Analyzing the Quantum Limits to Magnetic Resonance Microscopy: Insights from Claude Shannon, John von Neumann, and Richard Feynman

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Modern quantum information theory lends itself to descriptions of magnetic resonance microscopy in which microscopes act as receivers of information and sample spins act as transmitters. This formalism encourages questions like "What fundamental limits does quantum measurement theory impose upon the rate at which a single proton can transmit information to a single electron, via their dipole-dipole coupling, as a function of their separation distance?" Expressions for this limiting rate (known as the channel capacity) are derived for several forms of magnetic resonance microscopy. A canonical result is that a single proton can magnetically transmit about two hundred bits of information per second to a single continuously observed electron at a distance of one hundred nanometers. These expressions blend classic results of Claude Shannon in information theory, John von Neumann in measurement theory, and Richard Feynman in quantum simulation theory, and a brief historical overview of their contributions is provided. Some of the prospects and challenges of achieving atomic-resolution biomicroscopy by magnetic resonance are summarized.

John von Neumann and Richard Feynman both were greatly interested in quantum measurement theory and in atomic-resolution microscopy [1–3]. Recent advances in quantum information theory allow Claude Shannon's classic results in communication theory [4] to be linked to Feynman's and von Neumann's objective of atomic-resolution microscopy, and we show that useful design principles are thereby obtained.

To describe magnetic resonance microscopy in terms of communication theory, we imagine that Alice possesses means of continuously measuring the instantaneous spin axis of a spin-j_{A} particle (typically a single proton), and applying B-fields to control its orientation (Fig. 1). We further imagine that Bob is similarly equipped with a spin-j_{B} particle.

We model Alice's spin transmitter as a three-axis interferometer that provides a continuous (noisy) measurement $\hat{\boldsymbol{x}}^{\mathrm{M}}_{\mathrm{A}}(t)$ of her spin axis $\hat{\boldsymbol{x}}_{\mathrm{A}}(t)$. Then from standard quantum measurement theory—originating largely with von Neumann and Feynman—Alice's transmitter-spin dynamics are

$$\frac{\partial \hat{\boldsymbol{x}}_{\mathrm{A}}}{\partial t} = \frac{[\boldsymbol{I} - \hat{\boldsymbol{x}}_{\mathrm{A}} \otimes \hat{\boldsymbol{x}}_{\mathrm{A}}] \cdot (\hat{\boldsymbol{x}}^{\mathrm{M}}_{\mathrm{A}} - \hat{\boldsymbol{x}}_{\mathrm{A}})}{2 j_{\mathrm{A}} S_{x_{\mathrm{A}}}} - \gamma_{\mathrm{A}} \boldsymbol{B}_{\mathrm{A}} \times \hat{\boldsymbol{x}}_{\mathrm{A}}, \quad (1)$$

where $\boldsymbol{B}_{\mathrm{A}}(t)$ is the magnetic field seen by Alice's spin (including feedback control) and $S_{x_{\mathrm{A}}}$ is Alice's measurement noise (two-sided PSD). Bob's receiver-spin is described by $\mathrm{A} \to \mathrm{B}$.

Bob can linearize his receiver-spin's response by feedback, in which case Shannon's communication theory applies. In particular, Shannon's waterfilling theorem [4] yields the following channel capacity for Alice-to-Bob transmission:

$$\begin{aligned} C_{\mathrm{A}\to\mathrm{B}} &= 0.476 \cdot \gamma_{\mathrm{B}} B_{\mathrm{sig}} \sqrt{j_{\mathrm{B}}} && (2a) \\ \omega^{\lim}_{A}/(2\pi) &= 0.569 \cdot C_{\mathrm{A}\to\mathrm{B}} && (2b) \\ S_{x_{\mathrm{B}}} &= 0.456 \cdot (j_{\mathrm{B}} C_{\mathrm{A}\to\mathrm{B}})^{-1} && (2c) \end{aligned}$$

Here $C_{\mathrm{A}\to\mathrm{B}}$ is the maximal (classical) bits-per-second that Alice can transmit to Bob, B_{sig} is the rms B-field generated at Bob's spin by Alice's spin, γ_{B} (rad/s) is the gyromagnetic ratio of Bob's spin, $\omega^{\lim}_{A}$ (rad/s) is Alice's waterfilling bandwidth, and $S_{x_{\mathrm{B}}}$ is Bob's optimal measurement noise.

These quantum limits have considerable generality, in that (by appropriate scaling) they encompass multiple schemes for atomic-resolution magnetic resonance microscopy, including E-center microscopy and magnetic resonance force microscopy (MRFM), and also more speculative schemes such as ferromagnetic resonance microscopy (FMRM). There is a notable convergence among these methods: they all exhibit a single-proton channel capacity on the order of 10^2–10^4 bits-per-second over a distance of order one hundred nanometers.

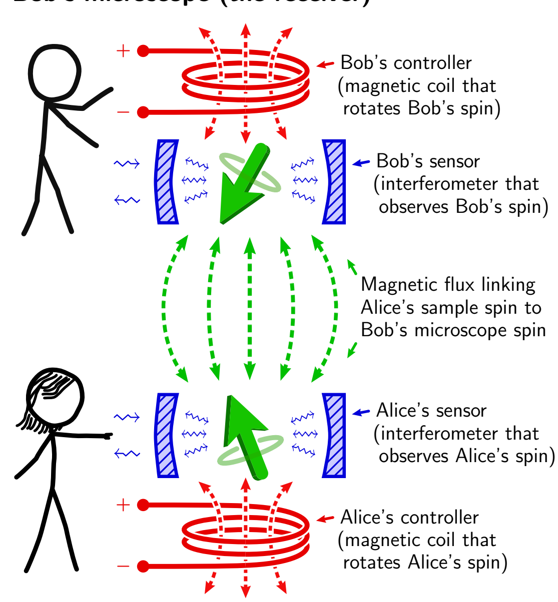

FIG. 1: Magnetic resonance microscopy regarded as an information channel. Alice modulates the orientation of her sample-spin (bottom) to transmit information to Bob's receiver-spon (top) [5].

This result suggests that a long-held dream of Feynman and von Neumann—"to see the individual atoms" in biological molecules—may be achievable by several approaches.

The concluding portion of the lecture will discuss how the same advances in quantum information science that lead us to consider microscopy as a problem in communication, also provide us with powerful new tools for simulating the spin physics in our samples, allowing microscopes and experiments to be designed with reduced risk and at a faster pace.

[1] R. P. Feynman, *There's plenty of room at the bottom*, J. Microelectromechanical Sys. **1**, 60 (1992), (1959 APS Lecture).
[2] R. P. Feynman, *Simulating physics with computers*, Int. J. Theor. Phys. **21**, 467 (1982).
[3] J. von Neumann, in *Proc. Norbert Wiener Centenary Congress, 1994*, (AMS, 1997), vol. 52 of *Proc. Symp. Appl. Math.*, pp. 506–512, a letter from John von Neumann to Norbert Weiner.
[4] C. Shannon, *Communication in the presence of noise*, Proc. Inst. Radio Eng. **37**, 10 (1949). The numerical coefficients in (2a–c) maximize the waterfilling integral of Shannon's eqs. 32–4 and fig. 8.
[5] The Alice and Bob stick-figures are by permission of `xkcd.com`.