# Strength-Based Reliability and Interference Theory

CEE518

Main reference: Rao, (1992)

$$\text{Reliability} = P(R > S) = P(R - S > 0) = \iint f_{RS}(r,s)\,dr\,ds$$

$R \equiv$ resistance or strength

$S \equiv$ load effect or stress

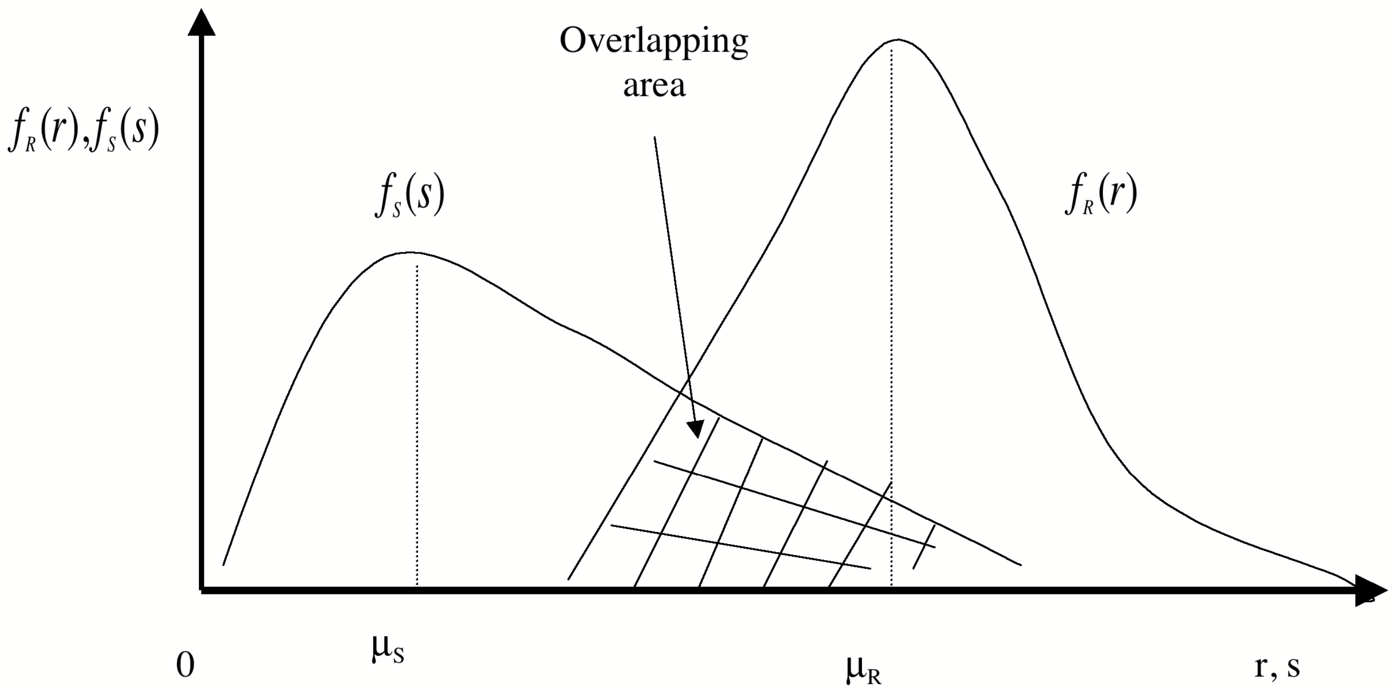
*Usually we don't know the joint density, so…*

Let $R$,$S$ be independent

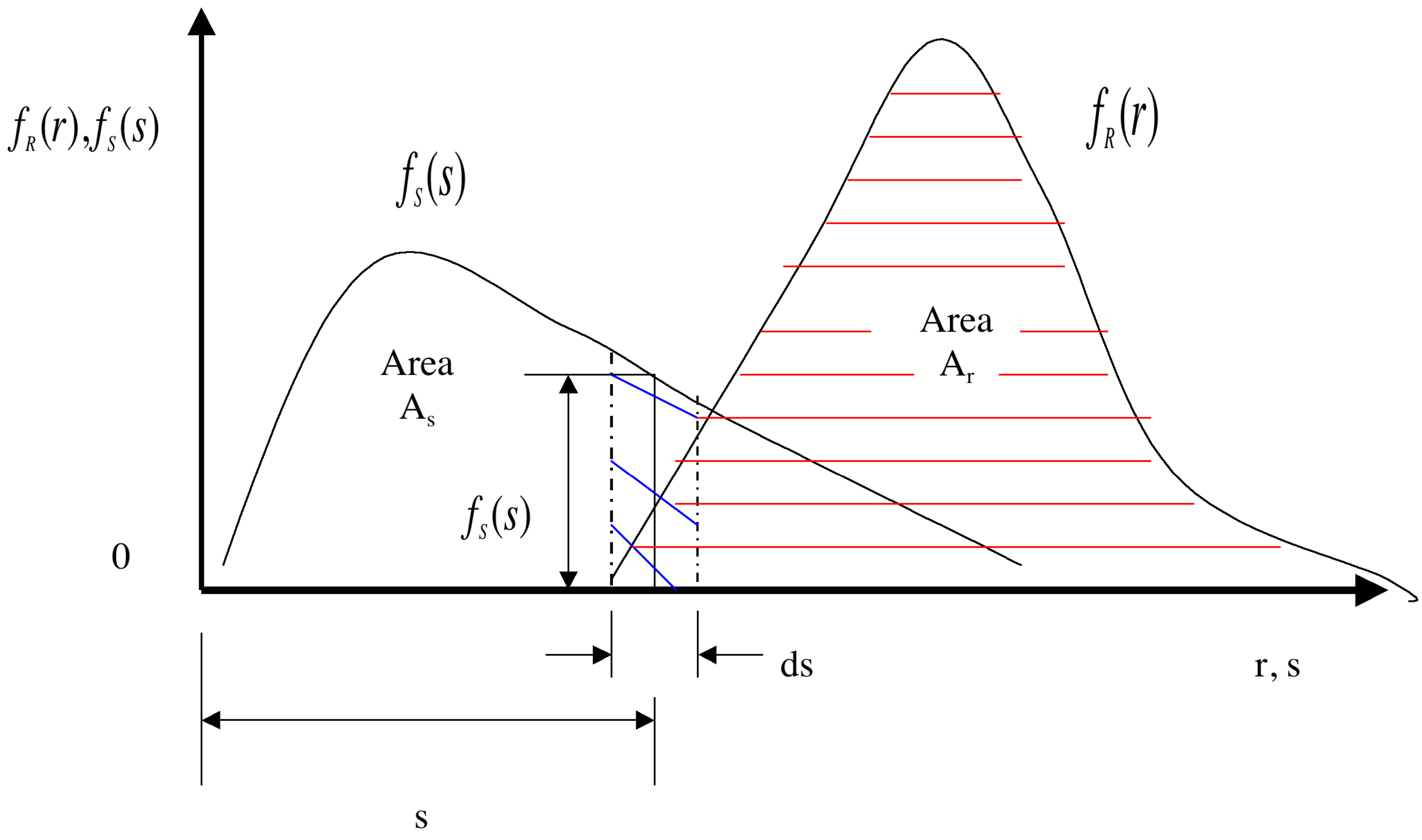
Overlapping region in densities gives a measure of $p_f$

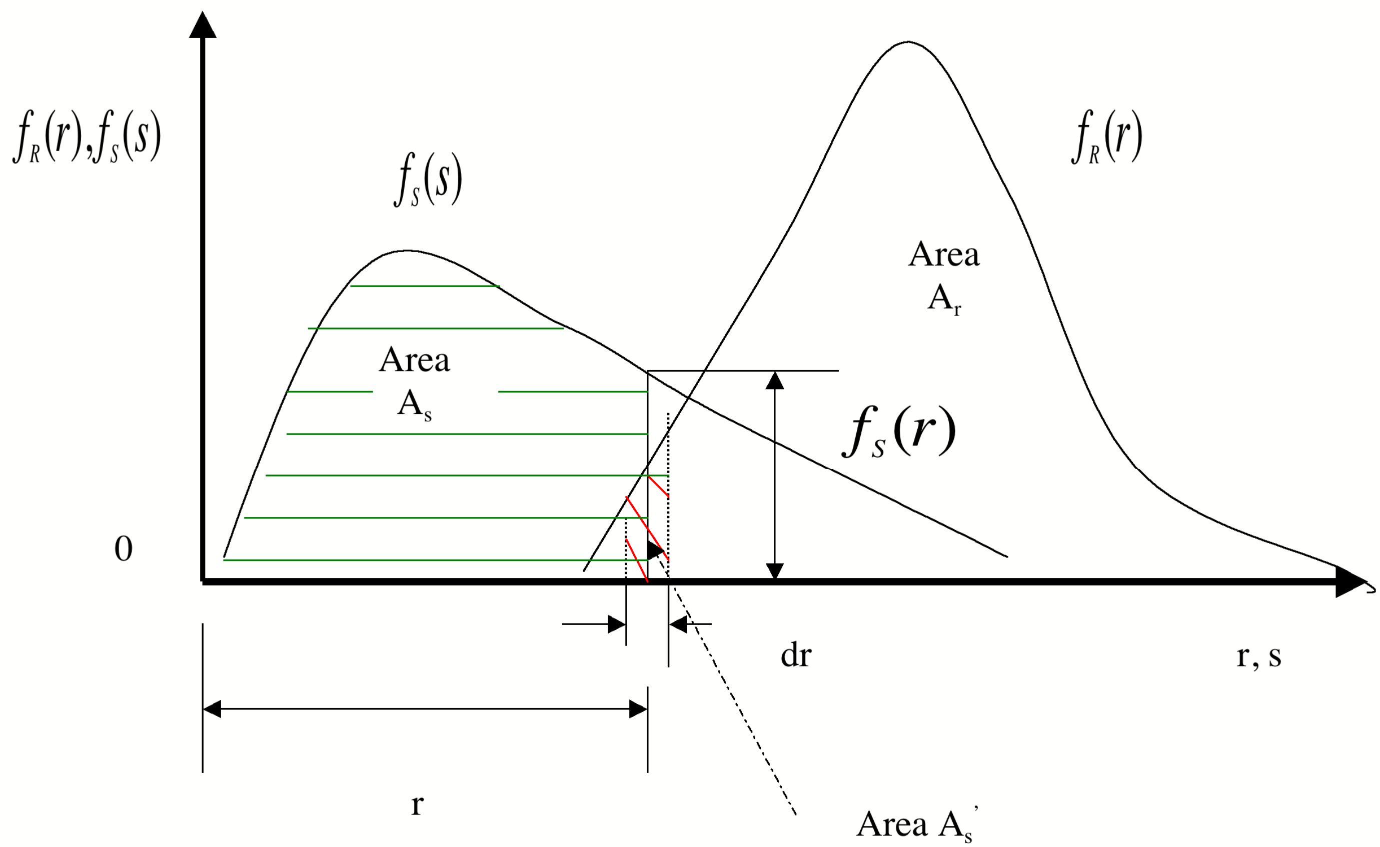
Probability of load $S$ falling in a small interval ds around s is

$$P\left(s - \frac{ds}{2} \le S \le s + \frac{ds}{2}\right) = f_S(s)\,ds$$



$f_R(r), f_S(s)$
Overlapping area
$f_S(s)$
$f_R(r)$
0
$\mu_S$
$\mu_R$
r, s


$f_R(r), f_S(s)$
$f_S(s)$
$f_R(r)$
Area
$A_s$
Area
$A_r$
$f_S(s)$
0
ds
r, s
s


$f_R(r), f_S(s)$
$f_S(s)$
$f_R(r)$
Area
$A_r$
Area
$A_s$
$f_S(r)$
0
dr
r, s
r
Area $A_s'$

$$P\left(R > s\right) = \int_{s}^{\infty} f_R(r)dr = 1 - F_R(s)$$

Reliability of the component

is the probability of the load value falling in the interval ds around s

AND

the strength value exceeding the value s

SIMULTANEOUSLY:

$$d(\text{Reliability}) = f_S(s)ds \int_{s}^{\infty} f_R(r)dr = f_S(s)ds\left[1 - F_R(s)\right]$$

*The reliability is given as*
*the probability of the resistance or strength R exceeding*
*the load S for all possible values of the load S:*

$$\text{Reliability} = \int d\text{Reliability} = \int_{-\infty}^{\infty} f_S(s)\left[\int_{s}^{\infty} f_R(r)\,dr\right]ds$$

$$= \int_{-\infty}^{\infty} f_S(s)\left[1 - \mathrm{F_R}(\mathrm{s})\right]ds$$

Alternative expression: find the probability of the load assuming a smaller value than the value of the strength:

$$\text{Reliability} = \int d\text{Reliability} = \int_{-\infty}^{\infty} f_R(r) \int_{-\infty}^{r} f_S(s)\,ds\,dr$$

$$= \int_{-\infty}^{\infty} f_R(r) F_S(r)\,dr$$

*Probability of failure*

$$P_f = P(R \le S) = 1 - P(S \le R) = 1 - \text{Reliability}$$

$$P_f = 1 - \int_{-\infty}^{\infty} f_R(r)\left[\int_{-\infty}^{r} f_S(s)ds\right]dr$$

$$= 1 - \int_{-\infty}^{\infty} f_R(r)F_S(r)dr$$

$$= \int_{-\infty}^{\infty} f_R(r)dr - \int_{-\infty}^{\infty} f_R(r)F_S(r)ds$$

$$= \int_{-\infty}^{\infty} \left[1 - F_S(r)\right] f_R(r)dr$$

$$P_f = 1 - \text{Reliability} = 1 - P(R > S)$$

$$= 1 - \int_{-\infty}^{\infty} f_S(s) \left[ \int_{s}^{\infty} f_R(r) dr \right] ds$$

$$= 1 - \int_{-\infty}^{\infty} f_S(s) \left[ 1 - F_R(s) \right] ds$$

$$= 1 - \int_{-\infty}^{\infty} f_S(s) ds + \int_{-\infty}^{\infty} f_S(s) F_R(s) ds$$

$$= \int_{-\infty}^{\infty} f_S(s) F_R(s) ds$$

# Reliability when both R and S follow Normal distributions

$$f_R(r) = \frac{1}{\sigma_R \sqrt{2\pi}} \exp\left\{ -\frac{1}{2} \left( \frac{r - \mu_R}{\sigma_R} \right)^2 \right\}$$

$$f_S(s) = \frac{1}{\sigma_S \sqrt{2\pi}} \exp\left\{ -\frac{1}{2} \left( \frac{s - \mu_S}{\sigma_S} \right)^2 \right\}$$

Use the limit state equation $Z = R - S$

If $R,S \sim N(\mu,\sigma)$, then $Z \sim N(\mu,\sigma)$.

$$\therefore f_Z(z) = \frac{1}{\sigma_Z\sqrt{2\pi}}\exp\left\{-\frac{1}{2}\left(\frac{z-\mu_Z}{\sigma_Z}\right)^2\right\}$$

$$\mu_Z = \mu_R - \mu_S$$

$$\sigma_Z = \sqrt{\sigma_R^2 + \sigma_S^2}$$

$$P(Z \geq 0) = \int_{0}^{\infty} f_Z(z)dz$$

Express z as a standard normal variable

$$z_1 = \frac{0\text{-}\mu_Z}{\sigma_Z} = -\left[\frac{\mu_R - \mu_S}{\sqrt{\sigma_R^2 + \sigma_S^2}}\right] \text{ when } z=0$$

$$\text{Reliability} = \int_{x=z_1}^{\infty} e^{-x^2/2} dx$$

We can use the standard normal tables.

Probability of Failure $P_f = P(Z < 0)$

$$P_f = \Phi\left[\frac{0-(\mu_R - \mu_S)}{\sqrt{\sigma_R^2+\sigma_S^2}}\right]$$

$$= 1-\Phi\left[\frac{(\mu_R - \mu_S)}{\sqrt{\sigma_R^2+\sigma_S^2}}\right]$$

$$1-P_f = \Phi\left[\frac{(\mu_R - \mu_S)}{\sqrt{\sigma_R^2+\sigma_S^2}}\right]$$

$$\Phi^{-1}\left(1-P_f\right) = \left[\frac{(\mu_R - \mu_S)}{\sqrt{\sigma_R^2+\sigma_S^2}}\right]$$

$$\mu_R \geq \mu_S + \Phi^{-1}\left(1-P_f\right)\sqrt{\sigma_R^2+\sigma_S^2}$$

$$\text{Define } \beta = \Phi^{-1}(1 - P_f)$$

$$\mu_R = \mu_S + \beta\sqrt{\sigma_R^2 + \sigma_S^2}$$

$$\beta = \Phi^{-1}(1 - P_f) = \frac{\mu_R - \mu_S}{\sqrt{\sigma_R^2 + \sigma_S^2}}$$

$$\beta \to \infty, \quad P_f \to 0$$

$$P_f = \Phi(-\beta) = 1 - \Phi(\beta)$$

For Normally and independently distributed R and S, Z is also Normally distributed. In this case,

$$\beta = \frac{\mu_Z}{\sigma_Z} \equiv \text{safety or reliability index}$$

First Order Second Moment [Cornell, 1969]

Usually denoted as "FOSM"

If R and S do not follow Normal distributions,
but are independent,
we must
modify our approach
to finding $\beta$.
This approach is called
the equivalent Normal variate
or
Advanced FOSM approach.