(- 5)	

Note:
Coefficient of variation:
Moment = 0.20
Load = 0.20

$\mu_{M_C}^{(1)'}$ = 7 K-ft; $\mu_{M_B}^{(1)'}$ = 5 Kip-ft

[a] Individual mechanism probability of failure determined by Pearson method with over-all failure probability found using approximate method of analysis.

RESULTS AND CONCLUSIONS

The results herein can be grouped into two major categories: (1) The effects of various statistical parameters on individual mode probability of failure; and (2) the effect of correlation between failure modes on overall structural reliability.

In the first category the mean as a measure of the factor of safety and the standard deviation as a measure of variability are compared as to their effect on failure probability. In general the variance of a particular statistical distribution is found to have as profound an effect on reliability as its mean value. The effect of different statistical distributions with the same mean and variance is also studied. Appreciable changes in failure probability are noted when the loads and moment resistances are of different statistical distributions or when their respective coefficients of variation differ greatly. As might be expected the differences between distributions are most pronounced at the low failure probabilities determined in the tail region extremes. However, for the particular distributions studied the distribution effect on failure probability is not nearly as pronounced as that determined from variation of the mean and standard deviation. In the high failure probability range the results tend to be distribution free. Skewness of distribution also effects failure probability. Extreme values in the direction of the skewness are more numerous than those found for symmetric distributions with the same variance and mean. For this reason skewness tends to accentuate shifts in failure probability between distributions. It should be emphasized that the conclusions are based on study of only normal, log normal and Weibull distributions. Other distributions which are not of the exponential type may yield different results.

The degree of correlation between failure modes can have a considerable effect on overall structural reliability. This effect is less pronounced at the low correlation or independence end of the correlation spectrum. When the maximum simple correlation between modes is less than 15 %, a number of trial examples indicate the modes can be assumed independent without introducing any significant error. However, at the other end of the spectrum correlation coefficients as high as 98 % still yield overall failure probabilities appreciably different from those values determined assuming complete dependence of 100 % correlation.

The various problems studied have assumed that the statistical distributions describing the member resistances and load phenomenon are known. For the most part this is not the case with analytical distributions being determined by a goodness-of-fit test on the sample data from a frequency table. It is in this application that the Pearson method is particularly useful because the raw sample data from a frequency table can be used to determine the first four statistical moments directly and thus allow the direct application of the Pearson distribution. This capability is particularly important in the solution of real problems where the true distribution of strength and load is not known.

The dependence on limited sample data also cast considerable doubt on failure probability predictions in the low probability region. This is a shortcoming that will continually plague structural reliability analysis in the desired low probability range unless statistical models can be developed which adequately describe the failure phenomenon and thus allow the analytical determination of statistical distributions.

The results of this study generally support the premise that reliability based analysis of framed structure explicitly considering correlation between failure modes is feasible.

As a result of examples studied the following conclusions have been reached:

1. Reliability based analysis permits a more consistant approach to structural safety by including the statistical variability of loads and strengths in the safety factor evaluation.

2. The additional computation for a reliability based design may be justified because it permits a quantification of safety which may be more useful in the design process.

3. The coefficient of variation which is usually ignored in conventional design has as great an effect on structural safety as does the mean values used in determining the factor of safety of conventional design.

4. Probability of failure is largely distribution free for values in the high probability range.

5. Changes in failure probability are significant as a function of statistical distributions and skewness when relative coefficients of variation differ considerably between loads and moment resistances for other than the high probability of failure range.

6. Evaluation of reliability allows one to formulate a rational design and optimization procedure. This was done and is reported elsewhere (15,18). It is furthermore necessary to include types of failures other than collapse in the reliability analysis to obtain an overall optimum design.

In terms of the design problem rather than analysis, a major advantage of reliability based design is the elimination of deterministic design restrictions on the individual members making up the structural system. The overriding design condition for any structure requires that the structure as a whole performs adequately and not that individual components meet some arbitrary restrictions. Reliability based design focuses on overall structural adequacy and thereby elimates the necessity of considering each component part separately. Another almost equally important advantage of reliability based design is the ability of portions of the structure which are overdesigned, hence have low probability of failure to help compensate for underdesigned high probability of failure portions. This compensating factor inherit in reliability based design should lead to more economical structures.

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APPENDIX II.—NOTATION

The following symbols are used in this paper:

A_{ki} = resistance coefficient determined by the position and condition of the ith point related to the kth failure mode;
B_{kj} = load coefficient determined by the position and magnitude of the jth load on the structure related to kth failure mode;
C_{ji} = coefficient of the ith statistical variable of the jth equation;
H = horizontal load on a joint;
M_i = independent continuous random variable denoting the plastic moment resistance of a structural member at the ith point in the structure;
P = vertical load on a joint;
$p(Z_k)$ = the probability density function of Z_k;
$\Pr(Z_k = t)$ = the probability density function of Z_k;
S_j = independent continuous random variable denoting the effect of the jth load on the structure;
Z_i = reserve strength of the ith failure mode;
$Z_i^\dagger$ = the ith failure mode that has survived which is the next most highly correlated with Z_k;

$Z_j^\dagger$ = the jth failure mode that has survived which is the most highly correlated with Z_k as determined from the correlation coefficient;
Z_k = continuous random variable denoting the reserve strength of the kth failure mode of the structure;
Z_k^* = the mean value of Z_k in the positive region;
$\mu^{(i)}_{C_{kj}x_j}$ = ith statistical central moment of the $C_{kj}x_j$ distribution;
$\mu^{(i)'}_{C_{kj}x_j}$ = the ith statistical ordinary moment of the $C_{kj}x_j$ distribution;
$\mu^{(i)}_{Z_k}$ = the ith statistical central moment of the Z_k distribution;
$\mu^{(2)}_{X_i}$ = the variance or second central moment of X_i;
$\mu^{(2)}_{Z_j, Z_k}$ = the covariance of $Z_{jk} Z_k$; and
ρ = the simple correlation coefficient between the Z.