Digital Communication Systems Engineering with Software-Defined Radio

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Lecture 12

Orthonormal Basis Functions

- Recall from Lecture 10 that $\{\phi_j(t)\}$ is an orthonormal set of functions on the time interval $[0, T]$ such that:

$$\int_0^T \phi_i(t)\phi_j(t)dt = \begin{cases} 1 & i = j \\ 0 & \text{otherwise} \end{cases}$$

- Furthermore, it is possible to represent a signal waveform $s_i(t)$ as the weighted sum of these orthonormal basis functions, i.e.:

$$s_i(t) = \sum_{k=1}^{N} s_{ik}\phi_k(t) \qquad (1)$$

- How do we determine the set of orthonormal basis functions $\{\phi_j(t)\}$?
  - Use *Gram-Schmidt Orthogonalization Procedure*

Orthonormal Basis Functions

- A complete orthonormal set of basis functions is needed for a set of M energy signals denoted by $s_1(t), \ldots, s_M(t)$
- Choose $s_1(t)$ and normalize it:

$$\phi_1(t) = \frac{s_1(t)}{\sqrt{E_{s_1}}}$$

where E_{s_1} is the energy of the signal $s_1(t)$

  - We can express $s_1(t)$ as:

$$s_1(t) = \sqrt{E_{s_1}}\phi_1(t) = s_{11}\phi_1(t)$$

where the coefficient $s_{11} = \sqrt{E_{s_1}}$ and $\phi_1(t)$ has unit energy as required

Setting Up $\phi_2(t)$

- Now using the signal $s_2(t)$, let us define the s_{21} coefficient as:

$$s_{21} = \int_0^T s_2(t)\phi_1(t)dt$$

- To help in getting the basis function $\phi_2(t)$, we define the intermediate function:

$$g_2(t) = s_2(t) - s_{21}\phi_1(t)$$

which is orthogonal to $\phi_1(t)$ over the interval $0 \leq t \leq T$

Creating the Second Basis Function

- The second basis function can be defined as:

$$\phi_2(t) = \frac{g_2(t)}{\sqrt{\int_0^T g_2^2(t)dt}} \tag{2}$$

which can be expanded to:

$$\phi_2(t) = \frac{s_2(t) - s_{21}\phi_1(t)}{\sqrt{E_{s_2} - s_{21}^2}} \tag{3}$$

where E_{s_2} is the energy of the signal $s_2(t)$.

- It is clear that the basis function $\phi_2(t)$ satisfies:

$$\int_0^T \phi_2^2(t)dt = 1 \quad \text{and} \quad \int_0^T \phi_1(t)\phi_2(t)dt = 0$$

General Expressions

- In general, we can define the following functions that can be employed in an iterative procedure for generating a set of orthonormal basis functions:

$$g_i(t) = s_i(t) - \sum_{j=1}^{i-1} s_{ij}\phi_j(t)$$

$$s_{ij} = \int_0^T s_i(t)\phi_j(t)dt, \; j = 1, 2, \dots, i-1$$

$$\phi_i(t) = \frac{g_i(t)}{\sqrt{\int_0^T g_i^2(t)dt}}, \; i = 1, 2, \dots, N$$

Problem Statement

- Carry out the Gram-Schmidt Orthogonalization procedure of the signals shown in the next figure in the order $s_3(t)$, $s_1(t)$, $s_4(t)$, $s_2(t)$ and thus obtain a set of orthonormal functions $\{\phi_m(t)\}$
- Then, determine the vector representation of the signals $\{s_n(t)\}$ by using the orthonormal functions $\{\phi_m(t)\}$

Signal Waveforms

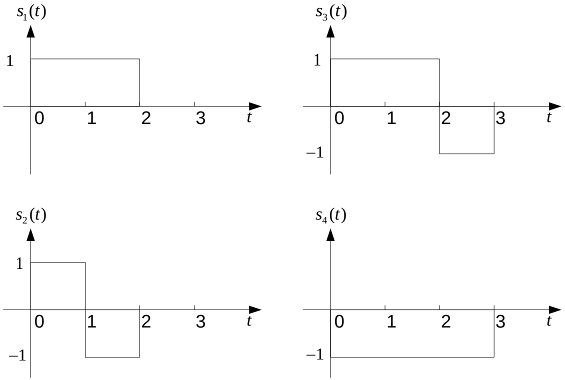

Figure : Example Signal Waveforms.

Determining Orthonormal Basis Functions – Part I

- For $s_3(t)$:

$$\phi_1(t) = \frac{s_3(t)}{\sqrt{E_{s_3}}} = \frac{s_3(t)}{\sqrt{3}}$$

- For $s_1(t)$:

$$g_2(t) = s_1(t) - s_{12}\phi_1(t) = s_1(t) - \frac{2}{3}s_3(t) = \begin{cases} 1/3, & 0 \leq t < 2 \\ 2/3, & 2 \leq t < 3 \\ 0, & t \geq 3 \end{cases}$$

$$\therefore \phi_2(t) = \frac{g_2(t)}{\sqrt{\int_0^T g_2^2(t)dt}} = \begin{cases} 1/\sqrt{6}, & 0 \leq t < 2 \\ 2/\sqrt{6}, & 2 \leq t < 3 \\ 0, & t \geq 3 \end{cases}$$

Determining Orthonormal Basis Functions – Part II

- For $s_4(t)$:

$$g_3(t) = s_4(t) - \sum_{j=1}^{2} s_{4j}\phi_j(t) = 0$$

$$\therefore\ \phi_3(t) = 0$$

- For $s_4(t)$:

$$g_4(t) = s_2(t) - \sum_{j=1}^{3} s_{2j}\phi_j(t) = 0$$

$$\therefore\ \phi_4(t) = \frac{g_4(t)}{\sqrt{\int_0^T g_4^2(t)dt}} = \frac{s_2(t)}{\sqrt{2}}$$

Vector Representations

- Using the orthonormal basis functions $\{\phi_1(t), \phi_2(t), \phi_4(t)\}$, we can express the four waveforms as:
  - $\mathbf{s}_1 = (2/\sqrt{3}, \sqrt{6}/3, 0)$
  - $\mathbf{s}_2 = (0, 0, \sqrt{2})$
  - $\mathbf{s}_3 = (\sqrt{3}, 0, 0)$
  - $\mathbf{s}_4 = (-1/\sqrt{3}, -4/\sqrt{6}, 0)$