Digital Communication Systems Engineering with Software-Defined Radio

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Lecture 11

Recall Simple Digital Transceiver Model

- Receiver only observes the corrupted version of $s(t)$ by $n(t)$, namely $r(t)$
- Usually $n(t)$ represents the culmination of all noise sources into a single variable
- Detection problem: Given $r(t)$ for $0 \leq t \leq T$, determine which $s_i(t)$, $i = 1, 2, \ldots, M$, is present in $r(t)$

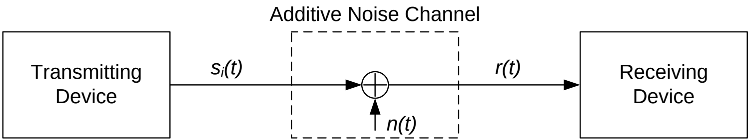

Figure : Simple Digital Transceiver Model.

Mathematical Formulation

- Decompose waveforms $s_i(t)$, $n(t)$, and $r(t)$ into a collection of weights applied to a set of orthonormal basis functions:

$$s_i(t) = \sum_{k=1}^{N} s_{ik}\phi_k(t), \quad r(t) = \sum_{k=1}^{N} r_k\phi_k(t), \quad n(t) = \sum_{k=1}^{N} n_k\phi_k(t)$$

- Thus, waveform model $r(t) = s_i(t) + n(t)$ now becomes

$$r(t) = s_i(t) + n(t)$$

$$\sum_{k=1}^{N} r_k\phi_k(t) = \sum_{k=1}^{N} s_{ik}\phi_k(t) + \sum_{k=1}^{N} n_k\phi_k(t)$$

$$\mathbf{r} = \mathbf{s}_i + \mathbf{n} \rightarrow \text{Vector Model}$$

$n(t)$ is Gaussian

- We know that the noise vector element n_k is equal to:

$$n_k = \int_0^T n(t)\phi_k(t)dt \tag{1}$$

- Since $n(t)$ is Gaussian and integration is a linear operation, then n_k is Gaussian as well
  - **n** is a Gaussian vector
- We need to determine the statistical characteristics of **n** in order to employ this knowledge in signal waveform detection

Calculating the Mean

- Applying the definition for the expectation:

$$
\begin{aligned}
E\{n_k\} &= E\left\{\int_0^T n(t)\phi_k(t)dt\right\} \\
&= \int_0^T E\{n(t)\}\phi_k(t)dt \\
&= 0 \text{ since } E\{n(t)\} = 0 \\
&\therefore E\{\mathbf{n}\} = \mathbf{0}
\end{aligned}
$$

Calculating the Variance

- Let $(\mathbf{nn}^T)_{kl} = n_k n_l$ be equal to the $(k,l)^{\text{th}}$ element of $\mathbf{nn}^T$
- Determine $E\{n_k n_l\}$, where:

$$n_k = \int_0^T n(t)\phi_k(t)dt, \quad n_l = \int_0^T n(\rho)\phi_l(\rho)d\rho$$

- Applying the definition for $E\{n_k n_l\}$ yields:

$$\begin{aligned} E\{n_k n_l\} &= E\left\{\left(\int_0^T n(t)\phi_k(t)dt\right)\left(\int_0^T n(\rho)\phi_l(\rho)d\rho\right)\right\} \\ &= E\left\{\int_0^T\int_0^T n(t)n(\rho)\phi_k(t)\phi_l(t)dtd\rho\right\} \end{aligned}$$

Solving $E\{n_k n_l\}$

$$E\{n_k n_l\} = \int_0^T \int_0^T E\{n(t)n(\rho)\}\phi_k(t)\phi_l(t)dtd\rho$$

$$= \int_0^T \int_0^T \frac{N_0}{2}\delta(t-\rho)\phi_k(t)\phi_l(t)dtd\rho \rightarrow \text{AWGN channel}$$

$$= \frac{N_0}{2}\int_0^T \phi_k(t)\phi_l(t)dt$$

$$= \frac{N_0}{2}\delta(k-l) \rightarrow \text{orthonormal functions } \phi_k(t) \text{ and } \phi_l(t)$$

$$\therefore \text{the matrix equivalent is } E\{\mathbf{n}\mathbf{n}^T\} = \frac{N_0}{2}\mathbf{I}_{N\times N}$$

Noise Properties

- Only for Gaussian random variables does *uncorrelated* implies *independence*
- By *central limit theorem*, if we sum up the outputs of several random variables possessing the same probability characteristics, they will yield a Gaussian distribution
  - $n(t)$ is usually composed of many individual sources
  - Superposition of these sources will yield a Gaussian distribution
  - Modeling $n(t)$ closely matches communication channel noise in several scenarios

Defining the Probability Density Function

- Given a vector of Gaussian random variables, we define the joint probability density function as:

$$p(\mathbf{n}) = p(n_1, n_2, \ldots, n_N) = \frac{1}{(2\pi\sigma^2)^{N/2}} \prod_{i=1}^{N} e^{-n_i^2/2\sigma^2}$$
$$= p(n_1)p(n_2)\ldots p(n_N)$$

  where $p(n_i) = \frac{1}{\sigma\sqrt{2\pi}} e^{-n_i^2/2\sigma^2}$
- Since $E\{n_k n_l\} = \frac{N_0}{2}\delta(k-l)$, then $E\{n_k^2\} = \frac{N_0}{2} = \sigma^2$
- Defining $\sum_{i=1}^{N} n_i^2 = ||\mathbf{n}||^2$ yields the following expression:

$$p(\mathbf{n}) = p(n_1, n_2, \ldots, n_N) = \frac{1}{(2\pi\sigma^2)^{N/2}} e^{-||\mathbf{n}||^2/2\sigma^2}$$

Probability of Correct Detection

- Our criterion for the receiver is:

$$\begin{aligned} \text{Minimize } P(\text{error}) &\rightarrow P(\hat{m}_i \neq m_i) \\ \text{Maximize } P(\text{correct}) &\rightarrow P(\hat{m}_i = m_i) \end{aligned} \tag{2}$$

  where $P(e) = P(\text{error})$, $P(c) = P(\text{correct})$, and $P(e) = 1 - P(c)$

- The overall probability of correct detection is equal to:

$$P(c) = \int_V P(c|\mathbf{r} = \rho)p(\rho)d\rho \tag{3}$$

  where $P(c|\mathbf{r} = \rho) \geq 0$ and $p(\rho) \geq 0$
  - Therefore $P(c)$ is maximum when $P(c|\mathbf{r} = \rho)$ is maximum

Decision Rule Formulation

- To maximize $P(c|\mathbf{r} = \rho)$, we use the *decision rule*:

$$P(\mathbf{s}_k|\rho) \geq P(\mathbf{s}_i|\rho), \ \text{for } i = 1, 2, \ldots, M \text{ and } i \neq k \tag{4}$$

  for $i = 1, 2, \ldots, M$ and $i \neq k$

- Declare $\mathbf{s}_k$ as present in ρ:

$$\rho = \mathbf{s}_k + \mathbf{n} \rightarrow \hat{m} = m_k \tag{5}$$

- Employ a mixed form of *Bayes Rule* that is composed of probability density functions and probabilities, namely:

$$P(\mathbf{s}_i|\mathbf{r} = \rho) = \frac{p(\rho|\mathbf{s}_i)P(\mathbf{s}_i)}{p(\rho)} \tag{6}$$

Optimal Detector

- Using the mixed form of Bayes Rule, and recalling how we want to maximize $P(c|\mathbf{r}=\rho)$, the optimal detector is equal to:

$$\max_{\mathbf{s}_i} P(\mathbf{s}_i|\mathbf{r}=\rho) = \max_{\mathbf{s}_i} \frac{p(\rho|\mathbf{s}_i)P(\mathbf{s}_i)}{p(\rho)} \tag{7}$$

  for $i = 1, 2, \ldots, M$

- Since $p(\rho)$ does not depend on $\mathbf{s}_i$, we can simply the optimal detector to:

$$\max_{\mathbf{s}_i} p(\rho|\mathbf{s}_i)P(\mathbf{s}_i) \tag{8}$$

  for $i = 1, 2, \ldots, M$

MAP and ML Detectors

- A *maximum a posteriori* (MAP) detector is equal to:

$$P(\mathbf{s}_i|\mathbf{r}=\rho) = \max_{\mathbf{s}_i} p(\rho|\mathbf{s}_i)P(\mathbf{s}_i) \quad (9)$$

  for $i = 1, 2, \ldots, M$
- A *maximum likelihood* (ML) detector is defined as:

$$P(\mathbf{s}_i|\mathbf{r}=\rho) = \max_{\mathbf{s}_i} p(\rho|\mathbf{s}_i) \quad (10)$$

  for $i = 1, 2, \ldots, M$, and assuming $P(\mathbf{s}_i) = \frac{1}{M}$
  - This implies that $P(\mathbf{s}_i)$ does not depend on $\mathbf{s}_i$