Digital Communication Systems Engineering with Software-Defined Radio

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Lecture 05

Definition

- *Noise* is an unwanted signal that is superimposed on top of a useful information signal
- Introduction of noise to useful information signal usually occurs prior to interception and decoding by receiver
- Noise can be modeled as a *random variable*
  - Different noise-generating phenomena can be modeled by different random variables
  - Although noise signal not known precisely, its statistical characteristics can be used in the communication system design process

Noise Sources

- Noise originates from multiple sources within the communication system transmission environment
  - Natural sources
    - Solar flare activity and atmospheric conditions, e.g., ionosphere
    - Johnson-Nyquist noise, i.e., thermal noise in conductive materials
    - Lightning
    - Cosmic background noise
  - Human-built sources
    - Microwave ovens (2.4 GHz)
    - Power transmission lines (60 Hz)
    - Other wireless transmissions

Noise Blurs Intercepted Signals

- Randomness of noise makes detection and decoding of useful signals difficult

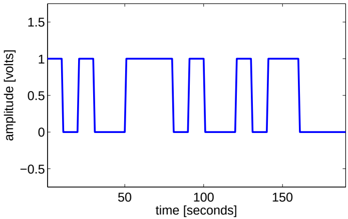

Figure : Signal Before Noise is Added. Notice Clear Pulse Shapes of Signal.

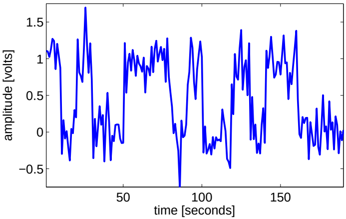

Figure : Signal After Noise is Added. Difficult for Receiver to Ascertain Pulse Shapes.

Gaussian Random Variable

- Often communication system engineers usually assume the noise to possess a *Gaussian* distribution
- Gaussian random variable $x \sim \mathcal{N}(\mu_x, \sigma_x^2)$ possesses a mean μ_x and variance σ_x^2
- Statistical behavior can be entirely characterize by the *probability density function* (PDF) of the random phenomenon
  - A Gaussian PDF is defined as:

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma_x^2}} e^{-\frac{(x-\mu_x)^2}{2\sigma_x^2}} \tag{1}$$

Computing Probabilities

- Probability density functions entirely capture the statistical behavior of a random variable
  - Although the exact random variable outcome is unknown, we can associate a probability to a specific outcome or range of outcomes occurring
- In general, to determine the probability that a random variable X will produce a value greater than or equal to a, solve:

$$P(X \geq a) = \int_a^\infty f_X(x)dx \tag{2}$$

- Since it is difficult to integrate a Gaussian PDF, we can alternatively use a *Q-function*, which is defined as:

$$Q(y) = \frac{1}{\sqrt{2\pi}} \int_y^\infty e^{-\frac{t^2}{2}} dt \tag{3}$$

  - Use look-up tables to find actual numerical value

"White" = Uncorrelated

- *White noise* refers to a noise signal whose power spectral density (PSD) is flat across the entire frequency range
- The term "white" is used since white light possesses a PSD that spans the entire frequency range of visible light
- Another interpretation of white noise is that the outcome at every time sample is uncorrelated with every other time sample except itself
  - As we will see in Lecture 6, the PSD and autocorrelation function are related to each other
  - PSD $S(f)$ = constant is related via inverse Fourier transform to autocorrelation $R(\tau) = c\delta(\tau)$
- Only for Gaussian random variables does *uncorrelated* also mean *independent*

Gaussian Noise Example

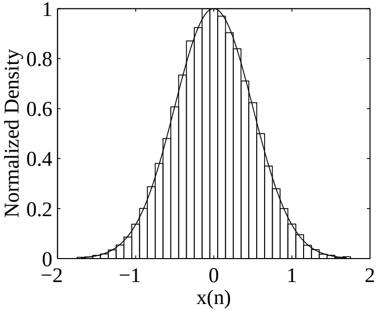

Figure : Example of a Gaussian Probability Density Function.

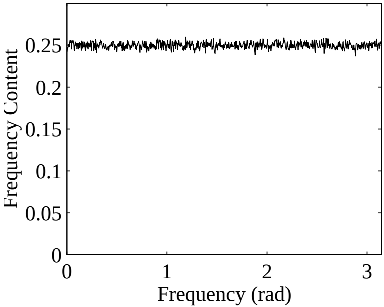

Figure : Example of a Gaussian Power Spectral Density.

Additive Noise Model

- Additive noise model is often used to describe how unwanted impairments are added to a useful information signal
- Usually $n(t)$ represents the culmination of all noise sources into a single variable
- Receiver only observes the corrupted version of $s(t)$ by $n(t)$, namely $r(t)$

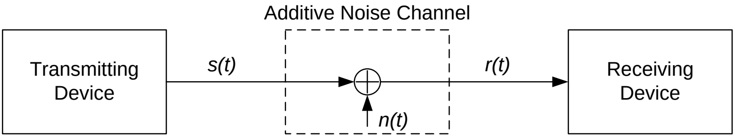

Figure : Simple Additive Noise Model.

Narrowband Noise Fundamentals

- Most transmissions are *bandlimited*
  - Constraint on amount of spectrum that can be used
  - Larger bandwidths yield more expensive wireless hardware
  - Narrowband filters at transmitter and receiver allow only modulated signal to pass
- A portion of the noise intercepted at the receiver also passes through the narrowband filter
  - If noise is white, then narrowband noise is shaped like a cosine-modulated bandpass filter response
  - How does this look like in the frequency domain? Time domain?

Mathematical Framework

- Need a convenient mathematical framework to represent narrowband noise
  - In-phase/Quadrature Representation
  - Envelope/Phase Representation
- Given $x = Ae^{j\phi} = a + jb$, where $x \in \mathbb{C}$ and $A, \phi, a, b \in \mathbb{R}$, we have the following relationships:
  - $A = \sqrt{a^2 + b^2}$ and $\phi = \tan^{-1}(b/a)$
  - $a = A\cos(\phi)$ and $b = A\sin(\phi)$

In-phase/Quadrature Representation

- Suppose we define the complex envelope (complex baseband) of a narrowband noise signal as:

$$\tilde{n}(t) = n_I(t) + jn_Q(t) \tag{4}$$

- By definition, the bandpass version of the narrowband noise signal is:

$$n(t) = \text{Real}\left\{\tilde{n}(t)e^{j2\pi f_c t}\right\} \tag{5}$$

- Using Euler's Relationship, namely $e^{j\omega} = \cos(\omega) + j\sin(\omega)$, we get the following expression:

$$n(t) = n_I(t)\cos(2\pi f_c t) - n_Q(t)\sin(2\pi f_c t) \tag{6}$$

- How do you generate/extract $n_I(t)$ and $n_Q(t)$?

Several Important Properties of I/Q Representation

- Both $n_I(t)$ and $n_Q(t)$ have zero mean
- If $n(t)$ is Gaussian, so are $n_I(t)$ and $n_Q(t)$
- If $n(t)$ is stationary, $n_I(t)$ and $n_Q(t)$ are jointly stationary
- The PSD of $n_I(t)$ and $n_Q(t)$ are equal to:

$$S_{N_I}(f) = S_{N_Q}(f) = \begin{cases} S_N(f - f_c) + S_N(f + f_c), & -B \leq f \leq B \\ 0, & \text{otherwise} \end{cases} \tag{7}$$

- Both $n_I(t)$ and $n_Q(t)$ have the same variance as $n(t)$
- If $n(t)$ is Gaussian and its PSD symmetric, then $n_I(t)$ and $n_Q(t)$ are statistically independent
- The cross spectral density between $n_I(t)$ and $n_Q(t)$ is purely imaginary, and for $-B \leq f \leq B$ it is equal to (zero otherwise):

$$S_{N_I N_Q}(f) = -S_{N_Q N_I}(f) = j(S_N(f + f_c) - S_N(f - f_c)) \tag{8}$$

Envelope/Phase Representation

- We could also express $n(t)$ in terms of envelope and phase:

$$n(t) = r(t)\cos(2\pi f_c t + \phi(t)) \tag{9}$$

where $r(t) = \sqrt{n_I(t) + n_Q(t)}$ is the envelope and $\phi(t) = \tan^{-1}(n_Q(t)/n_I(t))$ is the phase

- Given a joint PDF for $n_I(t)$ and $n_Q(t)$ equal to:

$$f_{N_I N_Q}(n_I, n_Q) = \frac{1}{2\pi\sigma^2} e^{-\frac{n_I^2 + n_Q^2}{2\sigma^2}}, \tag{10}$$

and using the relationships $n_I = r\cos(\phi)$ and $n_Q = r\sin(\phi)$ as well as a *Jacobian* in order to get:

$$f_{R\Phi}(r, \phi) = \begin{cases} \frac{r}{2\pi\sigma^2} e^{-\frac{r^2}{2\sigma^2}}, & r \geq 0 \text{ and } 0 \leq \phi \leq 2\pi \\ 0, & \text{otherwise} \end{cases} \tag{11}$$

- Notice the combination of *Rayleigh* and *Uniform* PDFs