Strength-Based Reliability and Interference Theory

CEE518

Reliability = \( P(R > S) = P(R - S > 0) = \int \int f_{RS}(r,s) dr ds \)

\( R \) = resistance or strength
\( S \) = load effect or stress

\[ \text{Usually we don’t know the joint density, so…} \]

Let \( R, S \) be independent

Overlapping region in densities gives a measure of \( p_f \)

Probability of load \( S \) falling in a small interval \( ds \) around \( s \) is

\[ P\left( s - \frac{ds}{2} \leq S \leq s + \frac{ds}{2} \right) = f_S(s) ds \]
Let $R, S$ be independent. Overlapping region in densities gives a measure of $p_f$.

Probability of load $S$ falling in a small interval $ds$ around $s$ is

$$P \left( s - \frac{ds}{2} \leq S \leq s + \frac{ds}{2} \right) = f_s(s) ds = \text{Area } A_s$$

$$P(R > s) = \int_s^{\infty} f_R(r) dr = 1 - F_R(s)$$
Reliability of the component is the probability of the load value falling in the interval $ds$ around $s$ and the strength value exceeding the value $s$ simultaneously:

$$d(\text{Reliability}) = f_S(s) ds \int_{s}^{\infty} f_R(r) dr = f_S(s) ds [1 - F_R(s)]$$

The reliability is given as the probability of the resistance or strength $R$ exceeding the load $S$ for all possible values of the load $S$:

$$\text{Reliability} = \int d\text{Reliability} = \int_{-\infty}^{\infty} f_S(s) \int_{s}^{\infty} f_R(r) dr ds$$

$$= \int_{-\infty}^{\infty} f_S(s) [1 - F_R(s)] ds$$

Alternative expression: find the probability of the load assuming a smaller value than the value of the strength:

$$\text{Reliability} = \int d\text{Reliability} = \int_{-\infty}^{\infty} f_R(r) \int_{-\infty}^{s} f_S(s) ds dr$$

$$= \int_{-\infty}^{\infty} f_R(r) F_{S}(r) dr$$

Probability of failure

$$P_f = P(R \leq S) = 1 - P(S \leq R) = 1 - \text{Reliability}$$
\[ P_f = 1 - \int_{-\infty}^{\infty} f_R(r) \left[ \int_{-\infty}^{r} f_S(s) ds \right] dr \]
\[ = 1 - \int_{-\infty}^{\infty} f_R(r) F_S(r) dr \]
\[ = \int_{-\infty}^{\infty} f_R(r) dr - \int_{-\infty}^{\infty} f_R(r) F_S(r) ds \]
\[ = \int_{-\infty}^{\infty} [1 - F_S(r)] f_R(r) dr \]

Reliability when both \( R \) and \( S \) follow Normal distributions

\[ f_R(r) = \frac{1}{\sigma_R \sqrt{2\pi}} \exp \left\{ -\frac{1}{2} \left( \frac{r - \mu_R}{\sigma_R} \right)^2 \right\} \]
\[ f_S(s) = \frac{1}{\sigma_S \sqrt{2\pi}} \exp \left\{ -\frac{1}{2} \left( \frac{s - \mu_S}{\sigma_S} \right)^2 \right\} \]
Use the limit state equation $Z = R - S$
If $R,S \sim N(\mu,\sigma)$, then $Z \sim N(\mu,\sigma)$.

$$f_Z(z) = \frac{1}{\sigma_Z \sqrt{2\pi}} \exp \left\{ -\frac{1}{2} \left( \frac{z - \mu_Z}{\sigma_Z} \right)^2 \right\}$$

$$\mu_Z = \mu_R - \mu_S$$
$$\sigma_Z = \sqrt{\sigma_R^2 + \sigma_S^2}$$

$$P(Z \geq 0) = \int_0^\infty f_Z(z) \, dz$$

Express $z$ as a standard normal variable
$$z_i = \frac{0 - \mu_Z}{\sigma_Z} = -\left[ \frac{\mu_R - \mu_S}{\sqrt{\sigma_R^2 + \sigma_S^2}} \right]$$

when $z = 0$

Reliability $= \int_{x=z_i}^\infty e^{-x^2/2} \, dx$
We can use the standard normal tables.

Define $\beta = \Phi^{-1}(1 - P_f)$

$$\mu_R = \mu_S + \beta \sqrt{\sigma_R^2 + \sigma_S^2}$$

$$\beta = \Phi^{-1}(1 - P_f) = \frac{\mu_R - \mu_S}{\sqrt{\sigma_R^2 + \sigma_S^2}}$$
The Reliability Index

\[ \beta \to \infty, \quad P_f \to 0 \]

\[ \beta \to 0, \quad P_f \to \infty \]

\[ P_f = \Phi(-\beta) = 1 - \Phi(\beta) \]

For Normally and independently distributed R and S, Z is also Normally distributed. In this case,

\[ \beta = \frac{\mu_z}{\sigma_z} = \text{safety or reliability index} \]

First Order Second Moment [Cornell, 1969]

Usually denoted as "FOSM"

If R and S do not follow Normal distributions, but are independent, we must modify our approach to finding \( \beta \).

This approach is called the equivalent Normal variate or Advanced FOSM approach.