Digital Communication Systems Engineering with Software-Defined Radio

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Lecture 10

Mathematical Framework

- Suppose that a signal $s_i(t)$, $i = 1, 2$, was transmitted across an AWGN channel with noise signal $n(t)$
- A receiver intercepts $r(t)$ and needs to determine whether $s_1(t)$ or $s_2(t)$ was sent
- This situation can be evaluated using *Hypothesis Theory*:

$$\begin{aligned} \mathcal{H}_1 &: r(t) = s_1(t) + n(t), \ 0 \leq t \leq T \\ \mathcal{H}_0 &: r(t) = s_2(t) + n(t), \ 0 \leq t \leq T \end{aligned}$$

where $\mathcal{H}_0$ and $\mathcal{H}_1$ are "Hypothesis 0" and "Hypothesis 1"

Decision Rule

- Suppose that $s_1(t)$ was transmitted
- In general, we can determine the level of correlation between two signals $x(t)$ and $y(t)$ over the time interval $0 \leq t \leq T$ using the expression:

$$\int_0^T x(t)y(t)dt$$

- Consequently, our decision rule on whether $s_1(t)$ or $s_2(t)$ was transmitted given that we observe $r(t)$ is defined as:

$$\int_0^T r(t)s_1(t)dt \geq \int_0^T r(t)s_2(t)dt \tag{1}$$

  where we assume that $s_1(t)$ was transmitted
  - The received signal $r(t)$ is more correlated to $s_1(t)$ than $s_2(t)$

Error Event

- Assuming $s_1(t)$ was transmitted, an *error event* occurs when:

$$\int_0^T r(t)s_1(t)dt \leq \int_0^T r(t)s_2(t)dt \tag{2}$$

- Since $r(t) = s_1(t) + n(t)$, we can substitute this into the error event in order to obtain:

$$\int_0^T s_1^2(t)dt + \int_0^T n(t)s_1(t)dt \leq \int_0^T s_1(t)s_2(t)dt + \int_0^T n(t)s_2(t)dt$$

$$E_{s_1} - \rho_{12} \leq \int_0^T n(t)(s_2(t) - s_1(t))dt$$

$$E_{s_1} - \rho_{12} \leq z$$

Characterizing z

- Since $n(t)$ is Gaussian $\rightarrow$ z is Gaussian
  - Integration is equivalent to a summation across an infinite number of samples
  - Summing up Gaussian random variables results in a Gaussian random variable
- Since $z \sim \mathcal{N}(0, \sigma^2)$, we need to calculate σ^2:

$$\sigma^2 = E\{z^2\} = \frac{N_0}{2} \int_0^T (s_1(t) - s_2(t))^2 dt$$

$$= \frac{N_0}{2}(E_{s_1} + E_{s_2} - 2\rho_{12}) \rightarrow \text{Assume } E_{s_1} = E_{s_2} = E$$

$$= N_0(E - \rho_{12})$$

where $E = E_i = \int_0^T s_i^2(t)dt$ and $\rho_{12} = \int_0^T s_1(t)s_2(t)dt$

Solving for the Probability of Error

- The probability of an error occurring given that a "1" was transmitted, i.e., $P(e|1)$, is equal to:

$$P(z \geq E - \rho_{12}) = Q\left(\frac{E - \rho_{12}}{\sigma}\right) \rightarrow \text{Since } z \sim \mathcal{N}(0, \sigma^2) \text{ and } E - \rho_{12} \text{ is constant}$$

$$= Q\left(\sqrt{\frac{(E - \rho_{12})^2}{\sigma^2}}\right) \rightarrow \text{Use } \sigma^2 = N_0(E - \rho_{12})$$

$$= Q\left(\sqrt{\frac{E - \rho_{12}}{N_0}}\right)$$

where the Q-function is defined as:

$$Q(x) = \frac{1}{\sqrt{2\pi}} \int_x^{\infty} e^{-t^2/2} dt \tag{3}$$

Optimizing the Probability of Error

- To minimize $P(e|1)$, we need to optimize ρ_{12}
  - Intuitively, the best choice we can make is $s_2(t) = -s_1(t)$, which gives us $\rho_{12} = -E$
  - This yields $P(e|1) = Q\left(\sqrt{\frac{2\bar{E}_b}{N_0}}\right)$
- As an exercise for the student, show that the total probability of error is equal to:

$$P_e = P(e|1)P(1) + P(e|0)P(0) = Q\left(\sqrt{\frac{E-\rho_{12}}{N_0}}\right) \quad (4)$$

- When $E_{s_1} \neq E_{s_2}$, then $d_{\min}^2 = E_{s_1} + E_{s_2} - 2\rho_{12}$, which yields:

$$P_e = Q\left(\sqrt{\frac{d_{\min}^2}{2N_0}}\right) \quad (5)$$

Signal Vectors

- Let $\phi_j(t)$ be an orthnormal set of functions on the time interval $[0, T]$ such that:

$$\int_0^T \phi_i(t)\phi_j(t)dt = \begin{cases} 1 & i = j \\ 0 & \text{otherwise} \end{cases}$$

- Let $s_i(t)$ be the modulation signal that we can represent in terms of the orthonormal functions:

$$s_i(t) = \sum_{k=1}^{N} s_{ik}\phi_k(t) \tag{6}$$

which can be represented by the vector:

$$s_i(t) \rightarrow \mathbf{s}_i = (s_{i1},\ s_{i2},\ s_{i3}, \ldots s_{iN}) \tag{7}$$

Vector Representation of $s_i(t)$

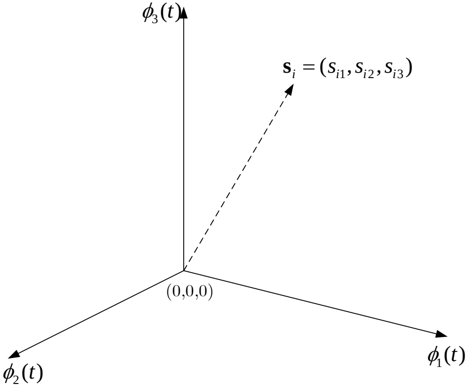

Figure : Sample Vector Representation of $s_i(t)$ in Three Dimensional Space using Basis Functions $\phi_1(t)$, $\phi_2(t)$, and $\phi_3(t)$.

Vector Manipulations

- To find s_{ik}, solve:

$$\int_0^T s_i(t)\phi_l(t)dt = \sum_{k=1}^{N} s_{ik} \int_0^T \phi_k(t)\phi_l(t)dt = s_{il} \tag{8}$$

- The vector dot product between $s_i(t)$ and $s_j(t)$ is equal to:

$$\int_0^T s_i(t)s_j(t)dt = \mathbf{s}_i \cdot \mathbf{s}_j = \rho_{ij} \tag{9}$$

while the energy of a signal $s_i(t)$ is equal to:

$$E_{s_i} = \int_0^T s_i^2(t)dt = \mathbf{s}_i \cdot \mathbf{s}_i = ||\mathbf{s}_i||^2 \tag{10}$$

Euclidean Distance

- To compute the *Euclidean Distance* using signal space vectors, we need to solve:

$$\begin{aligned} d_{\min}^2 &= \int_0^T \Delta s_{ij}^2(t)dt = \int_0^T (s_i(t) - s_j(t))^2 dt \\ &= ||\mathbf{s}_i - \mathbf{s}_j||^2 = (\mathbf{s}_i - \mathbf{s}_j) \cdot (\mathbf{s}_i - \mathbf{s}_j) \\ &= E_{s_i} + E_{s_j} - 2\rho_{ij} \end{aligned}$$

where:

$$\rho_{ij} = \int_0^T s_i(t)s_j(t)dt = \mathbf{s}_i \cdot \mathbf{s}_j \qquad (11)$$

Solving for the Power Efficiency

- Choose orthonormal basis functions $\phi_i(t)$, $i = 1, 2, \ldots, k$, where k is the dimension of the signal space
- Find $\mathbf{s}_i$, $i = 1, 2, \ldots, M$ where $\mathbf{s}_i = (s_{i1},\ s_{i2}, \ldots s_{ik})$ and $s_{ij} = \int_0^T s_i(t)\phi_j(t)dt$
- Consequently, solve for:

$$d^2_{\min} = \min_{i \neq j} ||\mathbf{s}_i - \mathbf{s}_j||^2$$

$$\bar{E}_s = \frac{1}{M} \sum_{i=1}^{M} ||\mathbf{s}_i||^2$$

$$\bar{E}_b = \bar{E}_s / \log_2(M)$$

$$\varepsilon_p = d^2_{\min} / \bar{E}_b$$