Strength-Based Reliability and Interference Theory

CEE518

Reliability = \( P(R > S) = P(R - S > 0) = \int \int f_{RS}(r,s)drds \)

\( R \equiv \) resistance or strength

\( S \equiv \) load effect or stress

\( Usually \ we \ don't \ know \ the \ joint \ density, \ so... \)

Let \( R, S \) be independent

Overlapping region in densities gives a measure of \( p_f \)

Probability of load \( S \) falling in a small interval \( ds \) around \( s \) is

\[
P\left( s - \frac{ds}{2} \leq S \leq s + \frac{ds}{2} \right) = f_S(s)ds
\]
\[ P(R > s) = \int_{s}^{\infty} f_R(r)dr = 1 - F_R(s) \]

Reliability of the component
is the probability of the load value falling in the interval \( ds \) around \( s \)
AND
the strength value exceeding the value \( s \)
SIMULTANEOUSLY:
\[ d(\text{Reliability}) = f_S(s)ds \int_{s}^{\infty} f_R(r)dr = f_S(s)ds \left[ 1 - F_R(s) \right] \]
The reliability is given as the probability of the resistance or strength $R$ exceeding the load $S$ for all possible values of the load $S$:

\[
\text{Reliability} = \int d\text{Reliability} = \int f_S(s) \left[ \int_{-\infty}^{\infty} f_R(r) \, dr \right] \, ds
\]

\[
= \int_{-\infty}^{\infty} f_S(s) \left[ 1 - F_R(s) \right] \, ds
\]
Alternative expression: find the probability of the load assuming a smaller value than the value of the strength:

\[
\text{Reliability} = \int d \text{Reliability} = \int_{-\infty}^{\infty} f_R(r) \int_{-\infty}^{r} f_S(s) ds dr
\]

\[
= \int_{-\infty}^{\infty} f_R(r) F_S(r) dr
\]
Probability of failure

\[ P_f = P(R \leq S) = 1 - P(S \leq R) = 1 - \text{Reliability} \]

\[ P_f = 1 - \int_{-\infty}^{\infty} f_R(r) \left[ \int_{-\infty}^{r} f_S(s) ds \right] dr \]

\[ = 1 - \int_{-\infty}^{\infty} f_R(r) F_S(r) dr \]

\[ = \int_{-\infty}^{\infty} f_R(r) dr - \int_{-\infty}^{\infty} f_R(r) F_S(r) ds \]

\[ = \int_{-\infty}^{\infty} [1 - F_S(r)] f_R(r) dr \]
$P_f = 1 - \text{Reliability} = 1 - P(R > S)$

$$= 1 - \int_{-\infty}^{\infty} f_S(s) \left( \int_{s}^{\infty} f_R(r)dr \right) ds$$

$$= 1 - \int_{-\infty}^{\infty} f_S(s) \left[ 1 - F_R(s) \right] ds$$

$$= 1 - \int_{-\infty}^{\infty} f_S(s) ds + \int_{-\infty}^{\infty} f_S(s) F_R(s) ds$$

$$= \int_{-\infty}^{\infty} f_S(s) F_R(s) ds$$
Reliability when both $R$ and $S$ follow Normal distributions

\[
f_R(r) = \frac{1}{\sigma_R \sqrt{2\pi}} \exp \left\{ -\frac{1}{2} \left( \frac{r - \mu_R}{\sigma_R} \right)^2 \right\}
\]

\[
f_S(s) = \frac{1}{\sigma_S \sqrt{2\pi}} \exp \left\{ -\frac{1}{2} \left( \frac{s - \mu_S}{\sigma_S} \right)^2 \right\}
\]
Use the limit state equation $Z = R - S$
If $R, S \sim N(\mu, \sigma)$, then $Z \sim N(\mu, \sigma)$.

$$f_Z(z) = \frac{1}{\sigma_Z \sqrt{2\pi}} \exp \left\{ -\frac{1}{2} \left( \frac{z - \mu_Z}{\sigma_Z} \right)^2 \right\}$$

$$\mu_Z = \mu_R - \mu_S$$

$$\sigma_Z = \sqrt{\sigma_R^2 + \sigma_S^2}$$
\[ P(Z \geq 0) = \int_{0}^{\infty} f_{Z}(z)dz \]

Express \( z \) as a standard normal variable

\[ z_{1} = \frac{0-\mu_{z}}{\sigma_{z}} = -\left[ \frac{\mu_{R} - \mu_{S}}{\sqrt{\sigma_{R}^{2} + \sigma_{S}^{2}}} \right] \text{ when } z=0 \]

Reliability = \( \int_{x=z_{1}}^{\infty} e^{-x^{2}/2} dx \)

We can use the standard normal tables.
Probability of Failure $P_f = P(Z < 0)$

$$P_f = \Phi \left[ \frac{0 - (\mu_R - \mu_S)}{\sqrt{\sigma_R^2 + \sigma_S^2}} \right]$$

$$= 1 - \Phi \left[ \frac{(\mu_R - \mu_S)}{\sqrt{\sigma_R^2 + \sigma_S^2}} \right]$$

$$1 - P_f = \Phi \left[ \frac{(\mu_R - \mu_S)}{\sqrt{\sigma_R^2 + \sigma_S^2}} \right]$$

$$\Phi^{-1}(1 - P_f) = \left[ \frac{(\mu_R - \mu_S)}{\sqrt{\sigma_R^2 + \sigma_S^2}} \right]$$

$$\mu_R \geq \mu_S + \Phi^{-1}(1 - P_f) \sqrt{\sigma_R^2 + \sigma_S^2}$$
Define $\beta = \Phi^{-1}(1 - P_f)$

$$\mu_R = \mu_S + \beta \sqrt{\sigma^2_R + \sigma^2_S}$$

$$\beta = \Phi^{-1}(1 - P_f) = \frac{\mu_R - \mu_S}{\sqrt{\sigma^2_R + \sigma^2_S}}$$

$$\beta \to \infty, \quad P_f \to 0$$
\[ P_f = \Phi(-\beta) = 1 - \Phi(\beta) \]

For Normally and independently distributed \( R \) and \( S \), \( Z \) is also Normally distributed. In this case,

\[ \beta = \frac{\mu_Z}{\sigma_Z} \equiv \text{safety or reliability index} \]

First Order Second Moment [Cornell, 1969]

Usually denoted as "FOSM"
If R and S do not follow Normal distributions, but are independent, we must modify our approach to finding $\beta$. This approach is called the equivalent Normal variate or Advanced FOSM approach.