Gaussian Plumes from “Point” Sources

- Time averaged vs instantaneous plumes
- Simplified steady-state plume model
- “Eddy” diffusion, advection/diffusion equation
- Gaussian point source plume model
- Plume sigma values vs stability and distance
- Plume reflection
- Non-gaussian plumes
- Plume Rise; plume trajectories
- Buoyancy-induced dispersion
- Stack downwash

View from below of “Coning” plume under neutral atmospheric conditions

Source: Slade et al “Meteorology and Atomic Energy, 1968”
Describing Plume Concentrations

Fig 4.3, p.44 in Martin et al
Tim’s Simple Plume Model

\[ \text{mass/time passing point 1} = \text{mass/time passing thru disk area 2} = \text{mass/time passing thru disk area 3} \]

\[ C_1 > C_2 > C_3 \]

*Pollutant is well mixed and confined within the cone
*Pollutant is continuously swept thru the cone by the wind

Concentration vs. distance downwind depends upon cone shape
Simple Model #1:

Concentration of air at 2 = \frac{\text{Mass emission rate}}{(\text{wind speed})(\text{area of disk 2})}

\[ \frac{\mu g}{m^3} = \frac{\mu g/sec}{(m/sec)(m^2)} \]

Disk shape depends upon stability category

More unstable and thus more pronounced vertical spreading
Perspective View of “Fanning” Plume in Very Stable Air

Less vertical motion

Source: Slade et al “Meteorology and Atomic Energy, 1968”
unstable

neutral

stable

More Detailed Plume Model

Mass is not uniformly distributed within the cone’s volume

Gaussian (normal) distribution occurs across AA due to changes in wind direction over averaging time

Shape is described by “plume sigmas”

Time-averaged concentration across AA

Most probable wind direction

wind

source
Simple Model #2:

\[
\text{Conc at } \frac{t}{2} = \frac{\text{Mass emission rate}}{(\text{wind speed})(\text{area of disk } 2)} \left\{ \text{Gaussian distribution function} \right\}
\]

\[
\frac{\mu g}{m^3} = \frac{\mu g/\text{sec}}{(m/\text{sec})(m^2)} [-]
\]

\(X\) is the time-averaged wind direction,  
\(Y\) is the cross-wind direction,  
\(Z\) is the vertical dimension.

Gaussian Plume Model

In order to derive an equation describing the distribution of mass within the plume, we must first consider the transport of mass within a small control volume.
*Transport of mass in x direction depends on the average horizontal wind

*Transport of mass in the y and z directions depends on turbulent motions

**“Eddy” Diffusion**

Consider two enclosed air volumes separated by a wall (arrows represent eddy motions; balls represent pollutant molecules)

Pollutant molecules are moving around in random directions due to random eddy motions

Random eddy motions in pollution-free air

Size of eddies > size of pollutant molecules

Now remove the wall between the enclosures
The rate of pollutant molecules crossing plane AA depends upon the concentration difference between the two sides. Specifically, this rate is \( K \frac{\delta c}{\delta x} \), where \( K \) is termed an “eddy diffusivity” with units of \( \text{m}^2/\text{sec} \). The magnitude of \( K \) depends upon the magnitude of the eddy motions.

Eventually in this case, the rate of net rate of pollution crossing AA is zero (an equal number of molecules cross in both directions). However the value of \( K \) remains constant throughout the “experiment.”

Changes in x Direction:

Net rate of change of mass flow =
(Mass flow rate in) - (Mass flow rate out)

Mass Flow Rate In = \( C \ u \ A_{yz} \) \( \{ \mu g/\text{m}^3 \} \ [\text{m/s}] \ [\text{m}^2] \})

Mass Flow Rate Out = \( C \ u \ A_{yz} + \frac{\partial}{\partial x} \left( C \ u \ A_{xz} \right) dx \)

Net Rate of Change = \( - \frac{\partial}{\partial x} \left( C \ u \ A_{yz} \right) dx = - \frac{\partial}{\partial x} \left( C \ u \right) V \)
Changes in z Direction via "Turbulent Diffusion":

Mass Flow Rate In = \(-A_{vy} \frac{\partial}{\partial z} \{ K_z C \}\) [m²][m⁻¹][µg m⁻³][m² sec⁻¹]

Net Rate of Change = \( \frac{\partial}{\partial z} \left\{ \frac{\partial}{\partial z} \left( K_z C \right) \right\} V \)

A similar result is obtained in the y direction. Given that the net rate of change in the volume \( V = \frac{\partial}{\partial t} \) is the change in all three directions, we obtain an overall expression in terms of x, y and z.

"Advection-Diffusion" Equation

\[ \frac{\partial C}{\partial t} = -u \frac{\partial C}{\partial x} + \frac{\partial}{\partial y} \left\{ K_y \frac{\partial C}{\partial y} \right\} + \frac{\partial}{\partial z} \left\{ K_z \frac{\partial C}{\partial z} \right\} \]

+ other losses due to deposition and chemical reactions

\( \approx 0 \) for steady-state models

Effect of turbulent "diffusion", i.e., exchange of polluted air parcel with surrounding air parcels. If the surrounding air is cleaner, \( \frac{\delta C}{\delta z} \) & \( \frac{\delta C}{\delta y} \) are negative. \( K \) is the "eddy diffusivity" and represents the intensity of turbulent motions and varies with stability.
The Gaussian plume equation is a particular solution to this more general equation under the following assumptions:

* Steady state conditions \( \frac{\partial C}{\partial t} = 0 \)
* Constant wind speed with height \( (u \) does not depend on \( z \))
* Constant eddy diffusivity \( (K \) does not depend on \( y \) or \( z \))

Define:

\[
\delta_z^2 = \frac{2 K_z x}{u} \quad \delta_y^2 = \frac{2 K_y x}{u}
\]

* Mass is conserved

\[
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} C dydz = Q \quad [\text{for } x > 0] \quad \{\mu g/\text{sec}\}
\]

**Gaussian “Point” Source Plume Model:**

\[
C(x, y, z) = \frac{Q}{2\pi \mu \sigma_x \sigma_z} \left\{ \exp \left( -\frac{(z - h)^2}{2\sigma_z^2} \right) + \exp \left( -\frac{(z + h)^2}{2\sigma_z^2} \right) \right\} \left\{ \exp \left( -\frac{y^2}{2\delta_y^2} \right) \right\}
\]

- Pollutant concentration as a function of downwind position \((x, y, z)\)
- Mass emission rate
- “Effective” stack height, including rise of the hot plume near the source
- Wind speed evaluated at “effective” release height
- Corresponds to disk area in simple model (values depend upon downwind distance, \( x \))
- Distribution of mass in vertical dimension \((z)\) at a given downwind distance, \( x \) (includes the effect of surface reflection)
- Distribution of mass in cross-wind dimension \((y)\) at a given downwind distance, \( x \)
Gaussian Plume
(Concentrations vary with x, y and z)

For a given x, the max conc. is at the plume centerline and decreases exponentially away from the centerline at a rate dependent upon the sigma values, $\sigma_y$ and $\sigma_z$.

$\sigma_y$ and $\sigma_z$ are functions of x

Concentration distribution in a Gaussian plume
($\sigma_y = 20$ m; $\sigma_z = 10$ m; centerline concentration = 1.0)

Note: theoretical plume has infinite extent in all directions!

Source: Hanna et al., 1981
Sigma-y

Sigma-z
Plume sigma formulas from EPA’s ISC Model

Vertical distribution:

\[ \sigma_z = a x^b \]

- \( x \) is in kilometers
- \( \sigma_z \) is in meters
- \( a, b \) depend on \( x \)

Cross-wind distribution:

\[ \sigma_y = 465.11628 x (\tan \Theta) \]

\[ \Theta = 0.017453293 (c - d \ln(x)) \]

- \( x \) is in kilometers
- \( \sigma_y \) is in meters
- \( \Theta \) is in radians

<table>
<thead>
<tr>
<th>Pasquill Stability Category</th>
<th>x (km)</th>
<th>a</th>
<th>b</th>
</tr>
</thead>
<tbody>
<tr>
<td>A*</td>
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<td>0.16 - 0.20</td>
<td>170.220</td>
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<td>0.21 - 0.25</td>
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<td>0.26 - 0.30</td>
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<td>0.31 - 0.40</td>
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<tr>
<td>&gt;3.11</td>
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<td>**</td>
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</table>

* If the calculated value of \( \sigma_z \) exceed 5000 m, \( \sigma_z \) is set to 5000 m.
## Pasquill Stability Category

### Pasquill Stability Category

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* If the calculated value of $\sigma_z$ exceed 5000 m, $\sigma_z$ is set to 5000 m.

** $\sigma_z$ is equal to 5000 m.

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\[ \Theta = 0.017453293(c - d \ln(x)) \]

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<th>d</th>
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<tr>
<td>F</td>
<td>4.167</td>
<td>0.36191</td>
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Plume “Reflection” off of the Ground
(*pollutant cannot penetrate the ground*)

Reflection is modeled by adding a “virtual” source contribution to the “real” one
A virtual" source can also be used to model the effect of hot, rising plumes

Physical stack height, \(h\)

Plume rise, \(\Delta h\)

"Effective" stack height, \(h_e\)

Concentration vs. \(x\) at ground level (\(z=0\))

(\(\text{note maximum at } x > 0\))

**Example Calculation**

*Given:*

\(Q = 10 \text{ grams/sec}; \ h = 50\text{ m}; \ x = 500 \text{ m} = 0.5 \text{ km}; \ u_{100} = 6 \text{ m/s}; \) Stability Class “D”

*Compute:*

\(C(500, 0, 0)\), i.e., the ground level concentration at plume centerline, 500 meters downwind.

\[\begin{align*}
\delta_z &= a x^b = 32.093 (0.5)^{0.81066} = 18.3 \text{m} \\
\Theta &= 0.017453293 (8.3330 - 0.72382 \ln (0.5)) = 0.1542 \text{ radians} \\
\delta_y &= 465.11628 x (\tan \Theta) = 465.11628 (0.5) (\tan (0.1542)) = 36.1 \text{m}
\end{align*}\]

\[
C(x, y, z) = \frac{Q}{2\pi \ u \sigma_y \sigma_z} \left\{ \exp \left( -\frac{(z - h)^2}{2\sigma_z^2} \right) + \exp \left( -\frac{(z + h)^2}{2\sigma_z^2} \right) \right\} \left\{ \exp \left( \frac{- (y)^2}{2\sigma_y^2} \right) \right\}
\]

\[
C(500, 0, 0) = \frac{10}{2\pi (6)(36.1)(18.3)} \left\{ \exp \left( -\frac{(0 - 50)^2}{2(18.3)^2} \right) + \exp \left( -\frac{(0 + 50)^2}{2(18.3)^2} \right) \right\} \left\{ \exp \left( -\left( \frac{0}{2(36.1)} \right)^2 \right) \right\}
\]

\[
C(500, 0, 0) = \frac{10}{2\pi (6)(36.1)(18.3)} [0.0479] = 1.92 \times 10^{-3} \text{g/m}^3 = 19.2 \mu g/m^3
\]
Virtual Source Also Used to Model Reflection Off of the Top of the Mixed Layer

Eventually well-mixed

Plume Reflection off of the top of the mixed layer

wind
Plume “Trapped” in Stable Layer Above Mixed Layer

Minimum Necessary Information
Needed to Implement a Simple
Atmospheric Dispersion Model

- Time of Day
- Date
- Latitude
- Land Use
- Cloud Cover
- Hourly Wind Speed
- Pre-dawn dθ/dz

Stability Category
Mixing Depth
Wind Speed Profile

Atmospheric Dispersion Model

Wind Direction

Source Emission Rate
Source Geometry
Non-Gaussian Plumes

Very Narrow Stable Plume over Water (as viewed from above)
Plume Fumigation During On-shore Flow

Plume “Trapped” in Building Wake
Plume “Looping” During Unstable Conditions
(large-scale vertical motions)

Extreme Departure From Gaussian

Instantaneous and corresponding ensemble-averaged plume in the CBL
Many tall industrial stacks release hot, effluents into the air. Hot air rises and cools. There are two well established results from the science of plume rise: the immediate downwind trajectories of “bent over” plumes and the maximum attained height of “vertical” plumes.

**Plume Rise in Neutral or Stable Air**

‘Bent over’ plume trajectory, \( z' \)

\[ (z' \text{ is a function of } x) \]

Vertical Plume Rise, \( \Delta h \)

Higher wind speeds in stable/neutral air

Low wind speeds in stable air
Near the stack (x < x_{final}), the plume trajectory is dominated by buoyant rise as shown in the equation below. The final plume rise, Δh, is a complex function of wind speed and stability and applies when x > x_{final}.

\[ z' = 1.6 \frac{3 \sqrt[3]{F_0 x^2}}{u} \]

\( F_0 \) is the initial buoyancy flux at the stack exit \([m^4 s^{-3}]\). The larger the stack radius, the more time it takes for the gas to cool by mixing with surrounding, cooler air.

There is also some rise due to momentum (stack gas velocity), but this is usually small compared with the effect of buoyancy.

\[ F_0 = \frac{g}{T} (T_0 - T) w_0 (R_0)^2 \]

\( T_0 \) = Stack gas exit temperature (K)
\( w_0 \) = Stack gas exit velocity (m/s)
\( R_0 \) = Stack radius at exit (m)
Dimensionless downwind parameter \((x/u)^{0.5}\)

Dimensionless plume rise parameter \((\Delta h/u s^{0.333} F_0^{0.333})\)

Observed Plume Trajectories

Source: Simon and Proudfit (1967); data from Ravenswood power plant, New York

Vertical Plume Rise Under Stable Conditions

\[ \Delta h = \frac{4(F_0)^{0.25}}{s^{0.375}} \]

Where \(s\) is the stability parameter, a continuous descriptor of the strength of the atmospheric restoring force under stable conditions

\[ s = \frac{g}{T} \left( 0.0098 - \frac{d\Theta}{dz} \right) \]
A virtual” source can be used to approximate the effect of plume rise on downwind concentrations.

Near the source, the rising plume entrains surrounding air as it rises. This dilution of the plume is not accounted for in the classic plume dispersion equations. To account for this extra dilution, an additional term is added to the “plume sigmas” (plume spreading parameters) that is a function of plume trajectory as follows:

\[
\left( \sigma^2_z \right)_{\text{effective}} = \left( \frac{z'}{3.5} \right)^2 + \sigma^2_z
\]

\[
\left( \sigma^2_y \right)_{\text{effective}} = \left( \frac{z'}{3.5} \right)^2 + \sigma^2_y
\]
Stack-Induced Plume Downwash

Wind flowing past a stack can create a region of lower pressure immediately downwind of the stack. If the vertical momentum of the stack gas is not sufficient, the plume will be drawn downward on the downwind side of the stack, lowering the effective stack height, h.

For \( w_0 < 1.5 \ u \)

\[
h_s' = h_s + 2D_0 \left( \frac{w_0}{u} - 1.5 \right)
\]

For \( w_0 \geq 1.5 \ u \)

\( h_s' = h_s \)

Where \( h_s' \) is the adjusted physical stack height (not including plume rise).