Digital Communication Systems Engineering with Software-Defined Radio

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Lecture 08

Modulation

- The process of modulation can be viewed as a mapping of b bits into a symbol
  - The message m_b is a b-vector of binary digits
- For each m_b (there are 2^b values of m_b available), we need a unique signal $s_i(t)$, $1 \leq i \leq 2^b$

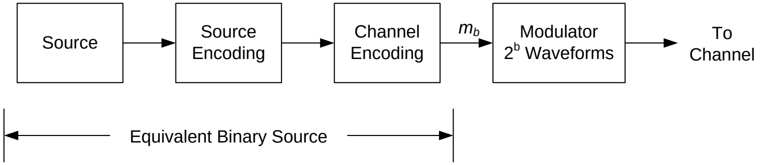

Figure : Modulation Process of Equivalent Binary Data.

Signal Constellation

- Modulation rule:
  - "1" $\rightarrow s_1(t)$
  - "0" $\rightarrow s_2(t)$
- The *bit rate* is defined as $R_b = 1/T$ bps
- The *symbol energy* is defined as $E_s = \int_{-\infty}^{\infty} s^2(t)dt$ [Joules]
- Suppose we have the following modulation rule given the waveform:

$$s(t) = A \cdot [u(t) - u(t - T)] \tag{1}$$

  where $u(t)$ is the unit step function, then:
  - "1" $\rightarrow s(t)$
  - "0" $\rightarrow -s(t)$

Average Bit Energy

- The symbol energy is then $E_s = E_{-s} = A^2 T = \frac{A^2}{R_b}$
  - Notice how E_s decreases as R_b increases
- We define the energy per bit as:

$$\bar{E}_b = P(1) \cdot \int_{-\infty}^{\infty} s_1^2(t)dt + P(0) \cdot \int_{-\infty}^{\infty} s_2^2(t)dt \qquad (2)$$

  where $P(1)$ is the probability that the bit is a “1”, and $P(0)$ is the probability that the bit is a “0”
- Suppose $s_1(t) = s(t)$ and $s_2(t) = -s(t)$, then:

$$\bar{E}_b = E_s\{P(1) + P(0)\} = E_s = \int_{-\infty}^{\infty} s^2(t)dt = A^2 T \qquad (3)$$

Euclidean Distance: $s_i(t)$ Versus $s_j(t)$

- We define the *Euclidean Distance* as:

$$d_{ij}^2 = \int_{-\infty}^{\infty} (s_i(t) - s_j(t))^2 dt = E_{\Delta s_{ij}} \tag{4}$$

where $\Delta s_{ij}(t) = s_i(t) - s_j(t)$

- For a signal, $s_i(t)$, used for modulation:

$$d_{min}^2 = \min_{s_i(t), s_j(t), i \neq j} \int_{-\infty}^{\infty} (s_i(t) - s_j(t))^2 dt \tag{5}$$

Power Efficiency ε_p

- The *Power Efficiency* of a signal set used for modulation is given by the expression:

$$\varepsilon_p = \frac{d_{\text{min}}^2}{\bar{E}_b} \tag{6}$$

- Suppose we would like to find the ε_p for Binary PAM
  - Given the waveforms:

$$\begin{aligned} s_1(t) &= A \cdot [u(t) - u(t-T)] = s(t) \\ s_2(t) &= -A \cdot [u(t) - u(t-T)] = -s(t) \end{aligned} \tag{7}$$

    where $u(t)$ is the unit step function
  - Compute the minimum Euclidean Distance: $d_{\text{min}}^2 = \int_{-\infty}^{\infty} \Delta s_{ij}^2(t)dt = 4A^2T$
  - Compute the average bit energy: $\bar{E}_b = E_s\{P(1) + P(0)\} = E_s = \int_{-\infty}^{\infty} s^2(t)dt = A^2T$
  - Compute the power efficiency: $\varepsilon_P = d_{\text{min}}^2/\bar{E}_b = 4$

Modulation

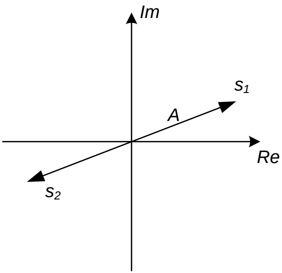

Figure : BPSK Signal Constellation.

- Modulation Rule:

$$\begin{aligned} \text{``1''} &\rightarrow s_1(t) = A \cdot \cos(\omega_c t + \theta) \\ \text{``0''} &\rightarrow s_2(t) = -A \cdot \cos(\omega_c t + \theta) \\ &= A \cdot \cos(\omega_c (t) + \theta + \pi) \\ &= -s_1(t) \end{aligned} \tag{8}$$

- As we have seen in the past few slides, determining ε_P is important
  - We will now show that $\varepsilon_p = 4$ is the **optimum**

Solving for $\varepsilon_{p,\mathrm{BPSK}}$: Compute d^2_{min}

- Employ the definition for d^2_{min} and solve

$$\begin{aligned}
d^2_{\mathrm{min}} &= \int_0^T (s_1(t) - s_2(t))^2 dt \\
&= 4A^2 \int_0^T \cos^2(\omega_c t + \theta) dt \\
&= \frac{4A^2 T}{2} + \frac{4A^2}{2} \int_0^T \cos(2\omega_c t + 2\theta) dt \\
&= 2A^2 T \rightarrow \text{double frequency term eliminated}
\end{aligned}$$

Solving for $\varepsilon_{p,\mathrm{BPSK}}$: Compute $\bar{E}_b$

- Employ the definition for $\bar{E}_b$ and solve

$$E_{s_1} = \int_0^T s_1^2(t)dt = A^2 \int_0^T \cos^2(\omega_c t + \theta)dt$$

$$= \frac{A^2 T}{2} + \frac{A^2}{2} \int_0^T \cos(2\omega_c t + 2\theta)dt$$

$$= \frac{A^2 T}{2} \rightarrow \text{double frequency term eliminated}$$

$$E_{s_2} = \frac{A^2 T}{2}$$

$$\therefore \quad \bar{E}_b = P(0) \cdot E_{s_2} + P(1) \cdot E_{s_1} = \frac{A^2 T}{2}$$

Solving for $\varepsilon_{p,\mathrm{BPSK}}$: The Finale

- Applying the definition for the power efficiency yields:

$$\varepsilon_{p,\mathrm{BPSK}} = \frac{d_{\min}^2}{\bar{E}_b} = 4 \qquad (9)$$

- This is suppose to be the largest possible value for ε_p for a modulation scheme employing all possible signal representations, i.e., $M = 2^b$ waveforms

Alternative Approach to Computing d^2_{min}

- Another way to compute d^2_{min} is to use the concept of *correlation* such that:

$$d^2_{\text{min}} = \int_0^T (s_2(t) - s_1(t))^2 dt = E_{s_1} + E_{s_2} - 2\rho_{12} \qquad (10)$$

where:

$$E_{s_i} = \int_0^T s_i^2(t)dt \quad \text{and} \quad \rho_{12} = \int_0^T s_1(t)s_2(t)dt$$

- In order to get a **large** ε_p, we need to maximize d^2_{min}, which means we want $\rho_{12} < 0$
  - $E_{s_1} = E_{s_2} = E = A^2T/2 \rightarrow d^2_{\text{min}} = 2(E - \rho_{12}) \rightarrow \rho_{12} = -E$

Several Examples

- Show that for the following signal waveforms:

$$s_1(t) = A \cdot \cos(\omega_c t + \theta)$$
$$s_2(t) = 0$$

the power efficiency is equal to $\varepsilon_p = 2$

  - This represents a 3 dB loss relative to BPSK!!
  - $\text{Loss} = 10 \cdot \log_{10}\left(\frac{\varepsilon_{p,\text{BPSK}}}{\varepsilon_{p,\text{other}}}\right)$

- Show that for the following signal waveforms:

$$s_1(t) = A \cdot \cos(\omega_c t + \theta)$$
$$s_2(t) = A \cdot \sin(\omega_c t + \theta)$$

the power efficiency is equal to $\varepsilon_p = 2$