Digital Communication Systems Engineering with Software-Defined Radio

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Lecture 14

Basic Problem

- We need some sort of way of detecting a pulse transmitted over a channel corrupted by noise
- Suppose we transmit a pulse signal $g(t)$ through an additive channel with noise $w(t)$, which produces the output $x(t)$:

$$x(t) = g(t) + w(t), \text{ for } 0 \leq t \leq T \tag{1}$$

- The signal $g(t)$ may represent a "1" or a "0" in a digital communication system
- The signal $w(t)$ is a sample function of a white noise process of zero mean and power spectral density of $\frac{N_0}{2}$
- Assume that the receiver has knowledge of all the possible waveforms $g(t)$

Detecting $g(t)$

- The function of the receiver is to detect the pulse signal $g(t)$ in an optimal manner given the received signal $x(t)$
- To achieve this, let us filter $x(t)$ in order to minimize the effects of the noise in some statistical sense
  - We want to enhance the probability of correct detection
- Suppose the output of the filter $h(t)$ is $y(t) = g_0(t) + n(t)$
  - Signal $g_0(t)$ is the filtered signal component and $n(t)$ is the filtered noise component
- Our goal is to maximize $g_0(t)$ with respect to $n(t)$ using peak pulse SNR

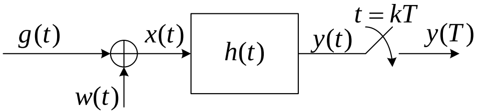


Figure : Filtering Process for Detecting $g(t)$.

Peak Pulse SNR

- Peak pulse SNR is defined as:

$$\eta = \frac{|g_0(T)|^2}{E\{n^2(t)\}} \tag{2}$$

  where $|g_0(T)|^2$ is the instantaneous power of the output signal at sampling instant T, and $E\{n^2(t)\}$ is the average power of the output noise
- How do we achieve the goal of maximizing $g_0(t)$ relative to $n(t)$?
  - Design a filter $h(t)$ that achieves the largest possible η

Designing $h(t)$

- Solving for $h(t)$ consists of evaluating the expression:

$$|g_0(t)|^2 = \left| \int_{-\infty}^{\infty} H(f)G(f)e^{j2\pi ft}df \right|^2, \qquad (3)$$

which is the magnitude squared of the inverse Fourier transform of $H(f)G(f) = \mathcal{F}\{h(t) * g(t)\}$

- Signal $w(t)$ is white with power spectral density $\frac{N_0}{2}$
  - Power spectral density of $n(t)$ is defined via EWK as $S_N(f) = \frac{N_0}{2}|H(f)|^2$
- Recall that the mean-square value of a waveform is equal to:

$$E\{n^2(t)\} = R_N(0) = \mathcal{F}^{-1}\{S_N(f)\}|_{\tau=0} = \frac{N_0}{2}\int_{-\infty}^{\infty}|H(f)|^2df$$

Solving for η

- Apply the definition for η:

$$\eta = \frac{|g_0(T)|^2}{E\{n^2(t)\}} = \frac{\left| \int\limits_{-\infty}^{\infty} H(f)G(f)e^{j2\pi ft}df \right|^2}{\frac{N_0}{2} \int\limits_{-\infty}^{\infty} |H(f)|^2 df} \tag{4}$$

- Now we need to solve for $H(f)$ that yields the largest possible η
  - Use *Schwarz's Inequality* to obtain a closed-form solution

Recall Schwarz's Inequality

- If we have two complex functions, $\phi_1(x)$ and $\phi_2(x)$, such that:

$$\int_{-\infty}^{\infty} |\phi_1(x)|^2 dx < \infty \quad \text{and} \quad \int_{-\infty}^{\infty} |\phi_2(x)|^2 dx < \infty$$

then:

$$\left| \int_{-\infty}^{\infty} \phi_1(x)\phi_2(x)dx \right|^2 \leq \left(\int_{-\infty}^{\infty} |\phi_1(x)|^2 dx \right) \cdot \left(\int_{-\infty}^{\infty} |\phi_1(x)|^2 dx \right)$$

- This becomes an equality when $\phi_1(x) = K \cdot \phi_2^*(x)$

Completing the Derivation

- Applying Schwarz's Inequality to this derivation, we obtain:

$$\left| \int_{-\infty}^{\infty} H(f)G(f)e^{j2\pi ft}df \right|^2 \leq \left(\int_{-\infty}^{\infty} |H(f)|^2 df \right) \cdot \left(\int_{-\infty}^{\infty} |G(f)|^2 df \right)$$

- Consequently, we get: $\eta \leq \frac{2}{N_0} \int_{-\infty}^{\infty} |G(f)|^2 df$
  - Thus, we get $H_{\text{opt}}(f) = K \cdot G^*(f)e^{-j2\pi fT}$
- To determine $H_{\text{opt}}(f)$, solve for the inverse Fourier transform:

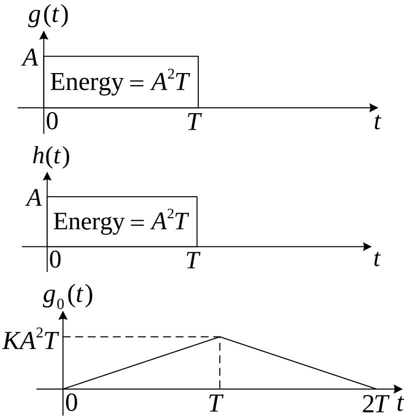
$$h_{\text{opt}}(t) = K \cdot \int_{-\infty}^{\infty} G^*(f)e^{-j2\pi fT}e^{-j2\pi ft}df = K \cdot g(T - t)$$

Why Call It "Matched Filter"?

- We convolve the time-flipped and time-shifted version of the transmitted pulse with the transmitted pulse itself
- We are trying to match the pulses→SNR maximizing!!
- This filter can be implemented using a *tapped delay line*

Matching $h(t)$ with $g(t)$

- Suppose we have a signal $g(t) \rightarrow$ What is $h(t)$?
  - Solve $h(t) = g(T - t)$
- What is the output of $h(t)$?
  - When we convolve $g(t)$ with $h(t)$, we get a triangular pulse



Matched Filter Realization Schematic

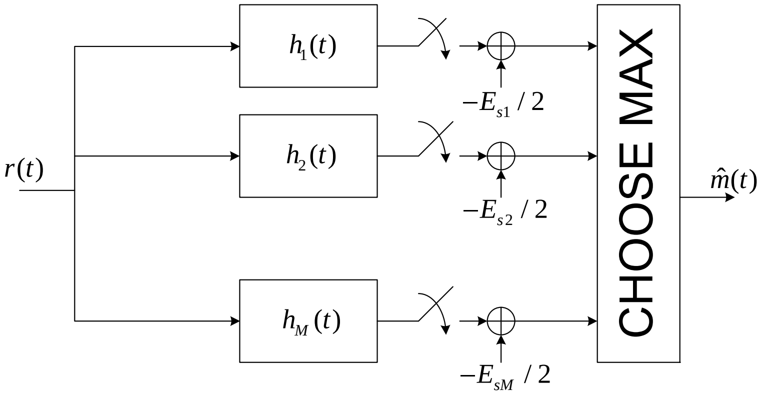


Figure : Schematic of Matched Filter Realization of Receiver Structure.