Sequences and Sums (Rosen, 6th edition, Sections 2.4) Carol Zander

This section is covered as a review and is not primary. In other words, you need to know this material to use in algorithm analysis.

Sequences are lists of elements. The book defines them as ordered, but often when the term sequence is used outside of discrete math, it means a sorted or unsorted list.

Definitions

Sequence is a function from a subset of the set of integers (either positive, nonnegative, or all integers) to a set S.

For example:

Positive integers, $a_n = n$ (start at 1): 1, 2, 3, ...
Positive even integers, $a_n = 2n$ (start at 0): 0, 2, 4, 6, ...
Squares, $a_n = n^2$ (start at 1): 1, 4, 9, 16, 25, 36, ...

Geometric progression is a sequence of the form:

$a, ar, ar^2, ar^3, ar^4, \ldots$

where a and the ratio r are real numbers.

For example:

$a = 2^0 = 1, r = 2$
Powers of two, $a_n = 2^n$ (start at 0): 1, 2, 4, 8, 16, 32, 64, ...

Arithmetic progression is a sequence of the form

a, a+d, a+2d, a+3d, ..., a+nd, ...

where a and the common difference d are real numbers.

For example:

a = 0, d = 10
0, 10, 20, 30, ...

a = 1, d = 10
1, 1+10, 1+20, 1+30, ... = 1, 11, 21, 31, ...

Summations

Summation notation is shorthand notation for a sum:

$a_k + a_{k+1} + a_{k+2} + \ldots + a_n$

where k, n are integers, $k \le n$.

The summation notation used a capital sigma:

```
 n
---
\
/   a_i
---
i=k
```

$$\sum_{i=k}^{n} a_i$$

The value k is called the lower limit, n is the upper limit. The sum is always incremented by one.

For example:

$$\sum_{i=1}^{n} i = 1 + 2 + 3 + ... + (n-1) + n$$

Properties of summations

Summations can be broken up across a sum or difference:

$$\sum_{i=k}^{n} (x_i + y_i) = \sum_{i=k}^{n} x_i + \sum_{i=k}^{n} y_i$$

For example:

$$\sum_{i=1}^{n} (2i + 10) = \sum_{i=1}^{n} 2i + \sum_{i=1}^{n} 10$$

```
From basic arithmetic, 2*1 + 10 + 2*2 + 10 + 2*3 + 10 + ... + 2*n + 10
is the same as
                       2*1 + 2*2 + 2*3 + ... + 2*n + 10 + 10 + 10 + 10 + ... + 10
```

You can factor out constants:

$$\sum_{i=k}^{n} c* x_i = c * \sum_{i=k}^{n} x_i$$

For example:

$$\sum_{i=1}^{n} 5i = 5 \sum_{i=1}^{n} i$$

```
From basic arithmetic,  5*1 + 5*2 + 5*3 + 5*4 + 5*5 + ... + 5*n
is the same as
                        5(1 + 2 + 3 + 4 + 5 + ... + n)
```

Nested summations are evaluated as in algebra, from inside out.
In the context of the inner summation, j is the variable, but i is constant.

```
 n      n          n
---    ---        ---
\      \          \
/      /   i*j = /   (i + 2i + 3i + ... + ni)
---    ---        ---
i=1    j=1        i=1

                 =   1 + 2*1 + 3*1 + ... + n*1
                   + 2 + 2*2 + 3*2 + ... + n*2
                   + 3 + 2*3 + 3*3 + ... + n*3
                   + ...
                   + n + 2*n + 3*n + ... + n*n
```

Two well-known summations frequently used in computer science are

```
 n
---
\
/   i = 1 + 2 + 3 + ... + (n-1) + n = n(n+1)
---                                   ------
i=1                                     2

 n
---
\    i                                  n   n+1
/   2  = 1 + 2 + 4 + 8 + 16 + 32 + ... + 2 = 2    - 1
---
i=0
```

Table 2 from the text includes these sums and has other useful summation formulas.

TABLE 2 Some Useful Summation Formulae.

Sum	*Closed Form*
$\sum_{k=0}^{n} ar^k \ (r \neq 0)$	$\frac{ar^{n+1} - a}{r - 1}, r \neq 1$
$\sum_{k=1}^{n} k$	$\frac{n(n+1)}{2}$
$\sum_{k=1}^{n} k^2$	$\frac{n(n+1)(2n+1)}{6}$
$\sum_{k=1}^{n} k^3$	$\frac{n^2(n+1)^2}{4}$
$\sum_{k=0}^{\infty} x^k, \lvert x \rvert < 1$	$\frac{1}{1-x}$
$\sum_{k=1}^{\infty}, kx^{k-1}, \lvert x \rvert < 1$	$\frac{1}{(1-x)^2}$