Induction

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The principle of mathematical induction

The principle of mathematical induction is an axiom of mathematics used to prove that some statement or propositional function is true on some countable sequence. Countable means there is a one-to-one correspondence with the positive integers, i.e., you can count them. A simple form can be stated as follows:

> Suppose we have a statement or propositional function P(n) that is either true or false on some countable sequence, n. The statement is true for all values of n in the sequence, if:
> 1. The statement P(1) is true. (Show true on the first value in the sequence. The base case)
> 2. Assume P(n) is true for every n up through some finite value, say k. (The induction hypothesis)
> 3. Then show P(k + 1) is true. (Show it is true on k+1. The induction step)

It's the domino principle. If you show it is true for the first one, then you can use that fact to show that it is true for next one. Now show, using that it is true on the first two, that it is true on the third one. If you do a general argument that shows you can go from anyone to the next one, only using that it is true on the previous ones, then it must be true for all of them. It's like recursion, backwards. In recursion, you break it down until you get to the base case. In induction, you start at the beginning and continually build up.

You typically do induction on countable sets such as the **nonnegative integers** (or an integer sequence starting at any number), **characters in a string**, **nodes in a graph**, the **number of bits**, etc.

Example 1

Prove the sum of the first n odd natural numbers is n^2 .

In other words, show P(n) = $\sum_{i=1}^{n} (2i-1) = n^2$ for all n ≥ 1 .

Recall that even integers are expressed as 2*i . Odd numbers are expressed either as 2i+1 or 2i-1, depending on where i starts. We use 2i -1 so we can start the summation at 1 .

Proof: By induction on n.

Base case: Show that P(1) is true, that is, show it is true when n = 1 .

Lefthand side (LHS) = $\sum_{i=1}^{1} (2i-1) = 2*1-1 = 2-1 = 1 = 1^2$ = Righthand side (RHS)

Induction hypothesis: Assume that P(n) is true up through some k≥1, P(k) is true, that is, $\sum_{i=1}^{k} (2i-1) = k^2$

Induction step: Show that P(k+1) is true, that is, show $\sum_{i=1}^{k+1} (2i-1) = (k+1)^2$.

Proof: LHS = $\sum_{i=1}^{k+1} (2i-1) = \left(\sum_{i=1}^{k} (2i-1)\right) + 2(k+1)-1$ Break last term out of sum

$= k^2 + 2(k+1)-1$ By the induction hypothesis

$= k^2 + 2k + 2 - 1$ Algebra

$= k^2 + 2k + 1$ Algebra

$= (k+1)^2$

Therefore, P(n) is true for all n ≥ 1 .

Example 2

Prove n! > n^2 for all n ≥ 4.

Proof: By induction on n.

Base case: Show that P(4) is true, that is, show it is true when n = 4 .

LHS = 4! = 24
RHS = 4^2 = 16
Since 24 > 16, P(4) is true.

Induction hypothesis: Assume that P(n) is true up through some value k≥4, P(k) is true, that is, k! > k^2

Induction step: Show that P(k+1) is true, that is, show (k+1)! > $(k+1)^2$

Proof: LHS = (k+1)! = (k+1) · k!	By definition of factorial
> (k+1) · k^2	By the induction hypothesis
> (k+1) · (k+1)	By your homework showing k^2 > k+1
= $(k+1)^2$	

Therefore, P(n) is true for all n ≥ 4 .

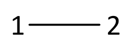
Notice the technique used in Example 2. It would be a straightforward proof if we knew that k^2 > k +1 .
It seems clear that it is true, so we do another proof by induction to prove it (in your homework).

Example 3

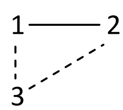
Prove that a graph with n nodes with one edge from each node to every other node (no edge to itself) has a total of n(n-1)/2 edges.

My thinking for the induction step:

two nodes:

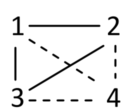


three nodes:



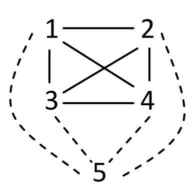
Dotted lines show edges added to go from 2 nodes to 3 nodes

four nodes:



Dotted lines show edges added to go from 3 nodes to 4 nodes

five nodes:



Dotted lines show edges added to go from 4 nodes to 5 nodes

Proof: By induction on the number of nodes, n.

Base case: Show that P(0) is true, that is, show it is true when n = 0 .

If n=0, then n(n-1)/2 = 0(0-1)/2 = 0. Since a graph with no nodes has zero edges, P(0) is true.

Induction hypothesis: Assume that P(n) is true up through some value k≥0, P(k) is true, that is, a graph with k nodes with one edge from each node to every other node has a total of k(k-1)/2 edges.

Induction step: Show that P(k+1) is true, that is, show that a graph with k+1 nodes has a total of (k+1)(k+1-1)/2 = k(k+1)/2 edges.

Proof: Consider a graph with k nodes. By the induction hypothesis, it has k(k-1)/2 edges. Add another node to the graph, say node x. Now the graph has k+1 nodes. The additional edges that must be added to the graph so that there is one edge from each node to every other node, go from node x to each of the original k nodes. There are k new edges added to the graph.

The total number of edges in the graph with k+1 nodes is k(k-1)/2 (the original edges from the k nodes) plus the k new edges that go from node x to the original k nodes. So the total is

	$k(k-1)/2 + k$	
=	$k(k-1)/2 + 2k/2$	Get common denominator for second term
=	$(k(k-1) + 2k)/2$	Make it one fraction, all over 2
=	$(k^2 - k + 2k)/2$	Multiply out
=	$(k^2 + k)/2$	Algebra
=	$k(k + 1)/2$	Factor

Therefore, P(n) is true for all n ≥ 0.

Example 4

Show that postage of ≥ 4 cents can be achieved by using only 2-cent and 5-cent stamps.

My thinking:

# cents:	4	5	6	7	8	9	10	10	11	12	12
2-cent:	2		3	1	4	2		5	3	1	6
5-cent:		1		1		1	2		1	2	

Proof: By induction on the number of cents.

Base case: Show that P(4) is true, that is, show it is true when n = 4 .

Since 4 = 2 + 2, you can use two 2-cent stamps.

Induction hypothesis: Assume that P(n) is true up through some value k≥4, P(k) is true, that is, you can form k cents from only 2-cent and 5-cent stamps.

Induction step: Show that P(k+1) is true, that is, show you can form k+1 cents from only 2-cent and 5-cent stamps.

Proof: This is a constructive proof showing how k+1 cents can be formed in postage. By the inductive hypothesis, k cents can be formed using only 2-cent and 5-cent stamps. So, either at least one 5-cent stamp is used, or all 2-cent stamps are used.

Case 1: k is an odd number: At least one 5-cent stamp is used to achieve k cents. Replace this 5-cent stamp with three 2-cent stamps. This will account for the extra cent of the k+1 cents.

Case 2: k is an even number: All 2-cent stamps can be used to achieve k cents. Since n ≥ 4, there are at least two 2-cent stamps. Replace these two stamps with one 5-cent stamp. This accounts for the extra cent.

Therefore, P(n) is true for all n ≥ 4 since in all cases, n+1 cents can be formed with 2- and 5-cent stamps.