Digital Communication Systems Engineering with Software-Defined Radio

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Lecture 13

AWGN Vector Channel

- Refer to Lecture 11 regarding optimal detection
  - The optimal detector is equal to:

$$\max_{\mathbf{s}_i} p(\rho|\mathbf{s}_i)P(\mathbf{s}_i), \text{ for } i = 1, 2, \ldots, M \tag{1}$$

  - When $P(\mathbf{s}_i) = \frac{1}{M}$ for all i, we get:

$$\max_{\mathbf{s}_i} p(\rho|\mathbf{s}_i), \text{ for } i = 1, 2, \ldots, M \tag{2}$$

- The received signal is equal to:

$$\mathbf{r} = \mathbf{s}_i + \mathbf{n} \tag{3}$$

and if we isolate for $\mathbf{n}$, we get:

$$\mathbf{n} = \mathbf{r} - \mathbf{s}_i \tag{4}$$

Manipulating the Gaussian Noise PDF

- Recall that the joint probability density function of the noise vector $\mathbf{N}$ is equal to:

$$p(\mathbf{n}) = p(n_1, n_2, \ldots, n_N) = \frac{1}{(2\pi\sigma^2)^{N/2}} e^{-||\mathbf{n}||^2/2\sigma^2} \quad (5)$$

- We can easily show that the conditional probability density function of ρ given $\mathbf{s}_i$ is equal to:

$$\begin{aligned} p(\rho|\mathbf{s}_i) &= p(\rho - \mathbf{s}_i) = p(\mathbf{n}) \\ &= \frac{1}{(2\pi\sigma^2)^{N/2}} e^{-||\rho - \mathbf{s}_i||^2/2\sigma^2} \end{aligned} \quad (6)$$

  - Vector $\mathbf{s}_i$ is a fixed quantity that is known
  - Vector ρ is simply the addition of $\mathbf{s}_i$ with a Gaussian noise vector $\mathbf{n}$

Uncorrelated Gaussian Noise Vector

- Suppose we consider a single element of these vectors, say the k^{th} element:

$$p(\rho_k|s_{ik}) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(\rho_k - s_{ik})^2/2\sigma^2} \tag{7}$$

- Since we assume that the AWGN vector elements are uncorrelated (i.e., independent), we have:

$$p(\rho|\mathbf{s}_i) = \prod_{k=1}^{N} p(\rho_k|s_{ik}), \text{ for } i = 1, 2, \ldots, M \tag{8}$$

- Consequently, this product of elemental probability density functions will yield:

$$p(\rho|\mathbf{s}_i) = \frac{1}{(2\pi\sigma^2)^{N/2}} e^{-||\rho - \mathbf{s}_i||^2/2\sigma^2} \tag{9}$$

Maximum Likelihood Detector

- For the maximum likelihood detector case, we want $\max_{\mathbf{s}_i} p(\rho|\mathbf{s}_i)$
- Suppose we take the expression for $p(\rho|\mathbf{s}_i)$, apply it to the ML detector, and take the natural logarithm, yielding:

$$\ln(p(\rho|\mathbf{s}_i)) = \frac{N}{2}\ln\left(\frac{1}{2\pi\sigma^2}\right) - \frac{||\rho - \mathbf{s}_i||^2}{2\sigma^2} \tag{10}$$

  - We take the natural logarithm in order to get rid of the exponential base in the expression
  - Results in a linear expression of the optimal decision rule
  - Natural logarithms are monotonic functions such that if $x_2 \geq x_1$ then $\ln(x_2) \geq \ln(x_1)$

Solving for a Linear Decision Rule

- Given the monotonic behavior of the natural logarithm:

$$\begin{aligned}\max_{\mathbf{s}_i} \ln(p(\rho|\mathbf{s}_i)) &= \max_{\mathbf{s}_i}\left(\frac{N}{2}\ln\left(\frac{1}{2\pi\sigma^2}\right) - \frac{||\rho - \mathbf{s}_i||^2}{2\sigma^2}\right)\\ &= \max_{\mathbf{s}_i}\left(-\frac{||\rho - \mathbf{s}_i||^2}{2\sigma^2}\right)\\ &= \max_{\mathbf{s}_i}\left(-||\rho - \mathbf{s}_i||^2\right)\\ &= \min_{\mathbf{s}_i} ||\rho - \mathbf{s}_i||\end{aligned}$$

- Specifically, we can define this decision rule as:

$$\mathbf{s}_k = \arg\min_{\mathbf{s}_i} ||\rho - \mathbf{s}_i|| \rightarrow \hat{\mathbf{m}} = \mathbf{m}$$

- We can interpret $||\rho - \mathbf{s}_i||$ as a distance
  - An ML detector is equivalent to a minimum distance detector

QPSK Example

- By inspection, we see that the vector ρ is closest to $\mathbf{s}_1$
- Consequently, our ML detector indicates that $\mathbf{s}_1$ was transmitted
  - Decision: $\hat{\mathbf{m}} = \mathbf{m}_1$
- our decision rule for a QPSK signal constellation is that we declare $\mathbf{s}_i$ as transmitted depending on which quadrant ρ is located in

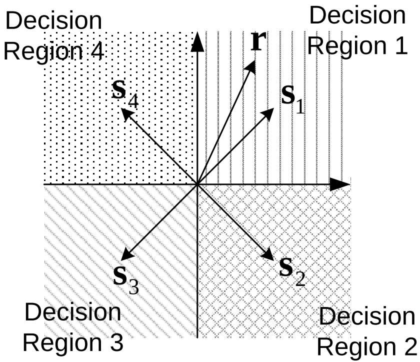

Figure : Decision Regions for QPSK Signal Constellation.

Receiver Realization

- Let us expand our decision rule:

$$\min_{\mathbf{s}_i} ||\rho - \mathbf{s}_i||^2 = \min_{\mathbf{s}_i} (\rho - \mathbf{s}_i) \cdot (\rho - \mathbf{s}_i)$$
$$= \rho \cdot \rho - 2\rho \cdot \mathbf{s}_i + \mathbf{s}_i \cdot \mathbf{s}_i$$

- Since $\rho \cdot \rho$ is common to all decision metrics for different $\mathbf{s}_i$, we can omit it, thus yielding:

$$\min_{\mathbf{s}_i} (-2\rho \cdot \mathbf{s}_i + \mathbf{s}_i \cdot \mathbf{s}_i) = \max_{\mathbf{s}_i} (2\rho \cdot \mathbf{s}_i - \mathbf{s}_i \cdot \mathbf{s}_i)$$

where

$$\rho \cdot \mathbf{s}_i = \int_0^T \rho(t) s_i(t) dt \qquad \mathbf{s}_i \cdot \mathbf{s}_i = \int_0^T s_i^2(t) dt = E_{s_i}$$

Correlator Realization

- We see that the waveform representation of $\rho \cdot \mathbf{s}_i$ is equal to the correlation of $r(t) = \rho(t)$ with respect to $s_i(t)$
- Thus, when $s_k(t)$ is present in $r(t)$, the optimal detector is equal to:

$$\mathbf{s}_k = \arg\max_i \left(\int_0^T \rho(t) s_i(t) dt - \frac{E_{s_i}}{2} \right)$$

Schematic of Correlator-based Receiver

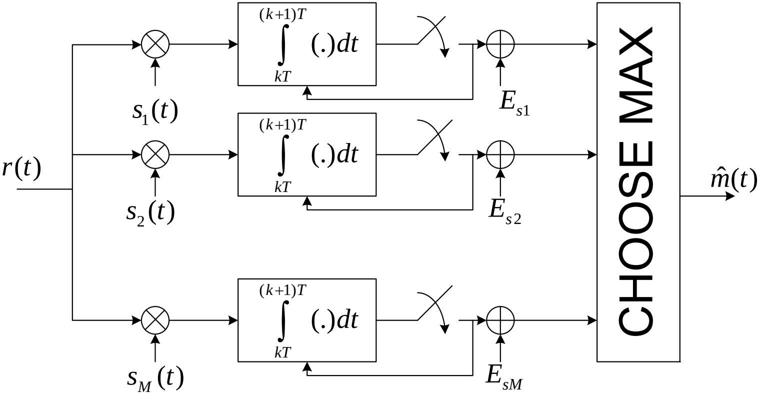

Figure : Correlator Realization of a Receiver Structure Assuming Perfect Synchronization.

Interpretation

- Given $r(t) = s_i(t) + n(t)$ and we observe only $r(t) = \rho(t)$ at the input to the receiver, which $s_i(t)$ for $i = 1, \ldots, M$ was sent?
- First we correlate $r(t)$ with $s_i(t)$ across all i
- Then we normalize the correlation result by the corresponding signal energy E_{s_i} in order to facilitate a fair comparison
  - Note that if all energy values are the same for each possible signal waveform, we can dispense with the energy normalization process since this will have no impact on the decision making