Detection and Estimation of Signals in Noise

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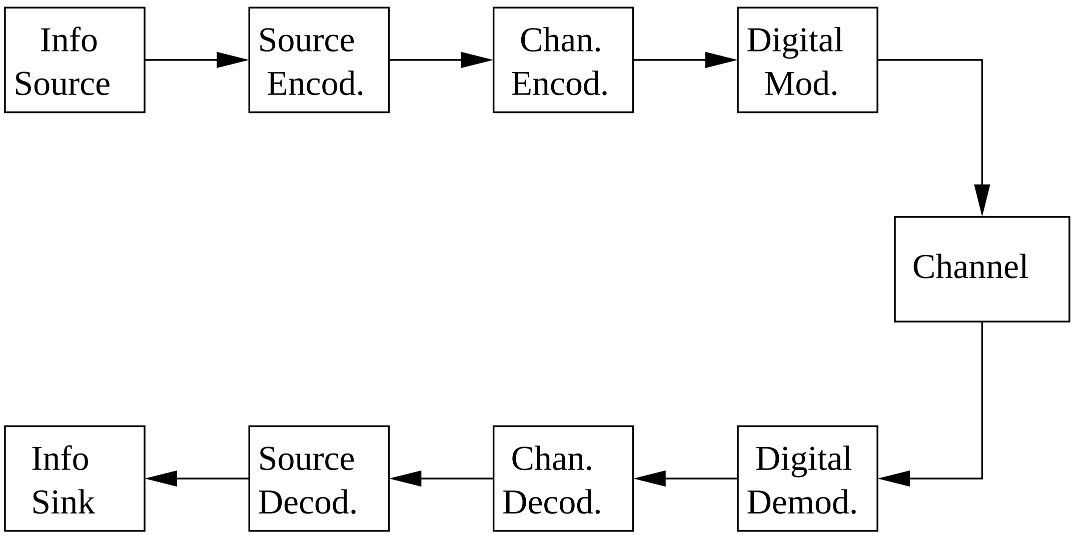
1 Basic Elements of a Digital Communication System



1.1 Transmitter

a) Information Source

- analog signal: e.g. audio or video signal
- digital signal: e.g. data, text

b) Source Encoder

Objective: Represent the source signal as efficiently as possible, i.e., with as few bits as possible
$\Rightarrow$ minimize the redundancy in the source encoder output bits

c) Channel Encoder

Objective: Increase reliability of received data

$\Rightarrow$ add redundancy in a controlled manner to information bits

d) Digital Modulator

Objective: Transmit most efficiently over the (physical) transmission channel

$\Rightarrow$ map the input bit sequence to a *signal waveform* which is suitable for the transmission channel

Examples: Binary modulation:

bit 0 $\longrightarrow s_0(t)$
bit 1 $\longrightarrow s_1(t)$
$\Rightarrow$ 1 bit per channel use

M–ary modulation:

we map b bits to one waveform
$\Rightarrow$ we need $M = 2^b$ different waveforms to represent all possible b–bit combinations
$\Rightarrow$ b bit/(channel use)

1.2 Receiver

a) Digital Demodulator

Objective: Reconstruct transmitted data symbols (binary or M–ary from channel–corrupted received signal

b) Channel Decoder

Objective: Exploit redundancy introduced by channel encoder to increase reliability of information bits

Note: In modern receivers demodulation and decoding is sometimes performed in an *iterative* fashion.

c) Source Decoder

Objective: Reconstruct original information signal from output of channel decoder

1.3 Communication Channels

1.3.1 Physical Channel

a) Types

- wireline
- optical fiber
- optical wireless channel
- wireless radio frequency (RF) channel
- underwater acoustic channel
- storage channel (CD, disc, etc.)

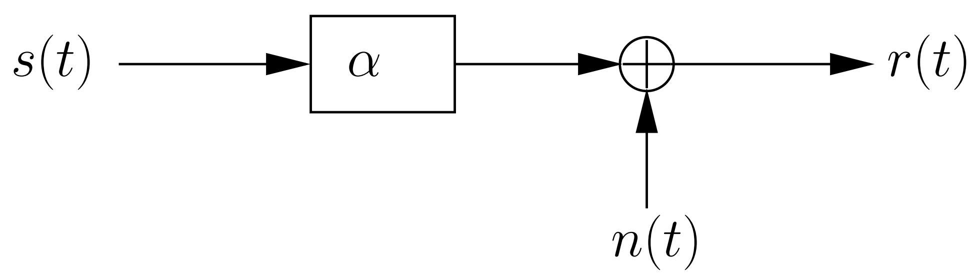
b) Impairments

- noise from electronic components in transmitter and receiver
- amplifier nonlinearities
- other users transmitting in same frequency band at the same time (co–channel or multiuser interference)
- linear distortions because of bandlimited channel
- time–variance in wireless channels

For the design of the transmitter and the receiver we need a *simple* mathematical model of the physical communication channel that captures its most important properties. This model will vary from one application to another.

1.3.2 Mathematical Models for Communication Channels

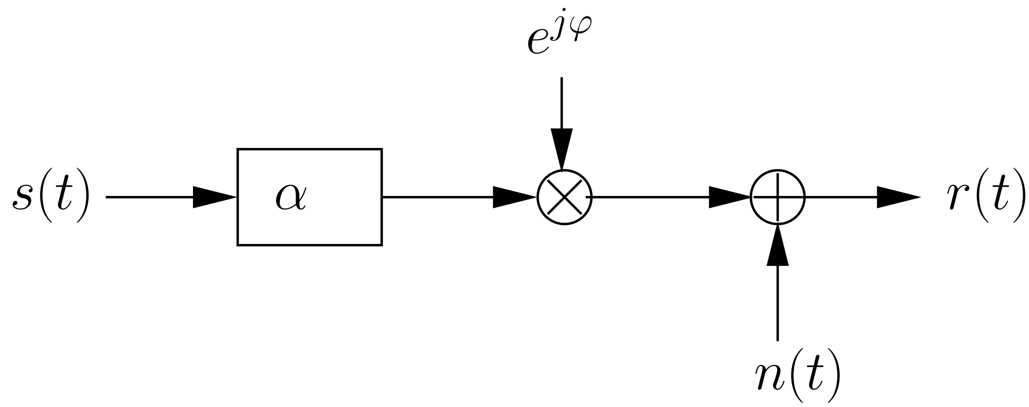
a) Additive White Gaussian Noise (AWGN) Channel



$$r(t) = \alpha\, s(t) + n(t)$$

The transmitted signal is only attenuated ($\alpha \leq 1$) and impaired by an additive white Gaussian noise (AWGN) process $n(t)$.

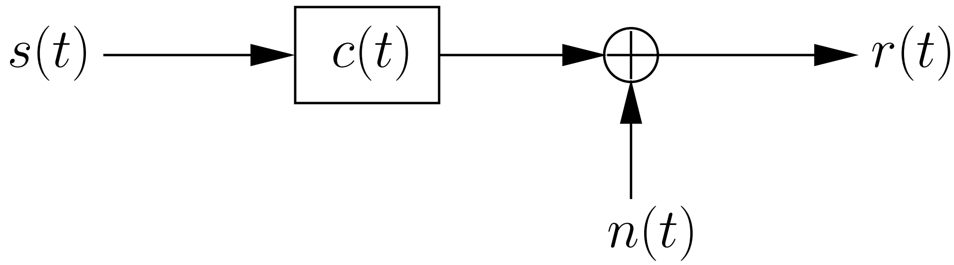
b) AWGN Channel with Unknown Phase



$$r(t) = \alpha\, \mathrm{e}^{j\varphi}\, s(t) + n(t)$$

In this case, the transmitted signal also experiences an *unknown* phase shift φ. φ is often modeled as a random variable, which is uniformly distributed in the interval $[-\pi,\, \pi)$.

c) Linearly Dispersive Channel (Linear Filter Channel)



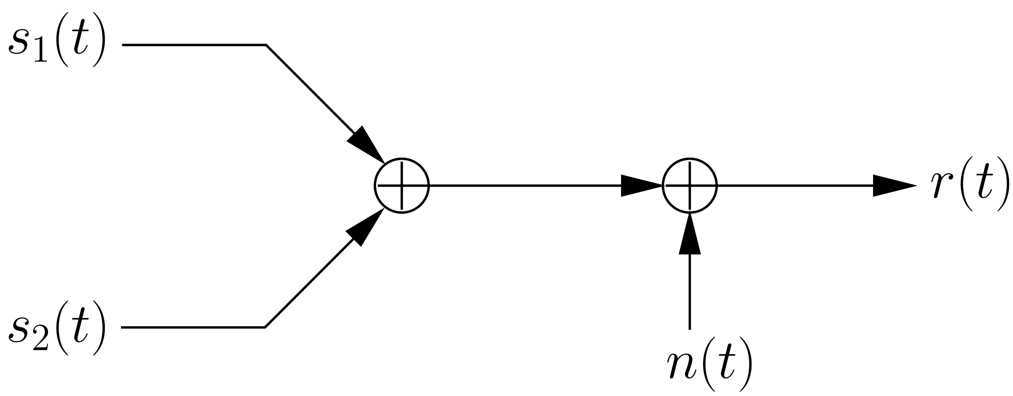
$c(t)$: channel impulse response; $*$: linear convolution

$$
\begin{aligned}
r(t) &= c(t) * s(t) + n(t) \\
&= \int_{-\infty}^{\infty} c(\tau)\, s(t-\tau)\mathrm{d}\tau + n(t)
\end{aligned}
$$

Transmit signal is *linearly* distorted by $c(t)$ and impaired by AWGN.

d) Multiuser Channel

Two users:



K–user channel:

$$
r(t) = \sum_{k=1}^{K} s_k(t) + n(t)
$$

e) Other Channels

- time–variant channels
- stochastic (random) channels
- fading channels
- multiple–input multiple–output (MIMO) channels
- ...

Some questions that we want to answer in this course:

- Which waveforms are used for digital communications?
- How are these waveforms demodulate/detect?
- What performance (= bit or symbol error rate) can be achieved?

2 Probability and Stochastic Processes

Motivation:

- Very important mathematical tools for the design and analysis of communication systems
- Examples:
  - The transmitted symbols are unknown at the receiver and are modeled as random variables.
  - Impairments such as noise and interference are also unknown at the receiver and are modeled as stochastic processes.

2.1 Probability

2.1.1 Basic Concepts

Given: *Sample space* S containing all possible outcomes of an experiment

Definitions:

- Events A and B are *subsets* of S, i.e., $A \subseteq S$, $B \subseteq S$
- The complement of A is denoted by $\bar{A}$ and contains all elements of S not included in A
- Union of two events: $D = A \cup B$ consists of all elements of A *and* B
  $\Rightarrow A \cup \bar{A} = S$

- Intersection of two elements: $E = A \cap B$
  Mutually exclusive events have as intersection the null element ϕ
  e.g. $A \cap \bar{A} = \phi$.
- Associated with each event A is its probability $P(A)$

Axioms of Probability

1. $P(S) = 1$ (certain event)
2. $0 \leq P(A) \leq 1$
3. If $A \cap B = \phi$ then $P(A \cup B) = P(A) + P(B)$

The entire theory of probability is based on these three axioms.

E.g. it can be proved that

- $P(\bar{A}) = 1 - P(A)$
- $A \cap B \neq \phi$ then $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

Example:

Fair die

- $S = \{1, 2, 3, 4, 5, 6\}$
- $A = \{1, 2, 5\}$, $B = \{3, 4, 5\}$
- $\bar{A} = \{3, 4, 6\}$
- $D = A \cup B = \{1, 2, 3, 4, 5\}$
- $E = A \cap B = \{5\}$
- $P(1) = P(2) = \ldots = P(6) = \frac{1}{6}$

– $P(A) = P(1) + P(2) + P(5) = \frac{1}{2}$, $P(B) = \frac{1}{2}$

– $P(D) = P(A) + P(B) - P(A \cap B) = \frac{1}{2} + \frac{1}{2} - \frac{1}{6} = \frac{5}{6}$

Joint Events and Joint Probabilities

- Now we consider two experiments with outcomes

$$A_i, \quad i = 1, 2, \ldots n$$

and

$$B_j, \quad j = 1, 2, \ldots m$$

- We carry out both experiments and assign the outcome (A_i, B_j) the probability $P(A_i, B_j)$ with

$$0 \le P(A_i, B_j) \le 1$$

- If the outcomes B_j, $j = 1, 2, \ldots, m$ are mutually exclusive we get

$$\sum_{j=1}^{m} P(A_i, B_j) = P(A_i)$$

A similar relation holds if the outcomes of $A_i, i = 1, 2, \ldots n$ are mutually exclusive.

- If all outcomes of *both* experiments are mutually exclusive, then

$$\sum_{i=1}^{n} \sum_{j=1}^{m} P(A_i, B_j) = 1$$

Conditional Probability

- Given: Joint event (A, B)
- *Conditional probability* $P(A|B)$: Probability of event A given that we have already observed event B
- Definition:
$$P(A|B) = \frac{P(A, B)}{P(B)}$$
($P(B) > 0$ is assumed, for $P(B) = 0$ we cannot observe event B)

  Similarly:
$$P(B|A) = \frac{P(A, B)}{P(A)}$$
- **Bayes' Theorem:**
  From
$$P(A,\, B) = P(A|B)\, P(B) = P(B|A)\, P(A)$$
we get
$$\boxed{P(A|B) = \frac{P(B|A)\, P(A)}{P(B)}}$$

Statistical Independence

If observing B does not change the probability of observing A, i.e.,

$$P(A|B) = P(A),$$

then A and B are *statistically independent.*

In this case:

$$\begin{aligned} P(A, B) &= P(A|B)\, P(B) \\ &= P(A)\, P(B) \end{aligned}$$

Thus, two events A and B are statistically independent if and only if

$$\boxed{P(A, B) = P(A)\, P(B)}$$

2.1.2 Random Variables

- We define a function $X(s)$, where $s \in S$ are elements of the sample space.
- The domain of $X(s)$ is S and its range is the set of real numbers.
- $X(s)$ is called a *random variable*.
- $X(s)$ can be continuous or discrete.
- We use often simply X instead of $X(s)$ to denote the random variable.

Example:

- Fair die: $S = \{1, 2, \dots, 6\}$ and

$$X(s) = \begin{cases} 1, & s \in \{1, 3, 5\} \\ 0, & s \in \{2, 4, 6\} \end{cases}$$

- Noise voltage at resistor: S is continuous (e.g. set of all real numbers) and so is $X(s) = s$

Cumulative Distribution Function (CDF)

- Definition: $F(x) = P(X \leq x)$

  The CDF $F(x)$ denotes the probability that the random variable (RV) X is smaller than or equal to x.

- Properties:

$$0 \leq F(x) \leq 1$$

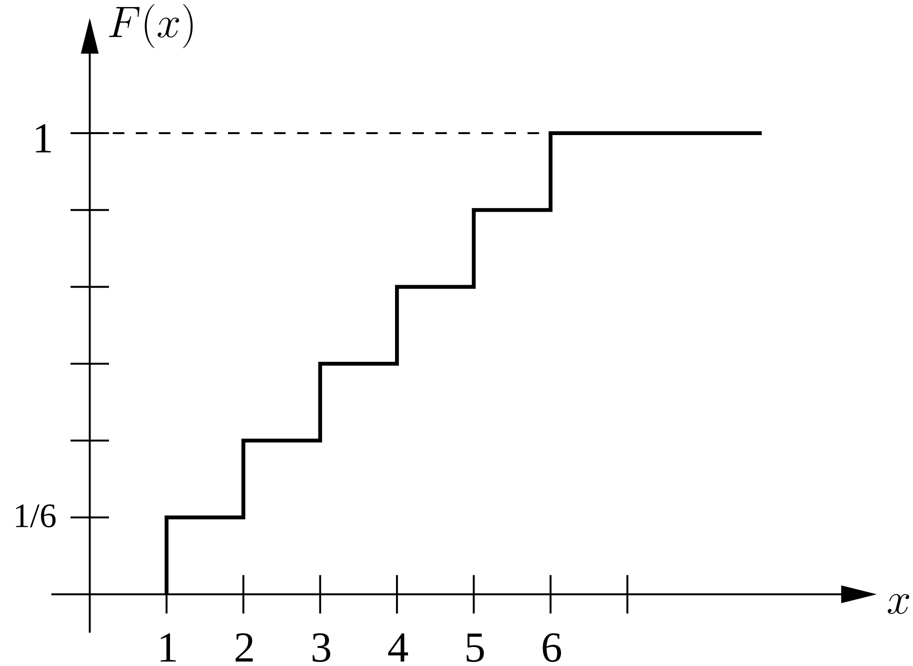
$$\lim_{x \to -\infty} F(x) = 0$$

$$\lim_{x \to \infty} F(x) = 1$$

$$\frac{\mathrm{d}}{\mathrm{d}x} F(x) \geq 0$$

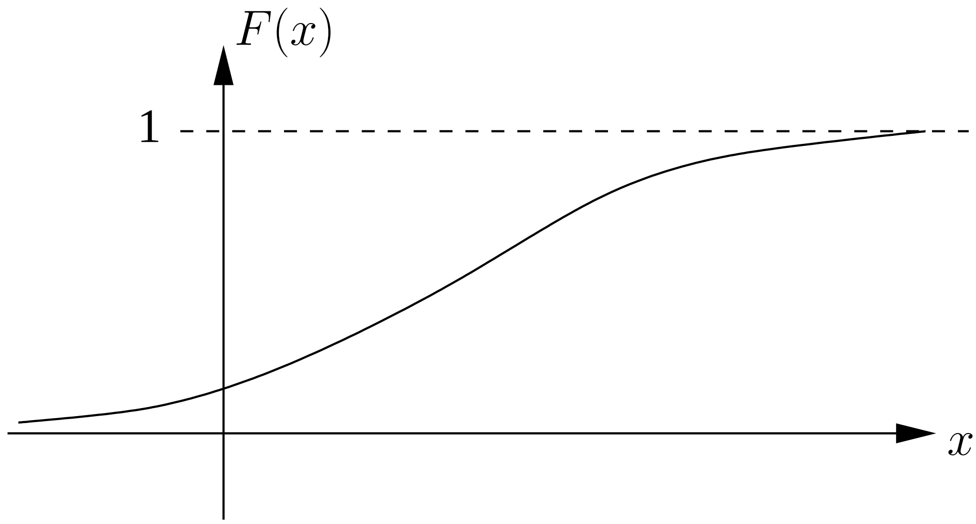
Example:

1. Fair die $X = X(s) = s$



Note: X is a *discrete* random variable.

2. Continuous random variable



Probability Density Function (PDF)

- Definition:

$$p(x) = \frac{\mathrm{d}F(x)}{\mathrm{d}x}, \qquad -\infty < x < \infty$$

- Properties:

$$p(x) \geq 0$$

$$F(x) = \int_{-\infty}^{x} p(u)\mathrm{d}u$$

$$\int_{-\infty}^{\infty} p(x)\mathrm{d}x = 1$$

- Probability that $x_1 \leq X \leq x_2$

$$P(x_1 \leq X \leq x_2) = \int_{x_1}^{x_2} p(x)\mathrm{d}x = F(x_2) - F(x_1)$$

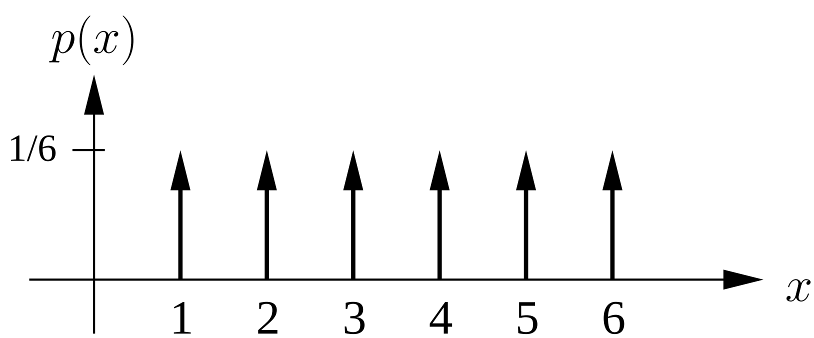
- Discrete random variables: $X \in \{x_1, x_2, \ldots, x_n\}$

$$p(x) = \sum_{i=1}^{n} P(X = x_i)\delta(x - x_i)$$

with the *Dirac impulse* $\delta(\cdot)$

Example:

Fair die



Joint CDF and Joint PDF

- Given: Two random variables X, Y

- Joint CDF:

$$F_{XY}(x,y) = P(X \leq x, Y \leq y) = \int_{-\infty}^{x} \int_{-\infty}^{y} p_{XY}(u,v) \, \mathrm{d}u \, \mathrm{d}v$$

  where $p_{XY}(x,y,)$ is the joint PDF of X and Y

- Joint PDF:

$$p_{XY}(x,y) = \frac{\partial^2}{\partial x \partial y} F_{XY}(x,y)$$

- Marginal densities

$$p_X(x) = \int_{-\infty}^{\infty} p_{XY}(x,y) \, \mathrm{d}y$$

$$p_Y(y) = \int_{-\infty}^{\infty} p_{XY}(x,y) \, \mathrm{d}x$$

- Some properties of $F_{XY}(x, y)$ and $p_{XY}(x, y)$

$$F_{XY}(-\infty, -\infty) = F_{XY}(x, -\infty) = F_{XY}(-\infty, y) = 0$$

$$F_{XY}(\infty, \infty) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} p_{XY}(x, y)\, \mathrm{d}x\, \mathrm{d}y = 1$$

- Generalization to n random variables X_1, X_2, $\ldots$, X_n: see textbook

Conditional CDF and Conditional PDF

- Conditional PDF

$$p_{X|Y}(x|y) = \frac{p_{XY}(x, y)}{p_Y(y)}$$

- Conditional CDF

$$\begin{aligned} F_{X|Y}(x|y) &= \int_{-\infty}^{x} p_{X|Y}(u|y)\, \mathrm{d}u \\ &= \frac{\int_{-\infty}^{x} p_{XY}(u, y)\, \mathrm{d}u}{p_Y(y)} \end{aligned}$$

Statistical Independence

X and Y are statistical independent if and only if

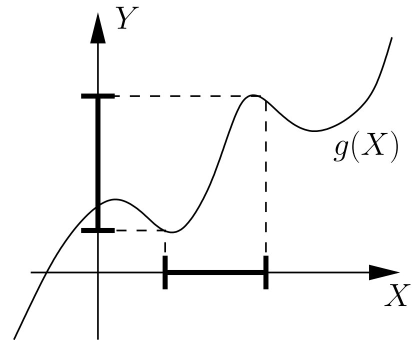
$$p_{XY}(x,y) = p_X(x)\,p_Y(y)$$
$$F_{XY}(x,y) = F_X(x)\,F_Y(y)$$

Complex Random Variables

- The *complex* RV $Z = X + jY$ consists of two real RVs X and Y
- Problem: $Z \leq z$ is not defined
- Solution: We treat Z as a tupel (vector) of its real components X and Y with joint PDF $p_{XY}(x,y)$
- CDF
$$F_Z(z) = P(X \leq x, Y \leq y) = F_{XY}(x,y)$$
- PDF
$$p_Z(z) = p_{XY}(x,y)$$

2.1.3 Functions of Random Variables

Problem Statement (one–dimensional case)



- Given:
  - RV X with $p_X(x)$ and $F_X(x)$
  - RV $Y = g(X)$ with function $g(\cdot)$
- Calculate: $p_Y(y)$ and $F_Y(y)$

Since a general solution to the problem is very difficult, we consider some important special cases.

Special Cases:

a) Linear transformation $Y = aX + b$, $a > 0$

– CDF

$$\begin{aligned}
F_Y(y) &= P(Y \leq y) = P(aX + b \leq y) = P\left(X \leq \frac{y-b}{a}\right) \\
&= \int_{-\infty}^{(y-b)/a} p_X(x)\,\mathrm{d}x \\
&= F_X\left(\frac{y-b}{a}\right)
\end{aligned}$$

– PDF

$$p_Y(y) = \frac{\partial}{\partial y} F_Y(y) = \frac{\partial}{\partial y} \frac{\partial x}{\partial x} F_X(x)\bigg|_{x=(y-b)/a}$$

$$= \frac{\partial x}{\partial y}\bigg|_{x=(y-b)/a} \frac{\partial}{\partial x} F_X(x)\bigg|_{x=(y-b)/a}$$

$$= \frac{1}{a} p_X\left(\frac{y-b}{a}\right)$$

b) $g(x) = y$ has real roots $x_1, x_2, \ldots, x_n$

– PDF

$$\boxed{p_Y(y) = \sum_{i=1}^{n} \frac{p_X(x_i)}{|g'(x_i)|}}$$

with $g'(x_i) = \frac{\mathrm{d}}{\mathrm{d}x} g(x)\Big|_{x=x_i}$

– CDF: Can be obtained from PDF by integration.

Example:

$Y = aX^2 + b$, $a > 0$

Roots:

$$ax^2 + b = y \qquad \Rightarrow \qquad x_{1/2} = \pm\sqrt{\frac{y-b}{a}}$$

$g'(x_i)$:

$$\frac{\mathrm{d}}{\mathrm{d}x} g(x) = 2ax$$

PDF:

$$p_Y(y) = \frac{p_X\left(\sqrt{\frac{y-b}{a}}\right)}{2a\sqrt{\frac{y-b}{a}}} + \frac{p_X\left(-\sqrt{\frac{y-b}{a}}\right)}{2a\sqrt{\frac{y-b}{a}}}$$

---

c) A simple multi–dimensional case

- Given:
  * RVs X_i, $1 \leq i \leq n$ with joint PDF $p_X(x_1, x_2, \ldots, x_n)$
  * Transformation: $Y_i = g_i(x_1, x_2, \ldots, x_n)$, $1 \leq i \leq n$
- Problem: Calculate $p_Y(y_1, y_2, \ldots, y_n)$
- Simplifying assumptions for $g_i(x_1, x_2, \ldots, x_n)$, $1 \leq i \leq n$
  * $g_i(x_1, x_2, \ldots, x_n)$, $1 \leq i \leq n$, have continuous partial derivatives

* $g_i(x_1, x_2, \dots, x_n)$, $1 \le i \le n$, are invertible, i.e.,

$$X_i = g_i^{-1}(Y_1, Y_2, \dots, Y_n), \qquad 1 \le i \le n$$

– PDF:

$$\boxed{p_Y(y_1, y_2, \dots, y_n) = p_X(x_1 = g_1^{-1}, \dots, x_n = g_n^{-1}) \cdot |\boldsymbol{J}|}$$

with

* $g_i^{-1} = g_i^{-1}(y_1, y_2, \dots, y_n)$
* Jacobian of transformation

$$\boldsymbol{J} = \begin{bmatrix} \frac{\partial\left(g_1^{-1}\right)}{\partial y_1} & \dots & \frac{\partial\left(g_n^{-1}\right)}{\partial y_1} \\ \vdots & & \vdots \\ \frac{\partial\left(g_1^{-1}\right)}{\partial y_n} & \dots & \frac{\partial\left(g_n^{-1}\right)}{\partial y_n} \end{bmatrix}$$

* $|\boldsymbol{J}|$: Determinant of matrix $\boldsymbol{J}$

d) Sum of two RVs X_1 and X_2

$$Y = X_1 + X_2$$

– Given: $p_{X_1X_2}(x_1, x_2)$

– Problem: Find $p_Y(y)$

– Solution:
From $x_1 = y - x_2$ we obtain the joint PDF of Y and X_2

$$\Rightarrow p_{YX_2}(y, x_2) = p_{X_1X_2}(x_1, x_2)\Big|_{x_1 = y - x_2} = p_{X_1X_2}(y - x_2, x_2)$$

$p_Y(y)$ is a marginal density of $p_{YX_2}(y, x_2)$:

$$\Rightarrow p_Y(y) = \int_{-\infty}^{\infty} p_{YX_2}(y, x_2)\,\mathrm{d}x_2 = \int_{-\infty}^{\infty} p_{X_1X_2}(y - x_2, x_2)\,\mathrm{d}x_2 = \int_{-\infty}^{\infty} p_{X_1X_2}(x_1, y - x_1)\,\mathrm{d}x_1$$

– Important special case: X_1 and X_2 are statistically independent

$$p_{X_1X_2}(x_1, x_2) = p_{X_1}(x_1)\, p_{X_2}(x_2)$$

$$\boxed{p_Y(y) = \int_{-\infty}^{\infty} p_{X_1}(x_1) p_{X_2}(y - x_1)\,\mathrm{d}x_1 = p_{X_1}(x_1) * p_{X_2}(x_2)}$$

The PDF of Y is simply the convolution of the PDFs of X_1 and X_2.

2.1.4 Statistical Averages of RVs

- Important for characterization of random variables.

General Case

- Given:
  - RV $Y = g(\boldsymbol{X})$ with random vector $\boldsymbol{X} = (X_1, X_2, \ldots, X_n)$
  - (Joint) PDF $p_{\boldsymbol{X}}(\boldsymbol{x})$ of $\boldsymbol{X}$

- Expected value of Y:

$$\mathcal{E}\{Y\} = \mathcal{E}\{g(\boldsymbol{X})\} = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} g(\boldsymbol{x})\, p_{\boldsymbol{X}}(\boldsymbol{x})\, \mathrm{d}x_1 \ldots \mathrm{d}x_n$$

$\mathcal{E}\{\cdot\}$ denotes statistical averaging.

Special Cases (one–dimensional): $\boldsymbol{X} = X_1 = X$

- Mean: $g(X) = X$

$$m_X = \mathcal{E}\{X\} = \int_{-\infty}^{\infty} x\, p_X(x)\, \mathrm{d}x$$

- nth moment: $g(X) = X^n$

$$\mathcal{E}\{X^n\} = \int_{-\infty}^{\infty} x^n\, p_X(x)\, \mathrm{d}x$$

- nth central moment: $g(X) = (X - m_X)^n$

$$\mathcal{E}\{(X - m_X)^n\} = \int\limits_{-\infty}^{\infty} (x - m_X)^n \, p_X(x) \, \mathrm{d}x$$

- Variance: 2nd central moment

$$\begin{aligned}
\sigma_X^2 &= \int\limits_{-\infty}^{\infty} (x - m_X)^2 \, p_X(x) \, \mathrm{d}x \\
&= \int\limits_{-\infty}^{\infty} x^2 \, p_X(x) \, \mathrm{d}x + \int\limits_{-\infty}^{\infty} m_X^2 \, p_X(x) \, \mathrm{d}x - 2 \int\limits_{-\infty}^{\infty} m_X \, x \, p_X(x) \, \mathrm{d}x \\
&= \int\limits_{-\infty}^{\infty} x^2 \, p_X(x) \, \mathrm{d}x - m_X^2 \\
&= \mathcal{E}\{X^2\} - (\mathcal{E}\{X\})^2
\end{aligned}$$

Complex case: $\sigma_X^2 = \mathcal{E}\{|X|^2\} - |\mathcal{E}\{X\}|^2$

- Characteristic function: $g(X) = e^{jvX}$

$$\psi(jv) = \mathcal{E}\{e^{jvX}\} = \int\limits_{-\infty}^{\infty} e^{jvx} \, p_X(x) \, \mathrm{d}x$$

Some properties of $\psi(jv)$:

– $\psi(jv) = G(-jv)$, where $G(jv)$ denotes the *Fourier transform* of $p_X(x)$

– $p_X(x) = \frac{1}{2\pi} \int\limits_{-\infty}^{\infty} \psi(jv)\, e^{-jvX}\, \mathrm{d}v$

– $\mathcal{E}\{X^n\} = (-j)^n \left. \frac{\mathrm{d}^n\, \psi(jv)}{\mathrm{d}v^n} \right|_{v=0}$

Given $\psi(jv)$ we can easily calculate the nth moment of X.

– **Application**: Calculation of PDF of sum $Y = X_1 + X_2$ of statistically independent RVs X_1 and X_2

 ∗ Given: $p_{X_1}(x_1)$, $p_{X_2}(x_2)$ or equivalently $\psi_{X_1}(jv)$, $\psi_{X_2}(jv)$

 ∗ Problem: Find $p_Y(y)$ or equivalently $\psi_Y(jv)$

 ∗ Solution:

$$\begin{aligned}
\psi_Y(jv) &= \mathcal{E}\{e^{jvY}\} \\
&= \mathcal{E}\{e^{jv(X_1+X_2)}\} \\
&= \mathcal{E}\{e^{jvX_1}\}\, \mathcal{E}\{e^{jvX_2}\} \\
&= \psi_{X_1}(jv)\, \psi_{X_2}(jv)
\end{aligned}$$

$\psi_Y(jv)$ is simply product of $\psi_{X_1}(jv)$ and $\psi_{X_2}(jv)$. This result is not surprising since $p_Y(y)$ is the convolution of $p_{X_1}(x_1)$ and $p_{X_1}(x_2)$ (see Section 2.1.3).

Special Cases (multi–dimensional)

- Joint higher order moments $g(X_1, X_2) = X_1^k X_2^n$

$$\mathcal{E}\{X_1^k X_2^n\} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1^k\, x_2^n\, p_{X_1X_2}(x_1, x_2)\,\mathrm{d}x_1\,\mathrm{d}x_2$$

  Special case $k = n = 1$: $\rho_{X_1X_2} = \mathcal{E}\{X_1X_2\}$ is called the correlation between X_1 and X_2.

- Covariance (complex case): $g(X_1, X_2) = (X_1 - m_{X_1})(X_2 - m_{X_2})^*$

$$\begin{aligned}\mu_{X_1X_2} &= \mathcal{E}\{(X_1 - m_{X_1})(X_2 - m_{X_2})^*\} \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x_1 - m_{X_1})\,(x_2 - m_{X_2})^*\, p_{X_1X_2}(x_1, x_2)\,\mathrm{d}x_1\,\mathrm{d}x_2 \\ &= \mathcal{E}\{X_1X_2^*\} - \mathcal{E}\{X_1\}\,\mathcal{E}\{X_2^*\}\end{aligned}$$

  m_{X_1} and m_{X_2} denote the means of X_1 and X_2, respectively.

  X_1 and X_2 are uncorrelated if $\mu_{X_1X_2} = 0$ is valid.

- Autocorrelation matrix of random vector $\boldsymbol{X} = (X_1,\, X_2,\, \ldots,\, X_n)^T$

$$\boldsymbol{R} = \mathcal{E}\{\boldsymbol{X}\,\boldsymbol{X}^H\}$$

  ^{H} is the Hermitian operator and means transposition and conjugation.

- Covariance matrix

$$\begin{aligned} \boldsymbol{M} &= \mathcal{E}\{(\boldsymbol{X}-\boldsymbol{m}_{\boldsymbol{X}})\,(\boldsymbol{X}-\boldsymbol{m}_{\boldsymbol{X}})^H\} \\ &= \boldsymbol{R}-\boldsymbol{m}_{\boldsymbol{X}}\boldsymbol{m}_{\boldsymbol{X}}^H \end{aligned}$$

with mean vector $\boldsymbol{m}_{\boldsymbol{X}} = \mathcal{E}\{\boldsymbol{X}\}$

- Characteristic function (two–dimensional case): $g(X_1, X_2) = e^{j(v_1X_1+v_2X_2)}$

$$\psi(jv_1, jv_2) = \int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} e^{j(v_1x_1+v_2x_2)}\, p_{X_1X_2}(x_1, x_2)\,\mathrm{d}x_1\,\mathrm{d}x_2$$

$\psi(jv_1, jv_2)$ can be applied to calculate the joint (higher order) moments of X_1 and X_2.

E.g. $\mathcal{E}\{X_1X_2\} = -\left.\frac{\partial^2\psi(jv_1, jv_2)}{\partial v_1\partial v_2}\right|_{v_1=v_2=0}$

2.1.5 Gaussian Distribution

The Gaussian distribution is the most important probability distribution in practice:

- Many physical phenomena can be described by a Gaussian distribution.
- Often we also *assume* that a certain RV has a Gaussian distribution in order to render a problem mathematical tractable.

Real One–dimensional Case

- PDF of Gaussian RV X with mean m_X and variance σ^2

$$p(x) = \frac{1}{\sqrt{2\pi}\sigma}\, e^{-(x-m_X)^2/(2\sigma^2)}$$

  Note: The Gaussian PDF is fully characterized by its first and second order moments!

- CDF

$$\begin{aligned} F(x) &= \int\limits_{-\infty}^{x} p(u)\,\mathrm{d}u = \frac{1}{\sqrt{2\pi}\sigma} \int\limits_{-\infty}^{x} e^{-(u-m_X)^2/(2\sigma^2)}\,\mathrm{d}u \\ &= \frac{1}{2}\frac{2}{\sqrt{\pi}} \int\limits_{-\infty}^{\frac{x-m_X}{\sqrt{2}\sigma}} e^{-t^2}\,\mathrm{d}t = \frac{1}{2} + \frac{1}{2}\,\mathrm{erf}\left(\frac{x-m_X}{\sqrt{2}\sigma}\right) \end{aligned}$$

with the *error function*

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int\limits_0^x e^{-t^2}\,\mathrm{d}t$$

Alternatively, we can express the CDF of a Gaussian RV in terms of the *complementary* error function:

$$F(x) = 1 - \frac{1}{2}\,\mathrm{erfc}\left(\frac{x - m_X}{\sqrt{2}\sigma}\right)$$

with

$$\begin{aligned} \mathrm{erfc}(x) &= \frac{2}{\sqrt{\pi}} \int\limits_x^\infty e^{-t^2}\,\mathrm{d}t \\ &= 1 - \mathrm{erf}(x) \end{aligned}$$

- Gaussian Q–function

  The integral over the tail $[x,\ \infty)$ of a *normal* distribution (= Gaussian distribution with $m_X = 0$, $\sigma^2 = 1$) is referred to as the Gaussian Q–function:

$$Q(x) = \frac{1}{\sqrt{2\pi}} \int\limits_x^\infty e^{-t^2/2}\,\mathrm{d}t$$

  The Q–function often appears in analytical expressions for error probabilities for detection in AWGN.

  The Q–function can be also expressed as

$$Q(x) = \frac{1}{2}\,\mathrm{erfc}\left(\frac{x}{\sqrt{2}}\right)$$

Sometimes it is also useful to express the Q–function as

$$Q(x) = \frac{1}{\pi} \int\limits_{0}^{\pi/2} \exp\left(-\frac{x^2}{2\sin^2\Theta}\right) \mathrm{d}\Theta.$$

The main advantage of this representation is that the integral has finite limits and does not depend on x. This is sometimes useful in error rate analysis, especially for fading channels.

- Characteristic function

$$\begin{aligned}
\psi(jv) &= \int\limits_{-\infty}^{\infty} e^{jvx}\, p(x)\, \mathrm{d}x \\
&= \int\limits_{-\infty}^{\infty} e^{jvx} \left[\frac{1}{\sqrt{2\pi}\sigma} e^{(x-m_X)^2/(2\sigma^2)}\right] \mathrm{d}x \\
&= e^{jvm_X - v^2\sigma^2/2}
\end{aligned}$$

- Moments

Central moments:

$$\mathcal{E}\{(X-m_X)^k\} = \mu_k = \begin{cases} 1 \cdot 3 \cdots (k-1)\sigma^k & \text{even}\, k \\ 0 & \text{odd}\, k \end{cases}$$

Non–central moments

$$\mathcal{E}\{X^k\} = \sum_{i=0}^{k} \binom{k}{i} m_X^i \mu_{k-i}$$

Note: All higher order moments of a Gaussian RV can be expressed in terms of its first and second order moments.

- Sum of n statistically independent RVs $X_1, X_2, \ldots, X_n$

$$Y = \sum_{i=1}^{n} X_i$$

X_i has mean m_i and variance σ_i^2.

Characteristic function:

$$\begin{aligned}\psi_Y(jv) &= \prod_{i=1}^{n} \psi_{X_i}(jv) \\ &= \prod_{i=1}^{n} e^{jvm_i - v^2\sigma_i^2/2} \\ &= e^{jvm_Y - v^2\sigma_Y^2/2}\end{aligned}$$

with

$$\begin{aligned}m_Y &= \sum_{i=1}^{n} m_i \\ \sigma_Y^2 &= \sum_{i=1}^{n} \sigma_i^2\end{aligned}$$

$\Rightarrow$ The sum of statistically independent Gaussian RVs is also a Gaussian RV. Note that the same statement is true for the sum of statistical dependent Gaussian RVs.

Real Multi–dimensional (Multi–variate) Case

- Given:
  - Vector $\boldsymbol{X} = (X_1, X_2, \dots, X_n)^T$ of n Gaussian RVs
  - Mean vector $\boldsymbol{m_X} = \mathcal{E}\{\boldsymbol{X}\}$
  - Covariance matrix $\boldsymbol{M} = \mathcal{E}\{(\boldsymbol{X} - \boldsymbol{m_X})(\boldsymbol{X} - \boldsymbol{m_X})^H\}$

- PDF

$$p(\boldsymbol{x}) = \frac{1}{(2\pi)^{n/2}\sqrt{|\boldsymbol{M}|}} \exp\left(-\frac{1}{2}(\boldsymbol{x} - \boldsymbol{m}_X)^T \boldsymbol{M}^{-1}(\boldsymbol{x} - \boldsymbol{m}_X)\right)$$

- Special case: $n = 2$

$$\boldsymbol{m_X} = \begin{bmatrix} m_1 \\ m_2 \end{bmatrix}, \qquad \boldsymbol{M} = \begin{bmatrix} \sigma_1^2 & \mu_{12} \\ \mu_{12} & \sigma_2^2 \end{bmatrix}$$

with the joint central moment

$$\mu_{12} = \mathcal{E}\{(X_1 - m_1)(X_2 - m_2)\}$$

Using the normalized covariance $\rho = \mu_{12}/(\sigma_1\sigma_2)$, $0 \le \rho \le 1$, we get

$$p(x_1, x_2) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}}$$

$$\cdot \exp\left(-\frac{\sigma_2^2(x_1-m_1)^2 - 2\rho\sigma_1\sigma_2(x_1-m_1)(x_2-m_2) + \sigma_1^2(x_2-m_2)^2}{2\sigma_1^2\sigma_2^2(1-\rho^2)}\right)$$

Observe that for $\rho = 0$ (X_1 and X_2 uncorrelated) the joint PDF can be factored into $p(x_1, x_2) = p_{X_1}(x_1) \cdot p_{X_2}(x_2)$. This means that two *uncorrelated* Gaussian RVs are also *statistically independent*. Note that this is *not* true for other distributions. On the other hand, statistically independent RVs are always uncorrelated.

- Linear transformation
  - Given: Linear transformation $\boldsymbol{Y} = \boldsymbol{A}\,\boldsymbol{X}$, where $\boldsymbol{A}$ denotes a non–singular matrix
  - Problem: Find $p_{\boldsymbol{Y}}(\boldsymbol{y})$
  - Solution: With $\boldsymbol{X} = \boldsymbol{A}^{-1}\boldsymbol{Y}$ and the Jacobian $\boldsymbol{J} = \boldsymbol{A}^{-1}$ of the linear transformation, we get (see Section 2.1.3)

$$\begin{aligned} p_{\boldsymbol{Y}}(\boldsymbol{y}) &= \frac{1}{(2\pi)^{n/2}\sqrt{|\boldsymbol{M}|}|\boldsymbol{A}|} \\ &\quad \cdot \exp\left(-\frac{1}{2}(\boldsymbol{A}^{-1}\boldsymbol{y} - \boldsymbol{m}_{\boldsymbol{X}})^T \boldsymbol{M}^{-1}(\boldsymbol{A}^{-1}\boldsymbol{y} - \boldsymbol{m}_{\boldsymbol{X}})\right) \\ &= \frac{1}{(2\pi)^{n/2}\sqrt{|\boldsymbol{Q}|}} \exp\left(-\frac{1}{2}(\boldsymbol{y} - \boldsymbol{m}_{\boldsymbol{Y}})^T \boldsymbol{Q}^{-1}(\boldsymbol{y} - \boldsymbol{m}_{\boldsymbol{Y}})\right) \end{aligned}$$

where vector $\boldsymbol{m_Y}$ and matrix $\boldsymbol{Q}$ are defined as

$$\begin{aligned}\boldsymbol{m_Y} &= \boldsymbol{A}\boldsymbol{m_X} \\ \boldsymbol{Q} &= \boldsymbol{A}\boldsymbol{M}\boldsymbol{A}^T\end{aligned}$$

We obtain the important result that a linear transformation of a vector of jointly Gaussian RVs results in another vector of jointly Gaussian RVs!

Complex One–dimensional Case

- Given: $Z = X + jY$, where X and Y are two Gaussian random variables with means m_X and m_Y, and variances σ_X^2 and σ_Y^2, respectively

- Most important case: X and Y are uncorrelated and $\sigma_X^2 = \sigma_Y^2 = \sigma^2$ (in this case, Z is also referred to as a *proper* Gaussian RV)

- PDF

$$\begin{aligned}p_Z(z) &= p_{XY}(x,y) = p_X(x)\, p_Y(y) \\ &= \frac{1}{\sqrt{2\pi}\sigma}\, e^{-(x-m_X)^2/(2\sigma^2)} \cdot \frac{1}{\sqrt{2\pi}\sigma}\, e^{-(y-m_Y)^2/(2\sigma^2)} \\ &= \frac{1}{2\pi\sigma^2}\, e^{-\left((x-m_X)^2+(y-m_Y)^2\right)/(2\sigma^2)} \\ &= \frac{1}{\pi\sigma_Z^2}\, e^{-|z-m_Z|^2/\sigma_Z^2}\end{aligned}$$

with

$$m_Z = \mathcal{E}\{Z\} = m_X + jm_Y$$

and

$$\sigma_Z^2 = \mathcal{E}\{|Z - m_Z|^2\} = \sigma_X^2 + \sigma_Y^2 = 2\sigma^2$$

Complex Multi–dimensional Case

- Given: Complex vector $\boldsymbol{Z} = \boldsymbol{X} + j\boldsymbol{Y}$, where $\boldsymbol{X}$ and $\boldsymbol{Y}$ are two real jointly Gaussian vectors with mean vectors $\boldsymbol{m}_X$ and $\boldsymbol{m}_Y$ and covariance matrices $\boldsymbol{M}_X$ and $\boldsymbol{M}_Y$, respectively

- Most important case: $\boldsymbol{X}$ and $\boldsymbol{Y}$ are uncorrelated and $\boldsymbol{M}_X = \boldsymbol{M}_Y$ (proper complex random vector)

- PDF

$$p_Z(\boldsymbol{z}) = \frac{1}{\pi^n\,|\boldsymbol{M}_Z|}\exp\left(-(\boldsymbol{z} - \boldsymbol{m}_Z)^H \boldsymbol{M}_Z^{-1}(\boldsymbol{z} - \boldsymbol{m}_Z)\right)$$

with

$$\boldsymbol{m}_Z = \mathcal{E}\{\boldsymbol{z}\} = \boldsymbol{m}_X + j\boldsymbol{m}_Y$$

and

$$\boldsymbol{M}_Z = \mathcal{E}\{(\boldsymbol{z} - \boldsymbol{m}_Z)(\boldsymbol{z} - \boldsymbol{m}_Z)^H\} = \boldsymbol{M}_X + \boldsymbol{M}_Y$$

2.1.6 Chernoff Upper Bound on the Tail Probability

- The "tail probability" (area under the tail of PDF) often has to be evaluated to determine the error probability of digital communication systems

- Closed–form results are often not feasible ⇒ the simple Chernoff upper bound can be used for system design and/or analysis

- Chernoff Bound

  The tail probability is given by

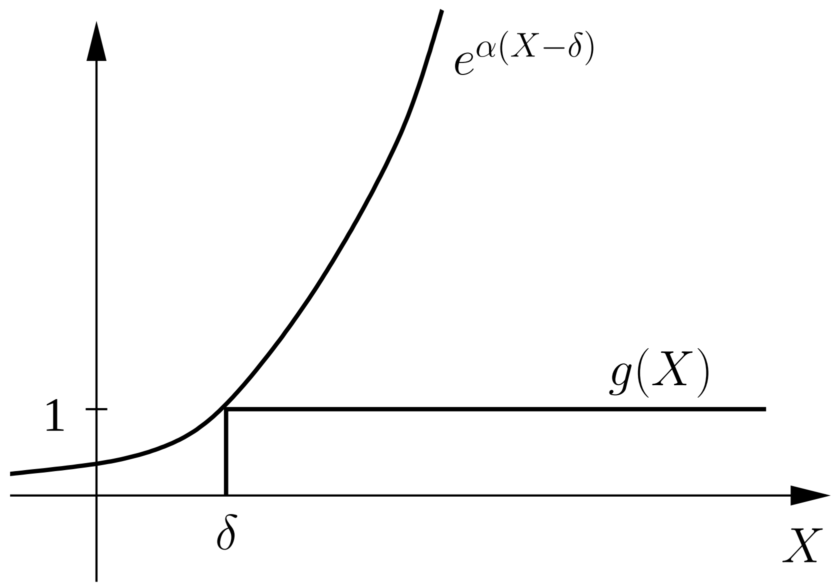
$$\begin{aligned} P(X \geq \delta) &= \int\limits_{\delta}^{\infty} p(x)\,\mathrm{d}x \\ &= \int\limits_{-\infty}^{\infty} g(x)\,p(x)\,\mathrm{d}x \\ &= \mathcal{E}\{g(X)\} \end{aligned}$$

  where we use the definition

$$g(X) = \begin{cases} 1, & X \geq \delta \\ 0, & X < \delta \end{cases}$$

  .

  Obviously $g(X)$ can be upper bounded by $g(X) \leq e^{\alpha(X-\delta)}$ with $\alpha \geq 0$.



Therefore, we get the bound

$$\begin{aligned} P(X \geq \delta) &= \mathcal{E}\{g(X)\} \\ &\leq \mathcal{E}\{e^{\alpha(X-\delta)}\} \\ &= e^{-\alpha\,\delta}\,\mathcal{E}\{e^{\alpha\,X}\} \end{aligned}$$

which is valid for *any* $\alpha \geq 0$. In practice, however, we are interested in the *tightest* upper bound. Therefore, we optimize α:

$$\frac{\mathrm{d}}{\mathrm{d}\alpha}\, e^{-\alpha\,\delta}\,\mathcal{E}\{e^{\alpha\,X}\} = 0$$

The optimum $\alpha = \alpha_{\mathrm{opt}}$ can be obtained from

$$\mathcal{E}\{X\, e^{\alpha_{\mathrm{opt}}\,X}\} - \delta\,\mathcal{E}\{e^{\alpha_{\mathrm{opt}}\,X}\} = 0$$

The solution to this equation gives the Chernoff bound

$$\boxed{P(X \geq \delta) \leq e^{-\alpha_{\mathrm{opt}}\,\delta}\,\mathcal{E}\{e^{\alpha_{\mathrm{opt}}\,X}\}}$$

2.1.7 Central Limit Theorem

- Given: n statistical independent and identically distributed RVs X_i, $i = 1, 2, \ldots, n$, with finite varuance. For simplicity, we assume that the X_i have zero mean and identical variances σ_X^2. Note that the X_i can have any PDF.

- We consider the sum

$$Y = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} X_i$$

- Central Limit Theorem

$$\boxed{\text{For } n \to \infty \; Y \text{ is a } \textit{Gaussian} \text{ RV with zero mean and variance } \sigma_X^2.}$$

  Proof: See Textbook

- In practice, already for small n (e.g. $n = 5$) the distribution of Y is very close to a Gaussian PDF.

- In practice, it is not necessary that all X_i have exactly the same PDF and the same variance. Also the statistical independence of different X_i is not necessary. If the PDFs and the variances of the X_i are similar, for sufficiently large n the PDF of Y can be approximated by a Gaussian PDF.

- The central limit theorem explains why many physical phenomena follow a Gaussian distribution.

2.2 Stochastic Processes

- In communications many phenomena (noise from electronic devises, transmitted symbol sequence, etc.) can be described as RVs $X(t)$ that depend on (continuous) time t. $X(t)$ is referred to as a *stochastic process*.

- A single realization of $X(t)$ is a *sample function*. E.g. measurement of noise voltage generated by a particular resistor.

- The collection of all sample functions is the *ensemble* of sample functions. Usually, the size of the ensemble is infinite.

- If we consider the specific time instants $t_1 > t_2 > \ldots > t_n$ with the arbitrary positive integer index n, the random variables $X_{t_i} = X(t_i)$, $i = 1, 2, \ldots, n$, are fully characterized by their joint PDF $p(x_{t_1}, x_{t_2}, \ldots, x_{t_n})$.

- Stationary stochastic process:

  Consider a second set $X_{t_i+\tau} = X(t_i+\tau)$, $i = 1, 2, \ldots, n$, of RVs, where τ is an arbitrary time shift. If X_{t_i} and $X_{t_i+\tau}$ have the same statistical properties, $X(t)$ is *stationary in the strict sense*. In this case,

  $$p(x_{t_1}, x_{t_2}, \ldots, x_{t_n}) = p(x_{t_1+\tau}, x_{t_2+\tau}, \ldots, x_{t_n+\tau})$$

  is true, where $p(x_{t_1+\tau}, x_{t_2+\tau}, \ldots, x_{t_n+\tau})$ denotes the joint PDF of the RVs $X_{t_i+\tau}$. If X_{t_i} and $X_{t_i+\tau}$ do not have the same statistical properties, the process $X(t)$ is *nonstationary*.

2.2.1 Statistical Averages

- Statistical averages (= ensemble averages) of stochastic processes are defined as averages with respect to the RVs $X_{t_i} = X(t_i)$.

- First order moment (mean):

$$m(t_i) = \mathcal{E}\{X_{t_i}\} = \int\limits_{-\infty}^{\infty} x_{t_i}\, p(x_{t_i})\, \mathrm{d}x_{t_i}$$

  For a stationary processes $m(t_i) = m$ is valid, i.e., the mean does not depend on time.

- Second order moment: *Autocorrelation function (ACF)* $\phi(t_1, t_2)$

$$\phi(t_1, t_2) = \mathcal{E}\{X_{t_1} X_{t_2}\} = \int\limits_{-\infty}^{\infty} \int\limits_{-\infty}^{\infty} x_{t_1}\, x_{t_2}\, p(x_{t_1}, x_{t_2})\, \mathrm{d}x_{t_1} \mathrm{d}x_{t_2}$$

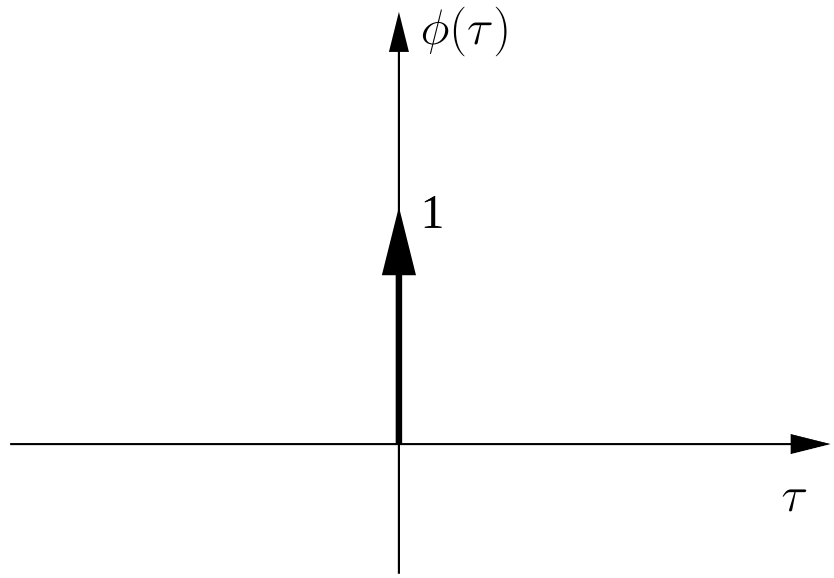
  For a stationary process $\phi(t_1, t_2)$ does not depend on the specific time instances t_1, t_2, but on the difference $\tau = t_1 - t_2$:

$$\mathcal{E}\{X_{t_1} X_{t_2}\} = \phi(t_1, t_2) = \phi(t_1 - t_2) = \phi(\tau)$$

  Note that $\phi(\tau) = \phi(-\tau)$ ($\phi(\cdot)$ is an even function) since $\mathcal{E}\{X_{t_1} X_{t_2}\} = \mathcal{E}\{X_{t_2} X_{t_1}\}$ is valid.

Example:

ACF of an uncorrelated stationary process: $\phi(\tau) = \delta(\tau)$



- Central second order moment: *Covariance function* $\mu(t_1, t_2)$

$$\begin{aligned} \mu(t_1, t_2) &= \mathcal{E}\{(X_{t_1} - m(t_1))(X_{t_2} - m(t_2))\} \\ &= \phi(t_1, t_2) - m(t_1)m(t_2) \end{aligned}$$

For a stationary processes we get

$$\mu(t_1, t_2) = \mu(t_1 - t_2) = \mu(\tau) = \phi(\tau) - m^2$$

- Stationary stochastic processes are asymptotically uncorrelated, i.e.,

$$\lim_{\tau \to \infty} \mu(\tau) = 0$$

- Average power of stationary process:

$$\mathcal{E}\{X_t^2\} = \phi(0)$$

- Variance of stationary processes:

$$\mathcal{E}\{(X_t - m)^2\} = \mu(0)$$

- Wide–sense stationarity:

  If the first and second order moments of a stochastic process are invariant to any time shift τ, the process is referred to as *wide–sense stationary* process. Wide–sense stationary processes are not necessarily stationary in the strict sense.

- Gaussian process:

  Since Gaussian RVs are fully specified by their first and second order moments, in this special case wide–sense stationarity automatically implies stationarity in the strict sense.

- Ergodicity:

  We refer to a process $X(t)$ as *ergodic* if its statistical averages can also be calculated as time–averages of sample functions. Only (wide–sense) stationary processes can be ergodic.

  For example, if $X(t)$ is ergodic and one of its sample functions (i.e., one of its realizations) is denoted as $x(t)$, the mean and the

ACF can be calulated as

$$m = \lim_{T\to\infty} \frac{1}{2T} \int\limits_{-T}^{T} x(t)\,\mathrm{d}t$$

and

$$\phi(\tau) = \lim_{T\to\infty} \frac{1}{2T} \int\limits_{-T}^{T} x(t)x(t+\tau)\,\mathrm{d}t,$$

respectively. In practice, it is usually *assumed* that a process is (wide–sense) stationary and ergodic. Ergodicity is important since in practice only *sample functions* of a stochastic process can be observed!

Averages for Jointly Stochastic Processes

- Let $X(t)$ and $Y(t)$ denote two stochastic processes and consider the RVs $X_{t_i} = X(t_i)$, $i = 1, 2, \ldots, n$, and $Y_{t'_j} = Y(t'_j)$, $j = 1, 2, \ldots, m$ at times $t_1 > t_2 > \ldots > t_n$ and $t'_1 > t'_2 > \ldots > t'_m$, respectively. The two stochastic processes are fully characterized by their joint PDF

$$p(x_{t_1}, x_{t_2}, \ldots, x_{t_n}, y_{t'_1}, y_{t'_2}, \ldots, y_{t'_m})$$

- Joint stationarity: $X(t)$ and $Y(t)$ are jointly stationary if their joint PDF is invariant to time shifts τ for all n and m.

- Cross–correlation function (CCF): $\phi_{XY}(t_1, t_2)$

$$\phi_{XY}(t_1, t_2) = \mathcal{E}\{X_{t_1} Y_{t_2}\} = \int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} x_{t_1}\, y_{t_2}\, p(x_{t_1}, y_{t_2})\, \mathrm{d}x_{t_1} \mathrm{d}y_{t_2}$$

If $X(t)$ and $Y(t)$ are jointly and individually stationary, we get

$$\phi_{XY}(t_1, t_2) = \mathcal{E}\{X_{t_1} Y_{t_2}\} = \mathcal{E}\{X_{t_2+\tau} Y_{t_2}\} = \phi_{XY}(\tau)$$

with $\tau = t_1 - t_2$. We can establish the symmetry relation

$$\phi_{XY}(-\tau) = \mathcal{E}\{X_{t_2-\tau} Y_{t_2}\} = \mathcal{E}\{Y_{t_2'+\tau} X_{t_2'}\} = \phi_{YX}(\tau)$$

- Cross–covariance function $\mu_{XY}(t_1, t_2)$

$$\begin{aligned} \mu_{XY}(t_1, t_2) &= \mathcal{E}\{(X_{t_1} - m_X(t_1))(Y_{t_2} - m_Y(t_2))\} \\ &= \phi_{XY}(t_1, t_2) - m_X(t_1) m_Y(t_2) \end{aligned}$$

If $X(t)$ and $Y(t)$ are jointly and individually stationary, we get

$$\mu_{XY}(t_1, t_2) = \mathcal{E}\{(X_{t_1} - m_X)(Y_{t_2} - m_Y)\} = \mu_{XY}(\tau)$$

with $\tau = t_1 - t_2$.

- Statistical independence

  Two processes $X(t)$ and $Y(t)$ are statistical independent if and only if

$$p(x_{t_1}, x_{t_2}, \dots, x_{t_n}, y_{t'_1}, y_{t'_2}, \dots, y_{t'_m}) = p(x_{t_1}, x_{t_2}, \dots, x_{t_n})\, p(y_{t'_1}, y_{t'_2}, \dots, y_{t'_m})$$

  is valid for all n and m.

- Uncorrelated processes

  Two processes $X(t)$ and $Y(t)$ are uncorrelated if and only if

$$\mu_{XY}(t_1, t_2) = 0$$

  holds.

Complex Stochastic Processes

- Given: Complex random process $Z(t) = X(t) + jY(t)$ with real random processes $X(t)$ and $Y(t)$

- Similarly to RVs, we treat $Z(t)$ as a tupel of $X(t)$ and $Y(t)$, i.e., the PDF of $Z_{t_i} = Z(t_i)$, 1, 2, $\dots$, n is given by

$$p_Z(z_{t_1}, z_{t_2}, \dots, z_{t_n}) = p_{XY}(x_{t_1}, x_{t_2}, \dots, x_{t_n}, y_{t_1}, y_{t_2}, \dots, y_{t_n})$$

- We define the *ACF* of a complex–valued stochastic process $Z(t)$

as

$$\begin{aligned}\phi_{ZZ}(t_1,t_2) &= \mathcal{E}\{Z_{t_1}Z^*_{t_2}\}\\ &= \mathcal{E}\{(X_{t_1}+jY_{t_1})(X_{t_2}-jY_{t_2})\}\\ &= \phi_{XX}(t_1,t_2)+\phi_{YY}(t_1,t_2)+j(\phi_{YX}(t_1,t_2)-\phi_{XY}(t_1,t_2))\end{aligned}$$

where $\phi_{XX}(t_1,t_2)$, $\phi_{YY}(t_1,t_2)$ and $\phi_{YX}(t_1,t_2)$, $\phi_{XY}(t_1,t_2)$ denote the ACFs and the CCFs of $X(t)$ and $Y(t)$, respectively.

Note that our definition of $\phi_{ZZ}(t_1,t_2)$ differs from the Textbook, where $\phi_{ZZ}(t_1,t_2)=\frac{1}{2}\mathcal{E}\{Z_{t_1}Z^*_{t_2}\}$ is used!

If $Z(t)$ is a stationary process we get

$$\phi_{ZZ}(t_1,t_2)=\phi_{ZZ}(t_2+\tau,t_2)=\phi_{ZZ}(\tau)$$

with $\tau=t_1-t_2$.

We can also establish the symmetry relation

$$\phi_{ZZ}(\tau)=\phi^*_{ZZ}(-\tau)$$

- CCF of processes $Z(t)$ and $W(t)$

$$\phi_{ZW}(t_1,t_2)=\mathcal{E}\{Z_{t_1}W^*_{t_2}\}$$

If $Z(t)$ and $W(t)$ are jointly and individually stationary we have

$$\phi_{ZW}(t_1,t_2)=\phi_{ZW}(t_2+\tau,t_2)=\phi_{ZW}(\tau)$$

Symmetry:

$$\phi^*_{ZW}(\tau)=\mathcal{E}\{Z^*_{t_2+\tau}W_{t_2}\}=\mathcal{E}\{W_{t'_2-\tau}Z^*_{t'_2}\}=\phi_{WZ}(-\tau)$$

2.2.2 Power Density Spectrum

- The Fourier spectrum of a random process does not exist.

- Instead we define the *power* spectrum of a stationary stochastic process as the Fourier transform $\mathcal{F}\{\cdot\}$ of the ACF

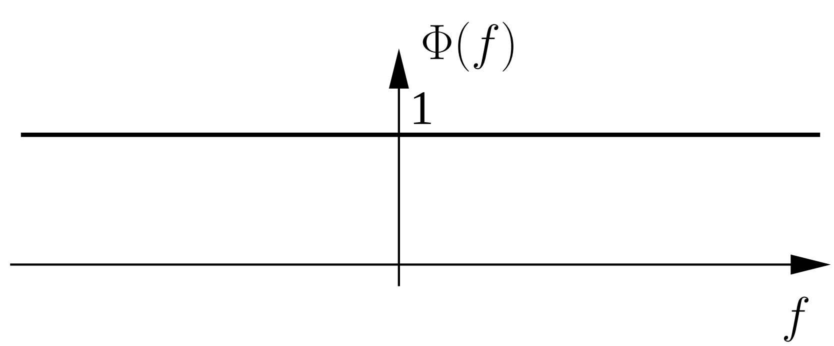
$$\boxed{\Phi(f) = \mathcal{F}\{\phi(\tau)\} = \int\limits_{-\infty}^{\infty} \phi(\tau)\, e^{-j2\pi f\tau}\, \mathrm{d}\tau}$$

Consequently, the ACF can be obtained from the power spectrum (also referred to as power spectral density) via inverse Fourier transform $\mathcal{F}^{-1}\{\cdot\}$ as

$$\boxed{\phi(\tau) = \mathcal{F}^{-1}\{\Phi(f)\} = \int\limits_{-\infty}^{\infty} \Phi(f)\, e^{j2\pi f\tau}\, \mathrm{d}f}$$

Example:

Power spectrum of an uncorrelated stationary process:
$\Phi(f) = \mathcal{F}\{\delta(\tau)\} = 1$



- The average power of a stationary stochastic process can be obtained as

$$\begin{aligned}\phi(0) &= \int_{-\infty}^{\infty} \Phi(f)\,\mathrm{d}f \\ &= \mathcal{E}\{|X_t|^2\} \geq 0\end{aligned}$$

Symmetry of power density spectrum:

$$\begin{aligned}\Phi^*(f) &= \int_{-\infty}^{\infty} \phi^*(\tau)\, e^{j2\pi f\tau}\,\mathrm{d}\tau \\ &= \int_{-\infty}^{\infty} \phi^*(-\tau)\, e^{-j2\pi f\tau}\,\mathrm{d}\tau \\ &= \int_{-\infty}^{\infty} \phi(\tau)\, e^{-j2\pi f\tau}\,\mathrm{d}\tau \\ &= \Phi(f)\end{aligned}$$

This means $\Phi(f)$ is a real–valued function.

- Cross–correlation spectrum

  Consider the random processes $X(t)$ and $Y(t)$ with CCF $\phi_{XY}(\tau)$. The cross–correlation spectrum $\Phi_{XY}(f)$ is defined as

$$\Phi_{XY}(f) = \int_{-\infty}^{\infty} \phi_{XY}(\tau)\, e^{-j2\pi f\tau}\, \mathrm{d}\tau$$

  It can be shown that the symmetry relation $\Phi^*_{XY}(f) = \Phi_{YX}(f)$ is valid. If $X(t)$ and $Y(t)$ are real stochastic processes $\Phi_{YX}(f) = \Phi_{XY}(-f)$ holds.

2.2.3 Response of a Linear Time–Invariant System to a Random Input Signal

- We consider a deterministic linear time–invariant system fully described by its impulse response $h(t)$, or equivalently by its frequency response

$$H(f) = \mathcal{F}\{h(t)\} = \int_{-\infty}^{\infty} h(t)\, e^{-j2\pi ft}\, \mathrm{d}t$$

- Let the signal $x(t)$ be the input to the system $h(t)$. Then the output $y(t)$ of the system can be expressed as

$$y(t) = \int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, \mathrm{d}\tau$$

In our case $x(t)$ is a sample function of a (stationary) stochastic process $X(t)$ and therefore, $y(t)$ is a sample function of a stochastic process $Y(t)$. We are interested in the mean and the ACF of $Y(t)$.

- Mean of $Y(t)$

$$\begin{aligned} m_Y &= \mathcal{E}\{Y(t)\} = \int\limits_{-\infty}^{\infty} h(\tau)\,\mathcal{E}\{X(t-\tau)\}\,\mathrm{d}\tau \\ &= m_X \int\limits_{-\infty}^{\infty} h(\tau)\,\mathrm{d}\tau = m_X\,H(0) \end{aligned}$$

- ACF of $Y(t)$

$$\begin{aligned} \phi_{YY}(t_1,t_2) &= \mathcal{E}\{Y_{t_1}Y_{t_2}^*\} \\ &= \int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} h(\alpha)\,h^*(\beta)\,\mathcal{E}\{X(t_1-\alpha)X^*(t_2-\beta)\}\,\mathrm{d}\alpha\,\mathrm{d}\beta \\ &= \int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} h(\alpha)\,h^*(\beta)\,\phi_{XX}(t_1-t_2+\beta-\alpha)\,\mathrm{d}\alpha\,\mathrm{d}\beta \\ &= \int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} h(\alpha)\,h^*(\beta)\,\phi_{XX}(\tau+\beta-\alpha)\,\mathrm{d}\alpha\,\mathrm{d}\beta \\ &= \phi_{YY}(\tau) \end{aligned}$$

Here, we have used $\tau = t_1 - t_2$ and the last line indicates that if the input to a linear time–invariant system is stationary, also the output will be stationary.

If we define the deterministic system ACF as

$$\phi_{hh}(\tau) = h(\tau) * h^*(-\tau) = \int_{-\infty}^{\infty} h^*(t)\, h(t+\tau)\, \mathrm{d}t,$$

where "$*$" is the convolution operator, then we can rewrite $\phi_{YY}(\tau)$ elegantly as

$$\boxed{\phi_{YY}(\tau) = \phi_{hh}(\tau) * \phi_{XX}(\tau)}$$

- Power spectral density of $Y(t)$

Since the Fourier transform of $\phi_{hh}(\tau) = h(\tau) * h^*(-\tau)$ is

$$\begin{aligned}
\Phi_{hh}(f) &= \mathcal{F}\{\phi_{hh}(\tau)\} = \mathcal{F}\{h(\tau) * h^*(-\tau)\} \\
&= \mathcal{F}\{h(\tau)\}\, \mathcal{F}\{h^*(-\tau)\} \\
&= |H(f)|^2,
\end{aligned}$$

it is easy to see that the power spectral density of $Y(t)$ is

$$\boxed{\Phi_{YY}(f) = |H(f)|^2\, \Phi_{XX}(f)}$$

Since $\phi_{YY}(0) = \mathcal{E}\{|Y_t|^2\} = \mathcal{F}^{-1}\{\Phi_{YY}(f)\}|_{\tau=0}$,

$$\phi_{YY}(0) = \int_{-\infty}^{\infty} \Phi_{XX}(f) |H(f)|^2\, \mathrm{d}f \geq 0$$

is valid.

As an example, we may choose $H(f) = 1$ for $f_1 \leq f \leq f_2$ and $H(f) = 0$ outside this interval, and obtain

$$\int\limits_{f_1}^{f_2} \Phi_{XX}(f)\,\mathrm{d}f \geq 0$$

Since this is only possible if $\Phi_{XX}(f) \geq 0$, $\forall f$, we conclude that power spectral densities are non–negative functions of f.

- CCF between $Y(t)$ and $X(t)$

$$\begin{aligned}
\phi_{YX}(t_1, t_2) &= \mathcal{E}\{Y_{t_1} X^*_{t_2}\} = \int\limits_{-\infty}^{\infty} h(\alpha)\,\mathcal{E}\{X(t_1 - \alpha) X^*(t_2)\}\,\mathrm{d}\alpha \\
&= \int\limits_{-\infty}^{\infty} h(\alpha)\,\phi_{XX}(t_1 - t_2 - \alpha)\,\mathrm{d}\alpha \\
&= h(\tau) * \phi_{XX}(\tau) \\
&= \phi_{YX}(\tau)
\end{aligned}$$

with $\tau = t_1 - t_2$.

- Cross–spectrum

$$\Phi_{YX}(f) = \mathcal{F}\{\phi_{YX}(\tau)\} = H(f)\,\Phi_{XX}(f)$$

2.2.4 Sampling Theorem for Band–Limited Stochastic Processes

- A *deterministic* signal $s(t)$ is called *band–limited* if its Fourier transform $S(f) = \mathcal{F}\{s(t)\}$ vanishes identically for $|f| > W$. If we sample $s(t)$ at a rate higher than $f_s \geq 2W$, we can reconstruct $s(t)$ from the samples $s(n/(2W))$, $n = 0, \pm 1, \pm 2, \ldots$, using an ideal low–pass filter with bandwidth W.

- A stationary *stochastic* process $X(t)$ is band–limited if its power spectrum $\Phi(f)$ vanishes identically for $|f| > W$, i.e., $\Phi(f) = 0$ for $|f| > W$. Since $\Phi(f)$ is the Fourier transform of $\phi(\tau)$, $\phi(\tau)$ can be reconstructed from the samples $\phi(n/(2W))$, $n = 0, \pm 1, \pm 2, \ldots$:

$$\phi(\tau) = \sum_{n=-\infty}^{\infty} \phi\left(\frac{n}{2W}\right) \frac{\sin\left[2\pi W(\tau - n/(2W))\right]}{2\pi W(\tau - n/(2W))}$$

$h(t) = \sin(2\pi W t)/(2\pi W t)$ is the impulse response of an ideal low–pass filter with bandwidth W.

If $X(t)$ is a band–limited stationary stochastic process, then we can represent $X(t)$ as

$$X(t) = \sum_{n=-\infty}^{\infty} X\left(\frac{n}{2W}\right) \frac{\sin\left[2\pi W(t - n/(2W))\right]}{2\pi W(t - n/(2W))},$$

where $X(n/(2W))$ are the samples of $X(t)$ at times $n = 0, \pm 1, \pm 2, \ldots$

2.2.5 Discrete–Time Stochastic Signals and Systems

- Now, we consider discrete–time (complex) stochastic processes $X[n]$ with discrete–time n which is an integer. Sample functions of $X[n]$ are denoted by $x[n]$. $X[n]$ may be obtained from a continuous–time process $X(t)$ by sampling $X[n] = X(nT)$, $T > 0$.

- $X[n]$ can be characterized in a similar way as the continuous–time process $X(t)$.

- ACF

$$\phi[n,k] = \mathcal{E}\{X_n X_k^*\} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_n x_k^* \, p(x_n, x_k)\, \mathrm{d}x_n \mathrm{d}x_k$$

If $X[n]$ is stationary, we get

$$\phi[\lambda] = \phi[n,k] = \phi[n, n-\lambda]$$

The average power of the stationary process $X[n]$ is defined as $\mathcal{E}\{|X_n|^2\} = \phi[0]$

- Covariance function

$$\mu[n,k] = \phi[n,k] - \mathcal{E}\{X_n\}\mathcal{E}\{X_k^*\}$$

If $X[n]$ is stationary, we get

$$\mu[\lambda] = \phi[\lambda] - |m_X|^2,$$

where $m_X = \mathcal{E}\{X_n\}$ denotes the mean of $X[n]$.

- Power spectrum

  The power spectrum of $X[n]$ is the (discrete–time) Fourier transform of the ACF $\phi[\lambda]$

$$\boxed{\Phi(f) = \mathcal{F}\{\phi[\lambda]\} = \sum_{\lambda=-\infty}^{\infty} \phi[\lambda] e^{-j2\pi f\lambda}}$$

  and the inverse transform is

$$\boxed{\phi[\lambda] = \mathcal{F}^{-1}\{\Phi(f)\} = \int_{-1/2}^{1/2} \Phi(f)\, e^{j2\pi f\lambda}\, \mathrm{d}f}$$

  Note that $\Phi(f)$ is periodic with a period $f_d = 1$, i.e., $\Phi(f+k) = \Phi(f)$ for $k = \pm 1, \pm 2, \ldots$

Example:

Consider a stochastic process with ACF

$$\phi[\lambda] = p\,\delta[\lambda+1] + \delta[\lambda] + p\,\delta[\lambda-1]$$

with constant p. The corresponding power spectrum is given by

$$\Phi(f) = \mathcal{F}\{\phi[\lambda]\} = 1 + 2\cos(2\pi f).$$

Note that $\phi[\lambda]$ is a valid ACF if and only if $-1/2 \leq p \leq 1/2$.

- Response of a discrete–time linear time–invariant system
  - Discrete–time linear time–invariant system is described by its impulse response $h[n]$
  - Frequency response

$$H(f) = \mathcal{F}\{h[n]\} = \sum_{n=-\infty}^{\infty} h[n]\, e^{-j2\pi f n}$$

  - Response $y[n]$ of system to sample function $x[n]$

$$y[n] = h[n] * x[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k]$$

where $*$ denotes now discrete–time convolution.

  - Mean of $Y[n]$

$$\begin{aligned} m_Y &= \mathcal{E}\{Y[n]\} = \sum_{k=-\infty}^{\infty} h[k]\, \mathcal{E}\{X[n-k]\} \\ &= m_X \sum_{k=-\infty}^{\infty} h[k] \\ &= m_X\, H(0) \end{aligned}$$

where m_X is the mean of $X[n]$.

– ACF of $Y[n]$

Using the deterministic "system" ACF

$$\phi_{hh}[\lambda] = h[\lambda] * h^*[-\lambda] = \sum_{k=-\infty}^{\infty} h^*[k]h[k+\lambda]$$

it can be shown that $\phi_{YY}[\lambda]$ can be expressed as

$$\phi_{YY}[\lambda] = \phi_{hh}[\lambda] * \phi_{XX}[\lambda]$$

– Power spectrum of $Y[n]$

$$\Phi_{YY}(f) = |H(f)|^2 \, \Phi_{XX}(f)$$

2.2.6 Cyclostationary Stochastic Processes

- An important class of nonstationary processes are *cyclostationary* processes. Cyclostationary means that the statistical averages of the process are *periodic*.

- Many digital communication signals can be expressed as

$$X(t) = \sum_{n=-\infty}^{\infty} a[n]\, g(t-nT)$$

where $a[n]$ denotes the transmitted symbol sequence and can be modeled as a (discrete–time) stochastic process with ACF $\phi_{aa}[\lambda] = \mathcal{E}\{a^*[n]a[n+\lambda]\}$. $g(t)$ is a deterministic function. T is the symbol duration.

- Mean of $X(t)$

$$
\begin{aligned}
m_X(t) &= \mathcal{E}\{X(t)\} \\
&= \sum_{n=-\infty}^{\infty} \mathcal{E}\{a[n]\}\, g(t-nT) \\
&= m_a \sum_{n=-\infty}^{\infty} g(t-nT)
\end{aligned}
$$

where m_a is the mean of $a[n]$. Observe that $m_X(t+kT) = m_X(t)$, i.e., $m_X(t)$ has period T.

- ACF of $X(t)$

$$
\begin{aligned}
\phi_{XX}(t+\tau, t) &= \mathcal{E}\{X(t+\tau)X^*(t)\} \\
&= \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \mathcal{E}\{a^*[n]a[m]\}\, g^*(t-nT)g(t+\tau-mT) \\
&= \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \phi_{aa}[m-n]\, g^*(t-nT)g(t+\tau-mT)
\end{aligned}
$$

Observe again that

$$
\phi_{XX}(t+\tau+kT, t+kT) = \phi_{XX}(t+\tau, t)
$$

and therefore the ACF has also period T.

- Time–average ACF

  The ACF $\phi_{XX}(t+\tau, t)$ depends on two parameters t and τ. Often we are only interested in the *time–average* ACF defined as

$$\bar{\phi}_{XX}(\tau) = \frac{1}{T} \int\limits_{-T/2}^{T/2} \phi_{XX}(t+\tau, t)\,\mathrm{d}t$$

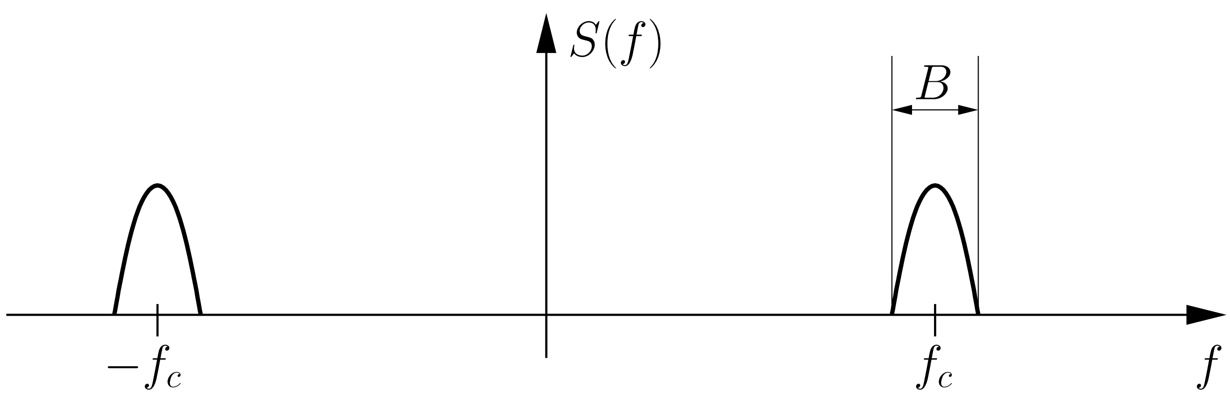
- Average power spectrum

$$\Phi_{XX}(f) = \mathcal{F}\{\bar{\phi}_{XX}(\tau)\} = \int\limits_{-\infty}^{\infty} \bar{\phi}_{XX}(\tau)\, e^{-j2\pi f\tau}\,\mathrm{d}\tau$$

3 Characterization of Communication Signals and Systems

3.1 Representation of Bandpass Signals and Systems

- Narrowband communication signals are often transmitted using some type of carrier modulation.
- The resulting transmit signal $s(t)$ has passband character, i.e., the bandwidth B of its spectrum $S(f) = \mathcal{F}\{s(t)\}$ is much smaller than the *carrier* frequency f_c.



- We are interested in a representation for $s(t)$ that is *independent* of the carrier frequency f_c. This will lead us to the so–called *equivalent (complex) baseband representation* of signals and systems.

3.1.1 Equivalent Complex Baseband Representation of Bandpass Signals

- **Given:** Real–valued bandpass signal $s(t)$ with spectrum

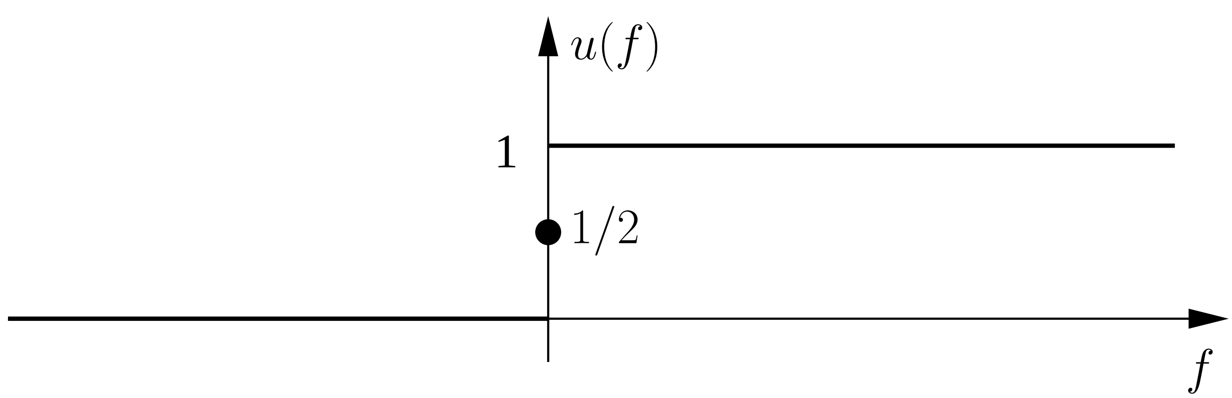
$$S(f) = \mathcal{F}\{s(t)\}$$

- **Analytic Signal** $s_+(t)$
In our quest to find the equivalent baseband representation of $s(t)$, we first suppress all negative frequencies in $S(f)$, since $S(f) = S(-f)$ is valid.
The spectrum $S_+(f)$ of the resulting so–called *analytic signal* $s_+(t)$ is defined as

$$S_+(f) = \mathcal{F}\{s_+(t)\} = 2\,u(f)S(f),$$

where $u(f)$ is the *unit step function*

$$u(f) = \begin{cases} 0, & f < 0 \\ 1/2, & f = 0 \\ 1, & f > 0 \end{cases}.$$



The analytic signal can be expressed as

$$\begin{aligned} s_+(t) &= \mathcal{F}^{-1}\{S_+(f)\} \\ &= \mathcal{F}^{-1}\{2\,u(f)S(f)\} \\ &= \mathcal{F}^{-1}\{2\,u(f)\} * \mathcal{F}^{-1}\{S(f)\} \end{aligned}$$

The inverse Fourier transform of $\mathcal{F}^{-1}\{2\,u(f)\}$ is given by

$$\mathcal{F}^{-1}\{2\,u(f)\} = \delta(t) + \frac{j}{\pi t}.$$

Therefore, the above expression for $s_+(t)$ can be simplified to

$$\begin{aligned} s_+(t) &= \left(\delta(t) + \frac{j}{\pi t}\right) * s(t) \\ &= s(t) + j\frac{1}{\pi t} * s(t) \end{aligned}$$

or

$$\boxed{s_+(t) = s(t) + j\hat{s}(t)}$$

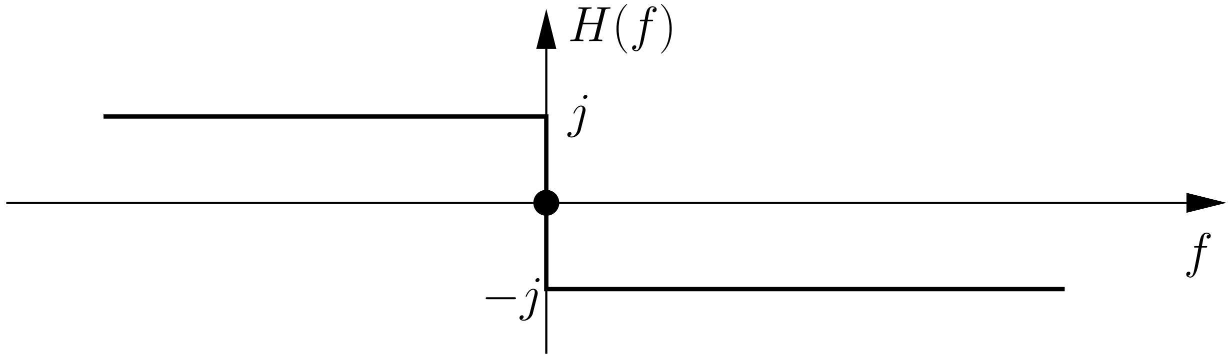
where

$$\boxed{\hat{s}(t) = \mathcal{H}\{s(t)\} = \frac{1}{\pi t} * s(t)\}}$$

denotes the *Hilbert transform* of $s(t)$.

We note that $\hat{s}(t)$ can be obtained by passing $s(t)$ through a linear system with impulse response $h(t) = 1/(\pi t)$. The frequency response, $H(f)$, of this system is the Fourier transform of $h(t) = 1/(\pi t)$ and given by

$$H(f) = \mathcal{F}\{h(t)\} = \begin{cases} j, & f < 0 \\ 0, & f = 0 \\ -j, & f > 0 \end{cases}.$$



The spectrum $\hat{S}(f) = \mathcal{F}\{\hat{s}(t)\}$ can be obtained from

$$\hat{S}(f) = H(f)S(f)$$

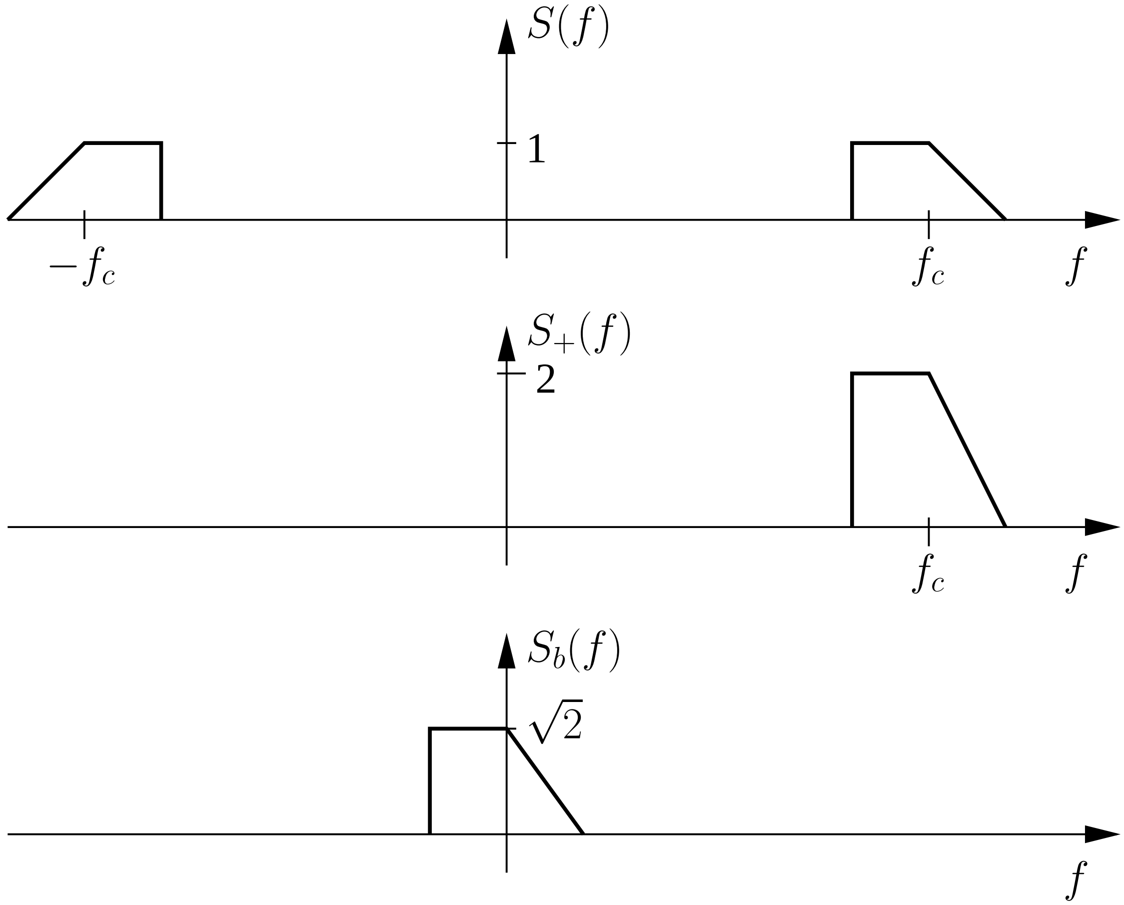
- **Equivalent Baseband Signal** $s_b(t)$
  We obtain the equivalent baseband signal $s_b(t)$ from $s_+(t)$ by frequency translation (and scaling), i.e., the spectrum $S_b(f) = \mathcal{F}\{s_b(t)\}$ of $s_b(t)$ is defined as

$$S_b(f) = \frac{1}{\sqrt{2}} S_+(f + f_c),$$

where f_c is an appropriately chosen translation frequency. In practice, if passband signal $s(t)$ was obtained through carrier modulation, it is often convenient to choose f_c equal to the carrier frequency.

Note: Our definition of the equivalent baseband signal is different from the definition used in the textbook. In particular, the factor $\frac{1}{\sqrt{2}}$ is missing in the textbook. Later on it will become clear why it is convenient to introduce this factor.

Example:



The equivalent baseband signal $s_b(t)$ (also referred to as *complex envelope* of $s(t)$) itself is given by

$$s_b(t) = \mathcal{F}^{-1}\{S_b(f)\} = \frac{1}{\sqrt{2}}\, s_+(t)\, e^{-j2\pi f_c t},$$

which leads to

$$\boxed{s_b(t) = \frac{1}{\sqrt{2}}\, [s(t) + j\hat{s}(t)]\; e^{-j2\pi f_c t}}$$

On the other hand, we can rewrite this equation as

$$s(t) + j\hat{s}(t) = \sqrt{2}\, s_b(t)\, e^{j2\pi f_c t},$$

and by realizing that both $s(t)$ and its Hilbert transform $\hat{s}(t)$ are real–valued signals, it becomes obvious that $s(t)$ can be obtained from $s_b(t)$ by taking the real part of the above equation

$$\boxed{s(t) = \sqrt{2}\,\mathsf{Re}\left\{s_b(t)\, e^{j2\pi f_c t}\right\}}$$

In general, the baseband signal $s_b(t)$ is complex valued and we may define

$$s_b(t) = x(t) + jy(t),$$

where $x(t) = \mathsf{Re}\{s_b(t)\}$ and $y(t) = \mathsf{Im}\{s_b(t)\}$ denote the real and imaginary part of $s_b(t)$, respectively. Consequently, the passband signal may be expressed as

$$s(t) = \sqrt{2}\, x(t)\mathsf{cos}(2\pi f_c t) - \sqrt{2}\, y(t)\mathsf{sin}(2\pi f_c t).$$

The equivalent complex baseband representation of a passband signal has both theoretical and practical value. From a theoretical point of view, operating at baseband simplifies the analysis (due to to independence of the carrier frequency) as well as the simulation (e.g. due to lower required sampling frequency) of passband signals. From a practical point of view, the equivalent complex baseband representation simplifies signal processing and gives insight into simple mechanisms for generation of passband signals. This application of the equivalent complex baseband representation is disscussed next.

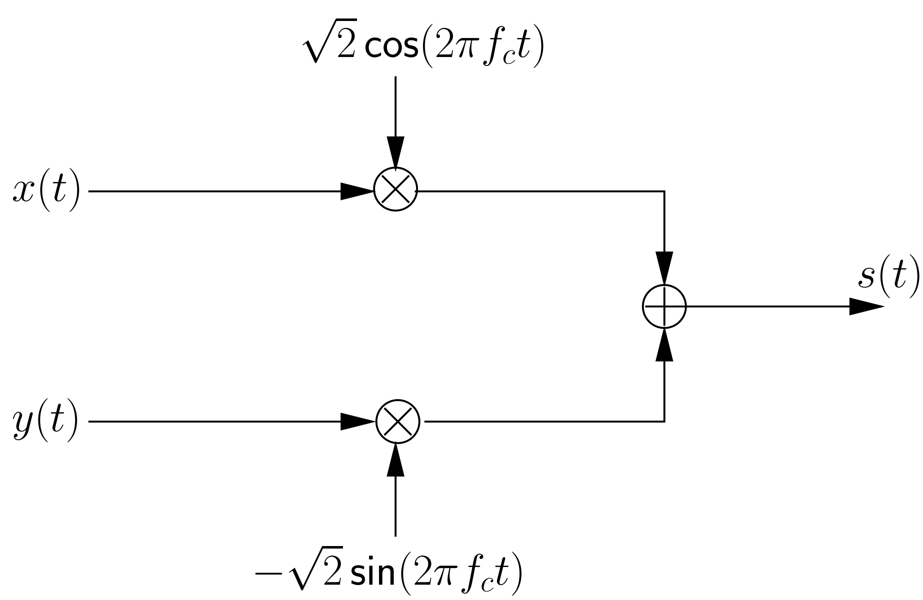
- **Quadrature Modulation**

  Problem Statement: Assume we have two real–valued lowpass signals $x(t)$ and $y(t)$ whose spectra $X(f) = \mathcal{F}\{x(t)\}$ and $Y(f) = \mathcal{F}\{y(t)\}$ are zero for $|f| > f_0$. We wish to transmit these lowpass signals over a passband channel in the frequency range $f_c - f_0 \leq |f| \leq f_c + f_0$, where $f_c > f_0$. How should we generate the corresponding passband signal?

  Solution: As shown before, the corresponding passband signal can be generated as

$$s(t) = \sqrt{2}\, x(t)\mathsf{cos}(2\pi f_c t) - \sqrt{2}\, y(t)\mathsf{sin}(2\pi f_c t).$$

  The lowpass signals $x(t)$ and $y(t)$ are modulated using the (orthogonal) carriers $\mathsf{cos}(2\pi f_c t)$ and $\mathsf{sin}(2\pi f_c t)$, respectively. $x(t)$ and $y(t)$ are also referred to as the *inphase* and *quadrature component*, respectively, and the modulation scheme is called *quadrature modulation*. Often $x(t)$ and $y(t)$ are also jointly addressed as the quadrature components of $s_b(t)$.



■ Demodulation

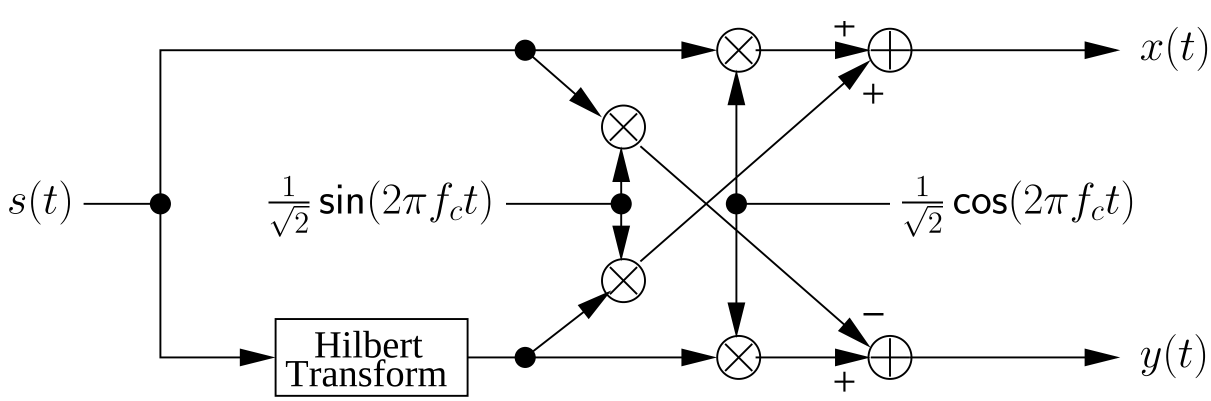
At the receiver side, the quadrature components $x(t)$ and $y(t)$ have to be extracted from the passband signal $s(t)$.

Using the relation

$$s_b(t) = x(t) + jy(t) = \frac{1}{\sqrt{2}}\,[s(t) + j\hat{s}(t)]\;e^{-j2\pi f_c t},$$

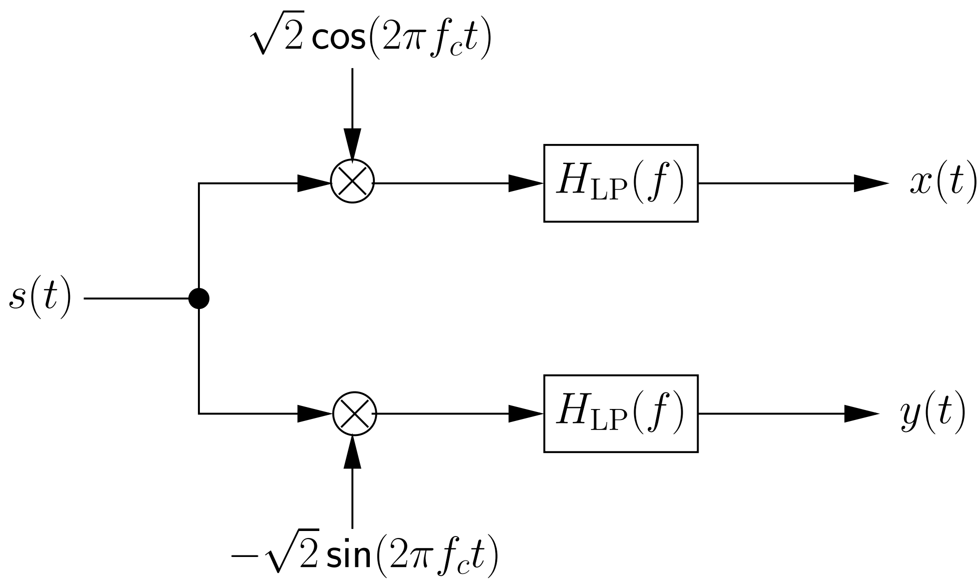
we easily get

$$\begin{aligned} x(t) &= \frac{1}{\sqrt{2}}[s(t)\cos(2\pi f_c t) + \hat{s}(t)\sin(2\pi f_c t)] \\ y(t) &= \frac{1}{\sqrt{2}}[\hat{s}(t)\cos(2\pi f_c t) - s(t)\sin(2\pi f_c t)] \end{aligned}$$

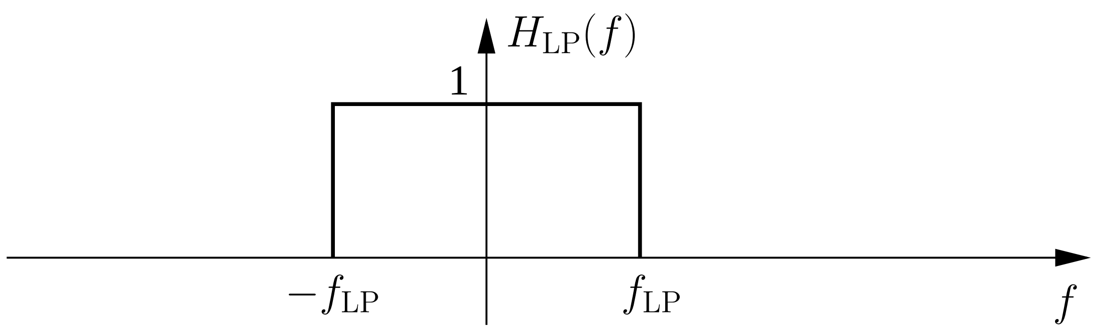


Unfortunately, the above structure requires a Hilbert transformer which is difficult to implement.

Fortunately, if $S_b(f) = 0$ for $|f| > f_0$ and $f_c > f_0$ are valid, i.e., if $x(t)$ and $y(t)$ are bandlimited, $x(t)$ and $y(t)$ can be obtained from $s(t)$ using the structure shown below.



$H_{\mathrm{LP}}(f)$ is a lowpass filter. With cut–off frequency f_{LP}, $f_0 \leq f_{\mathrm{LP}} \leq 2f_c - f_0$. The above structure is usually used in practice to transform a passband signal into (complex) baseband.



- **Energy of $s_b(t)$**
  For calculation of the energy of $s_b(t)$, we need to express the spectrum $S(f)$ of $s(t)$ as a function of $S_b(f)$:

$$\begin{aligned} S(f) &= \mathcal{F}\{s(t)\} \\ &= \int\limits_{-\infty}^{\infty} \mathsf{Re}\left\{\sqrt{2}s_b(t)e^{j2\pi f_c t}\right\} e^{-j2\pi f t}\,\mathrm{d}t \\ &= \frac{\sqrt{2}}{2}\int\limits_{-\infty}^{\infty} \left(s_b(t)e^{j2\pi f_c t} + s_b^*(t)e^{-j2\pi f_c t}\right) e^{-j2\pi f t}\,\mathrm{d}t \\ &= \frac{1}{\sqrt{2}}\left[S_b(f-f_c) + S_b^*(-f-f_c)\right] \end{aligned}$$

Now, using *Parsevals Theorem* we can express the energy of $s(t)$ as

$$\begin{aligned} E &= \int\limits_{-\infty}^{\infty} s^2(t)\,\mathrm{d}t = \int\limits_{-\infty}^{\infty} |S(f)|^2\,\mathrm{d}f \\ &= \int\limits_{-\infty}^{\infty} \left|\frac{1}{\sqrt{2}}\left[S_b(f-f_c) + S_b^*(-f-f_c)\right]\right|^2 \mathrm{d}f \\ &= \frac{1}{2}\int\limits_{-\infty}^{\infty} \Big[|S_b(f-f_c)|^2 + |S_b^*(-f-f_c)|^2 + \\ &\qquad S_b(f-f_c)S_b(-f-f_c) + S_b^*(f-f_c)S_b^*(-f-f_c)\Big]\,\mathrm{d}f \end{aligned}$$

It is easy to show that

$$\int_{-\infty}^{\infty} |S_b(f - f_c)|^2 \, \mathrm{d}f = \int_{-\infty}^{\infty} |S_b^*(-f - f_c)|^2 \, \mathrm{d}f = \int_{-\infty}^{\infty} |S_b(f)|^2 \, \mathrm{d}f$$

is valid. In addition, if the spectra $S_b(f - f_c)$ and $S_b(-f - f_c)$ do not overlap, which will be usually the case in practice since the bandwidth B of $S_b(f)$ is normally much smaller than f_c, $S_b(f - f_c)S_b(-f - f_c) = 0$ is valid.

Using these observations the energy E of $s(t)$ can be expressed as

$$\boxed{E = \int_{-\infty}^{\infty} |S_b(f)|^2 \, \mathrm{d}f = \int_{-\infty}^{\infty} |s_b(t)|^2 \, \mathrm{d}t}$$

To summarize, we have show that energy of the baseband signal is identical to the energy of the corresponding passband signal

$$E = \int_{-\infty}^{\infty} s^2(t) \, \mathrm{d}t = \int_{-\infty}^{\infty} |s_b(t)|^2 \, \mathrm{d}t$$

Note: This identity does not hold for the baseband transformation used in the text book. Since the factor $1/\sqrt{2}$ is missing in the definition of the equivalent baseband signal in the text book, there the energy of $s_b(t)$ is twice that of the passband signal $s(t)$.

3.1.2 Equivalent Complex Baseband Representation of Bandpass Systems

- The equivalent baseband representation of systems is similar to that of signals. However, there are a few minor but important differences.

- **Given:** Bandpass system with impulse response $h(t)$ and transfer function $H(f) = \mathcal{F}\{h(t)\}$.

- **Analytic System**
  The transfer function $H_+(f)$ and impulse response $h_+(t)$ of the analytic system are respectively defined as

$$H_+(f) = 2u(f)H(f)$$

  and

$$h_+(t) = \mathcal{F}^{-1}\{H_+(f)\},$$

  which is identical to the corresponding definitions for analytic signals.

- **Equivalent Baseband System**
  The transfer function $H_b(f)$ of the equivalent baseband *system* is defined as

$$H_b(f) = \frac{1}{2}H_+(f + f_c).$$

  Note: This definition differs from the definition of the equivalent baseband signal. Here, we have the factor $1/2$, whereas we had $1/\sqrt{2}$ in the definition of the the equivalent baseband signal.

We may express transfer function of the baseband system, $H_b(f)$, in terms of the transfer function of the passband system, $H(f)$, as

$$H_b(f - f_c) = \begin{cases} H(f), & f \geq 0 \\ 0, & f < 0 \end{cases}$$

Using the symmetry relation $H(f) = H^*(-f)$, which holds since $h(t)$ is real valued, we get

$$H(f) = H_b(f - f_c) + H_b^*(-f - f_c)$$

Finally, taking the Fourier transform of the above equation results in

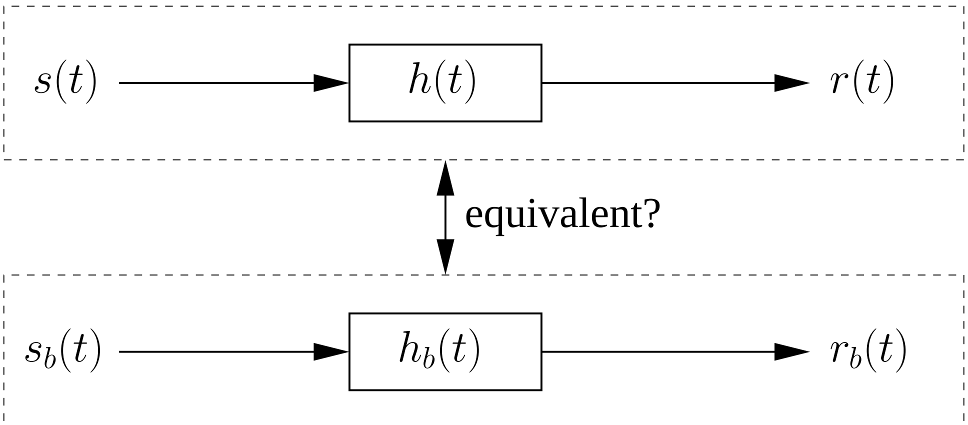
$$h(t) = h_b(t)e^{j2\pi f_c t} + h_b^*(t)e^{-j2\pi f_c t}$$

or

$$\boxed{h(t) = 2\mathsf{Re}\left\{h_b(t)e^{j2\pi f_c t}\right\}}$$

3.1.3 Response of a Bandpass Systems to a Bandpass Signal

Objective: In this section, we try to find out whether linear filtering in complex baseband is *equivalent* to the same operation in passband.



Obviously, we have

$$\begin{aligned} R(f) &= \mathcal{F}\{r(t)\} = H(f)S(f) \\ &= \frac{1}{\sqrt{2}}\left[S_b(f-f_c)+S^*(-f-f_c)\right]\left[H_b(f-f_c)+H_b^*(-f-f_c)\right] \end{aligned}$$

Since both $s(t)$ and $h(t)$ have narrowband character

$$\begin{aligned} S_b(f-f_c)H_b^*(-f-f_c) &= 0 \\ S_b^*(-f-f_c)H_b(f-f_c) &= 0 \end{aligned}$$

is valid, and we get

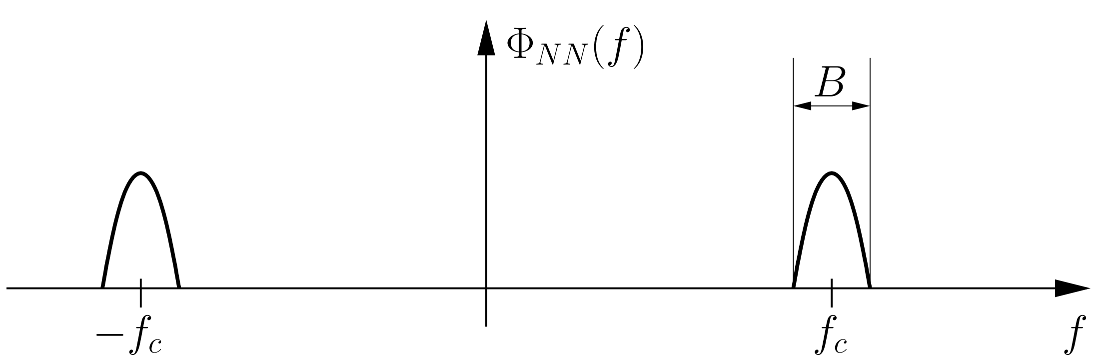
$$\begin{aligned} R(f) &= \frac{1}{\sqrt{2}}\left[S_b(f-f_c)H_b(f-f_c)+S^*(-f-f_c)H_b^*(-f-f_c)\right] \\ &= \frac{1}{\sqrt{2}}\left[R_b(f-f_c)+R_b^*(-f-f_c)\right] \end{aligned}$$

This result shows that we can get the output $r(t)$ of the linear system $h(t)$ by transforming the output $r_b(t)$ of the equivalent baseband system $h_b(t)$ into passband. This is an important result since it shows that, without loss of generality, we can perform linear filtering operations always in the *equivalent baseband* domain. This is very convenient for example if we want to simulate a communication system that operates in passband using a computer.

3.1.4 Equivalent Baseband Representation of Bandpass Stationary Stochastic Processes

Given: Wide–sense stationary noise process $n(t)$ with zero–mean and power spectral density $\Phi_{NN}(f)$. In particular, we assume a narrowband bandpass noise process with bandwidth B and center (carrier) frequency f_c, i.e.,

$$\Phi_{NN}(f) \begin{cases} \neq 0, & f_c - B/2 \leq |f| \leq f_c + B/2 \\ = 0, & \text{otherwise} \end{cases}$$



Equivalent Baseband Noise

Defining the equivalent complex baseband noise process

$$z(t) = x(t) + jy(t),$$

where $x(t)$ and $y(t)$ are real–valued baseband noise processes, we can express the passband noise $n(t)$ as

$$\begin{aligned} n(t) &= \sqrt{2}\mathrm{Re}\left\{z(t)e^{j2\pi f_c t}\right\} \\ &= \sqrt{2}\left[x(t)\cos(2\pi f_c t) - y(t)\sin(2\pi f_c t)\right]. \end{aligned}$$

Stationarity of $n(t)$

Since $n(t)$ is assumed to be wide–sense stationary, the correlation functions $\phi_{XX}(\tau) = \mathcal{E}\{x(t+\tau)x^*(t)\}$, $\phi_{YY}(\tau) = \mathcal{E}\{y(t+\tau)y^*(t)\}$, and $\phi_{XY}(\tau) = \mathcal{E}\{x(t+\tau)y^*(t)\}$ have to fulfill certain conditions as will be shown in the following.

The ACF of $n(t)$ is given by

$$\begin{aligned} \phi_{NN}(\tau, t+\tau) &= \mathcal{E}\{n(t)n(t+\tau)\} \\ &= \mathcal{E}\left\{2\left[x(t)\cos(2\pi f_c t) - y(t)\sin(2\pi f_c t)\right]\right. \\ &\qquad \left.\left[x(t+\tau)\cos(2\pi f_c(t+\tau)) - y(t+\tau)\sin(2\pi f_c(t+\tau))\right]\right\} \\ &= \left[\phi_{XX}(\tau) + \phi_{YY}(\tau)\right]\cos(2\pi f_c \tau) \\ &\quad + \left[\phi_{XX}(\tau) - \phi_{YY}(\tau)\right]\cos(2\pi f_c(2t+\tau)) \\ &\quad - \left[\phi_{YX}(\tau) - \phi_{XY}(\tau)\right]\sin(2\pi f_c \tau) \\ &\quad - \left[\phi_{YX}(\tau) + \phi_{XY}(\tau)\right]\sin(2\pi f_c(2t+\tau)), \end{aligned}$$

where we have used the trigonometric relations

$$\begin{aligned}
\cos(\alpha)\cos(\beta) &= \frac{1}{2}[\cos(\alpha-\beta)+\cos(\alpha+\beta)] \\
\sin(\alpha)\sin(\beta) &= \frac{1}{2}[\cos(\alpha-\beta)-\cos(\alpha+\beta)] \\
\sin(\alpha)\sin(\beta) &= \frac{1}{2}[\sin(\alpha-\beta)-\sin(\alpha+\beta)]
\end{aligned}$$

For $n(t)$ to be a wide–sense stationary passband process, the equivalent baseband process has to fulfill the conditions

$$\begin{aligned}
\phi_{XX}(\tau) &= \phi_{YY}(\tau) \\
\phi_{YX}(\tau) &= -\phi_{XY}(\tau)
\end{aligned}$$

Hence, the ACF of $n(t)$ can be expressed as

$$\phi_{NN}(\tau) = 2\left[\phi_{XX}(\tau)\cos(2\pi f_c\tau) - \phi_{YX}(\tau)\sin(2\pi f_c\tau)\right]$$

ACF of $z(t)$

Using the above conditions, the ACF $\phi_{ZZ}(\tau)$ of $z(t)$ is easily calculated as

$$\begin{aligned}
\phi_{ZZ}(\tau) &= \mathcal{E}\{z(t+\tau)z^*(t)\} \\
&= \phi_{XX}(\tau)+\phi_{YY}(\tau)+j(\phi_{YX}(\tau)-\phi_{XY}(\tau)) \\
&= 2\phi_{XX}(\tau)+j2\phi_{YX}(\tau)
\end{aligned}$$

As a consequence, we can express $\phi_{NN}(\tau)$ as

$$\boxed{\phi_{NN}(\tau) = \mathrm{Re}\{\phi_{ZZ}(\tau)e^{j2\pi f_c\tau}\}}$$

Power Spectral Density of $n(t)$

The power spectral densities of $z(t)$ and $n(t)$ are given by

$$\begin{aligned}\Phi_{NN}(f) &= \mathcal{F}\{\phi_{NN}(\tau)\} \\ \Phi_{ZZ}(f) &= \mathcal{F}\{\phi_{ZZ}(\tau)\}\end{aligned}$$

Therefore, we can represent $\Phi_{NN}(f)$ as

$$\begin{aligned}\Phi_{NN}(f) &= \int_{-\infty}^{\infty} \mathrm{Re}\{\phi_{ZZ}(\tau) e^{j2\pi f_c \tau}\}\, e^{-j2\pi f \tau}\, \mathrm{d}\tau \\ &= \frac{1}{2}\left[\Phi_{ZZ}(f - f_c) + \Phi_{ZZ}(-f - f_c)\right],\end{aligned}$$

where we have used the fact that $\Phi_{ZZ}(f)$ is a real–valued function.

Properties of the Quadrature Components

From the stationarity of $n(t)$ we derived the identity

$$\phi_{YX}(\tau) = -\phi_{XY}(\tau)$$

On the other hand, the relation

$$\phi_{YX}(\tau) = \phi_{XY}(-\tau)$$

holds for any wide–sense stationary stochastic process. If we combine these two relations, we get

$$\phi_{XY}(\tau) = -\phi_{XY}(-\tau),$$

i.e., $\phi_{XY}(\tau)$ is an odd function in τ, and $\phi_{XY}(0) = 0$ holds always.

If the quadrature components $x(t)$ and $y(t)$ are uncorrelated, their correlation function is zero, $\phi_{XY}(\tau) = 0$, $\forall \tau$. Consequently, the ACF of $z(t)$ is real valued

$$\phi_{ZZ}(\tau) = 2\phi_{XX}(\tau)$$

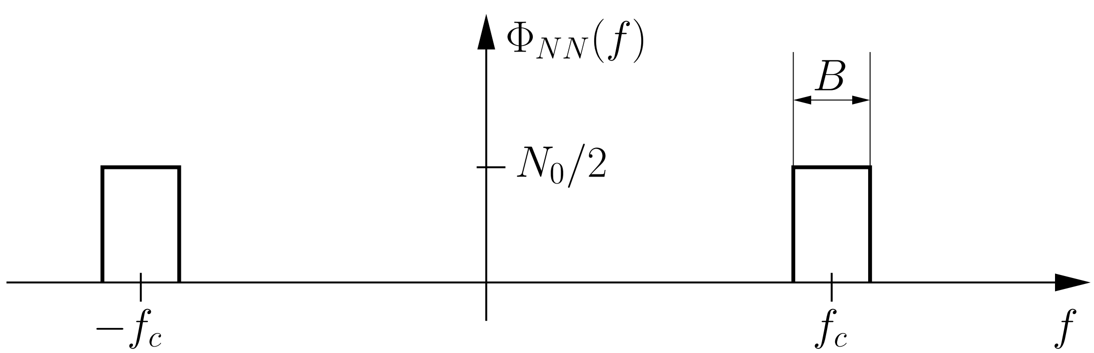
From this property we can conclude that for uncorrelated quadrature components the power spectral density of $z(t)$ is symmetric about $f = 0$

$$\boxed{\Phi_{ZZ}(f) = \Phi_{ZZ}(-f)}$$

White Noise

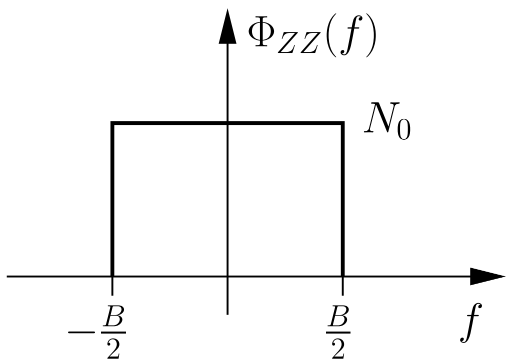
In the region of interest (i.e., where the transmit signal has non–zero frequency components), $\Phi_{NN}(f)$ can often be approximated as flat, i.e.,

$$\Phi_{NN}(f) = \begin{cases} N_0/2, & f_c - B/2 \leq |f| \leq f_c + B/2 \\ 0, & \text{otherwise} \end{cases}$$



The power spectral density of the corresponding baseband noise process $z(t)$ is given by

$$\Phi_{ZZ}(f) = \begin{cases} N_0, & |f| \leq B/2 \\ 0, & \text{otherwise} \end{cases}$$



and the ACF of $z(t)$ is given by

$$\phi_{ZZ}(\tau) = N_0 \frac{\sin(\pi B \tau)}{\pi \tau}$$

In practice, it is often convenient to assume that $B \to \infty$ is valid, i.e., that the spectral density of $z(t)$ is constant for all frequencies. Note that as far as the transmit signal is concerned increasing B does not change the properties of the noise process $z(t)$. For $B \to \infty$ we get

$$\begin{aligned} \phi_{ZZ}(\tau) &= N_0\, \delta(\tau) \\ \Phi_{ZZ}(f) &= N_0 \end{aligned}$$

A noise process with flat spectrum for all frequencies is also called a *white* noise process. The power spectral density of the process is typically denoted by N_0. Since $\Phi_{ZZ}(f)$ is symmetric about $f = 0$, the quadrature components of $z(t)$ are uncorrelated, i.e., $\phi_{XY}(\tau) = 0$, $\forall \tau$. In addition, we have

$$\phi_{XX}(\tau) = \phi_{YY}(\tau) = \frac{1}{2}\phi_{ZZ}(\tau) = \frac{N_0}{2}\, \delta(\tau)$$

This means that $x(t)$ and $y(t)$ are mutually uncorrelated, white processes with equal variances.

Note that although white noise is convenient for analysis, it is not physically realizable since it would have an infinite variance.

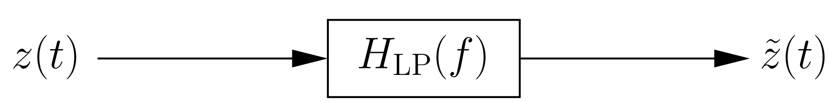
White Gaussian Noise

For most applications it can be assumed that the channel noise $n(t)$ is not only stationary and white but also *Gaussian* distributed. Consequently, the quadrature components $x(t)$ and $y(t)$ of $z(t)$ are mutually uncorrelated white Gaussian processes.

Observed through a lowpass filter with bandwidth B, $x(t)$ and $y(t)$ have equal variances $\sigma^2 = \phi_{XX}(0) = \phi_{YY}(0) = N_0B/2$. The PDF of corresponding filtered complex process $\tilde{z}(t_0) = \tilde{z} = \tilde{x} + j\tilde{y}$ is given by

$$\begin{aligned} p_Z(\tilde{z}) &= p_{XY}(\tilde{x}, \tilde{y}) \\ &= \frac{1}{2\pi\sigma^2} \exp\left(-\frac{\tilde{x}^2 + \tilde{y}^2}{2\sigma^2}\right) \\ &= \frac{1}{\pi\sigma_Z^2} \exp\left(-\frac{|\tilde{z}|^2}{\sigma_Z^2}\right), \end{aligned}$$

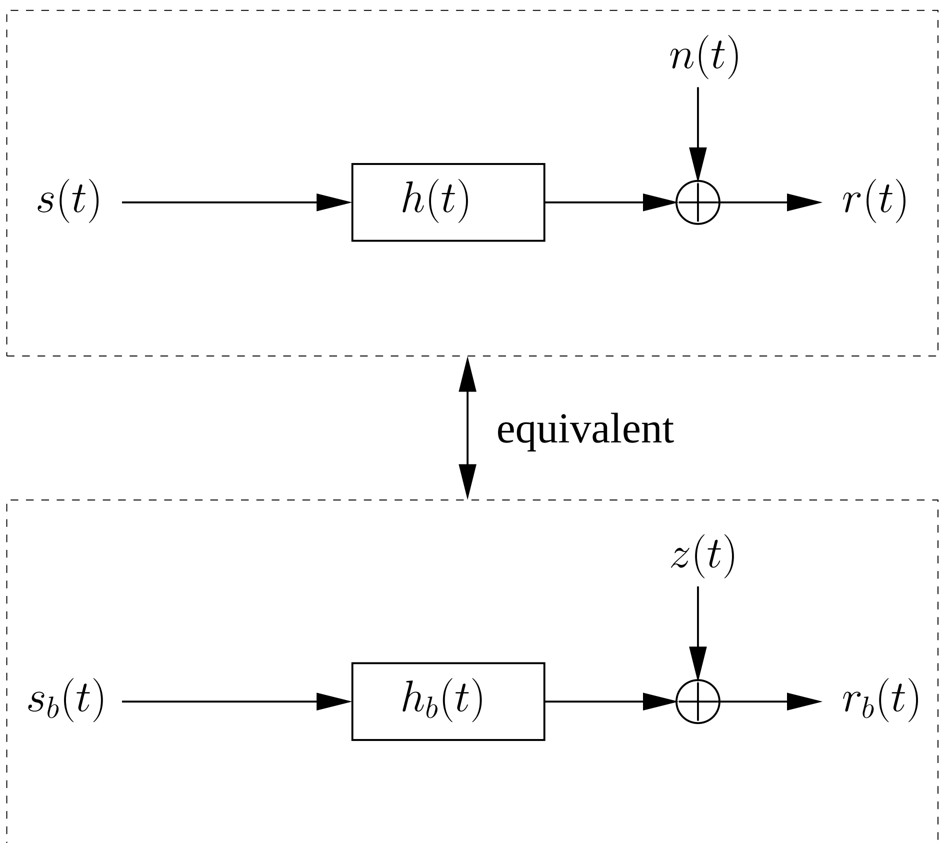
where $\sigma_Z^2 = 2\sigma^2$ is valid. Since this PDF is rotationally symmetric, the corresponding equivalent baseband noise process is also referred to as *circularly symmetric complex Gaussian noise*.



From the above considerations we conclude that if we want to analyze or simulate a passband communication system that is impaired by stationary white Gaussian noise, the corresponding equivalent baseband noise has to be circularly symmetric white Gaussian noise.

Overall System Model

From the considerations in this section we can conclude that a passband system, including linear filters and wide–sense stationary noise, can be equivalently represented in complex baseband. The baseband representation is useful for simulation and analysis of passband systems.



3.2 Signal Space Representation of Signals

- Signals can be represented as vectors over a certain basis.
- This description allows the application of many well–known tools (inner product, notion of orthogonality, etc.) from Linear Algebra to signals.

3.2.1 Vector Space Concepts – A Brief Review

- **Given:** n–dimensional vector

$$\begin{aligned} \boldsymbol{v} &= [v_1\ v_2\ \ldots\ v_n]^T \\ &= \sum_{i=1}^{n} v_i \boldsymbol{e}_i, \end{aligned}$$

with *unit* or *basis* vector

$$\boldsymbol{e}_i = [0\ \ldots\ 0\ 1\ 0\ \ldots\ 0]^T,$$

where 1 is the ith element of $\boldsymbol{e}_i$.

- **Inner Product $\boldsymbol{v}_1 \bullet \boldsymbol{v}_2$**
We define the (complex) vector $\boldsymbol{v}_j$, $j \in \{1, 2\}$, as

$$\boldsymbol{v}_j = [v_{j1}\ v_{j2}\ \ldots\ v_{jn}]^T.$$

The inner product between $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ is defined as

$$\begin{aligned} \boldsymbol{v}_1 \bullet \boldsymbol{v}_2 &= \boldsymbol{v}_2^H \boldsymbol{v}_1 \\ &= \sum_{i=1}^{n} v_{2i}^* v_{1i} \end{aligned}$$

Two vectors are *orthogonal* if and only if their inner product is zero, i.e.,

$$\boldsymbol{v}_1 \bullet \boldsymbol{v}_2 = 0$$

- **L_2–Norm of a Vector**
  The L_2–norm of a vector $\boldsymbol{v}$ is defined as

$$||\boldsymbol{v}|| = \sqrt{\boldsymbol{v} \bullet \boldsymbol{v}} = \sqrt{\sum_{i=1}^{n} |v_i|^2}$$

- **Linear Independence**
  Vector $\boldsymbol{x}$ is *linearly independent* of vectors $\boldsymbol{v}_j$, $1 \leq j \leq n$, if there is *no* set of coefficients a_j, $1 \leq j \leq n$, for which

$$\boldsymbol{x} = \sum_{j=1}^{n} a_j \boldsymbol{v}_j$$

  is valid.

- **Triangle Inequality**
  The *triangle inequality* states that

$$||\boldsymbol{v}_1 + \boldsymbol{v}_2|| \leq ||\boldsymbol{v}_1|| + ||\boldsymbol{v}_2||,$$

  where equality holds if and only if $\boldsymbol{v}_1 = a\,\boldsymbol{v}_2$, where a is positive real valued, i.e., $a \geq 0$.

- **Cauchy–Schwarz Inequality**
  The *Cauchy–Schwarz inequality* states that

$$|\boldsymbol{v}_1 \bullet \boldsymbol{v}_2| \leq ||\boldsymbol{v}_1||\,||\boldsymbol{v}_2||$$

  is true. Equality holds if $\boldsymbol{v}_1 = b\,\boldsymbol{v}_2$, where b is an arbitrary (complex–valued) scalar.

■ Gram–Schmidt Procedure

– Enables construction of an *orthonormal basis* for a given set of vectors.

– **Given:** Set of n–dimensional vectors $\boldsymbol{v}_j$, $1 \leq j \leq m$, which span an n_1–dimensional vector space with $n_1 \leq \mathsf{max}\{n, m\}$.

– **Objective:** Find orthonormal basis vectors $\boldsymbol{u}_j$, $1 \leq j \leq n_1$.

– **Procedure:**

1. *First Step:* Normalize first vector of set (the order is arbitrary).

$$\boldsymbol{u}_1 = \frac{\boldsymbol{v}_1}{||\boldsymbol{v}_1||}$$

2. *Second Step:* Identify that part of $\boldsymbol{v}_2$ which is orthogonal to $\boldsymbol{u}_1$.

$$\boldsymbol{u}_2' = \boldsymbol{v}_2 - (\boldsymbol{v}_2 \bullet \boldsymbol{u}_1)\, \boldsymbol{u}_1,$$

where $(\boldsymbol{v}_2 \bullet \boldsymbol{u}_1)\, \boldsymbol{u}_1$ is the *projection* of $\boldsymbol{v}_2$ onto $\boldsymbol{u}_1$. Therefore, it is easy to show that $\boldsymbol{u}_2' \bullet \boldsymbol{u}_1 = 0$ is valid. Thus, the second basis vector $\boldsymbol{u}_2$ is obtained by normalization of $\boldsymbol{u}_2'$

$$\boldsymbol{u}_2 = \frac{\boldsymbol{u}_2'}{||\boldsymbol{u}_2'||}$$

3. *Third Step:*

$$\boldsymbol{u}_3' = \boldsymbol{v}_3 - (\boldsymbol{v}_3 \bullet \boldsymbol{u}_1)\, \boldsymbol{u}_1 - (\boldsymbol{v}_3 \bullet \boldsymbol{u}_2)\, \boldsymbol{u}_2$$

$$\boldsymbol{u}_3 = \frac{\boldsymbol{u}_3'}{||\boldsymbol{u}_3'||}$$

4. Repeat until $\boldsymbol{v}_m$ has been processed.

– **Remarks:**

1. The number n_1 of basis vectors is smaller or equal $\mathsf{max}\{n,\, m\}$, i.e., $n_1 \leq \mathsf{max}\{n,\, m\}$.
2. If a certain vector $\boldsymbol{v}_j$ is linearly dependent on the previously found basis vectors $\boldsymbol{u}_i$, $1 \leq i \leq j-1$, $\boldsymbol{u}'_j = 0$ results and we proceed with the next element $\boldsymbol{v}_{j+1}$ of the set of vectors.
3. The found set of basis vectors is *not* unique. Different sets of basis vectors can be found e.g. by changing the processing order of the vectors $\boldsymbol{v}_j$, $1 \leq j \leq m$.

3.2.2 Signal Space Concepts

Analogous to the vector space concepts discussed in the last section, we can use similar concepts in the so–called signal space.

- **Inner Product**

  The inner product of two (complex–valued) signals $x_1(t)$ and $x_2(t)$ is defined as

$$< x_1(t),\, x_2(t) > = \int_a^b x_1(t) x_2^*(t)\, \mathrm{d}t, \qquad b \geq a,$$

  where a and b are real valued scalars.

- **Norm**

  The norm of a signal $x(t)$ is defined as

$$||x(t)|| = \sqrt{< x(t),\, x(t) >} = \sqrt{\int_a^b |x(t)|^2\, \mathrm{d}t}.$$

- **Energy**

  The energy of a signal $x(t)$ is defined as

$$E = ||x(t)||^2 = \int_{-\infty}^{\infty} |x(t)|^2 \, \mathrm{d}t.$$

- **Linear Independence**

  m signals are *linearly independent* if and only if no signal of the set can be represented as a linear combination of the other $m-1$ signals.

- **Triangle Inequality**

  Similar to the vector space case the triangle inequality states

$$||x_1(t) + x_2(t)|| \leq ||x_1(t)|| + ||x_2(t)||,$$

  where equality holds if and only if $x_1(t) = a x_2(t)$, $a \geq 0$.

- **Cauchy–Schwarz Inequality**

$$| < x_1(t), \, x_2(t) > | \leq ||x_1(t)|| \, ||x_2(t)||,$$

  where equality holds if and only if $x_1(t) = b x_2(t)$, where b is arbitrary complex.

3.2.3 Orthogonal Expansion of Signals

For the design and analysis of communication systems it is often necessary to represent a signal as a sum of *orthogonal* signals.

- **Given:**
  - (Complex–valued) signal $s(t)$ with finite energy

$$E_s = \int_{-\infty}^{\infty} |s(t)|^2 \, \mathrm{d}t$$

  - Set of K orthonormal functions $\{f_n(t),\ n = 1,\ 2,\ \ldots,\ K\}$

$$< f_n(t),\ f_m(t) > = \int_{-\infty}^{\infty} f_n(t) f_m^*(t) \, \mathrm{d}t = \begin{cases} 1, & n = m \\ 0 & n \neq m \end{cases}$$

- **Objective:**
  Find "best" approximation for $s(t)$ in terms of $f_n(t)$, $1 \leq n \leq K$. The approximation $\hat{s}(t)$ is given by

$$\hat{s}(t) = \sum_{n=1}^{K} s_n f_n(t),$$

  with coefficients s_n, $1 \leq n \leq K$. The optimality criterion adopted for the approximation of $s(t)$ is the energy of the error,

$$e(t) = s(t) - \hat{s}(t),$$

  which is to be minimized.

The error energy is given by

$$\begin{aligned} E_e &= \int\limits_{-\infty}^{\infty} |e(t)|^2 \, \mathrm{d}t \\ &= \int\limits_{-\infty}^{\infty} |s(t) - \sum_{n=1}^{K} s_n f_n(t)|^2 \, \mathrm{d}t \end{aligned}$$

- **Optimize Coefficients** s_n

In order to find the optimum coefficients s_n, $1 \leq n \leq K$, which minimize E_e, we have to differentiate E_e with respect to s_n^*, $1 \leq n \leq K$,

$$\frac{\partial E_e}{\partial s_n^*} = \int\limits_{-\infty}^{\infty} \left[s(t) - \sum_{k=1}^{K} s_k f_k(t) \right] f_n^*(t) \, \mathrm{d}t = 0, \qquad n = 1, \ldots, K,$$

where we have used the following rules for *complex* differentiation (z is a complex variable):

$$\frac{\partial z^*}{\partial z^*} = 1, \quad \frac{\partial z}{\partial z^*} = 0, \quad \frac{\partial |z|^2}{\partial z^*} = z$$

Since the $f_n(t)$ are orthonormal, we get

$$s_n = \int\limits_{-\infty}^{\infty} s(t) f_n^*(t) \, \mathrm{d}t = < s(t), \, f_n(t) >, \qquad n = 1, \ldots, K.$$

This means that we can interpret $\hat{s}(t)$ simply as the *projection* of $s(t)$ onto the K–dimensional subspace spanned by the functions $\{f_n(t)\}$.

- **Minimum Error Energy** $E_{\min}$

  Before, we calculate the minimum error energy, we note that for the optimum coefficients s_n, $n = 1, \ldots, K$, the identity

$$\int_{-\infty}^{\infty} e(t)\hat{s}^*(t)\,\mathrm{d}t = 0$$

  holds. Using this result, we obtain for $E_{\min}$

$$\begin{aligned}
E_{\min} &= \int_{-\infty}^{\infty} |e(t)|^2\,\mathrm{d}t \\
&= \int_{-\infty}^{\infty} e(t)s^*(t)\,\mathrm{d}t \\
&= \int_{-\infty}^{\infty} |s(t)|^2\,\mathrm{d}t - \int_{-\infty}^{\infty} \sum_{k=1}^{K} s_k f_k(t) s^*(t)\,\mathrm{d}t \\
&= E_s - \sum_{n=1}^{K} |s_n|^2
\end{aligned}$$

- $E_{\min} = 0$

  If $E_{\min} = 0$ is valid, we can represent $s(t)$ as

$$s(t) = \sum_{k=1}^{K} s_k f_k(t),$$

  where equality holds in the sense of zero mean–square error energy. In this case, the set of functions $\{f_n(t)\}$ is a *complete* set.

■ Gram–Schmidt Procedure

– **Given:** M signal waveforms $\{s_i(t),\ i = 1,\ \ldots,\ M\}$.

– **Objective:** Construct a set of orthogonal waveforms $\{f_i(t)\}$ from the original set $\{s_i(t)\}$.

– **Procedure**

1. *First Step:*

$$f_1(t) = \frac{s_1(t)}{\sqrt{E_1}},$$

where the energy E_1 of signal $s_1(t)$ is given by

$$E_1 = \int_{-\infty}^{\infty} |s_1(t)|^2 \, \mathrm{d}t$$

2. *Second Step:*
Calculate the inner product of $s_2(t)$ and $f_1(t)$

$$c_{12} = \int_{-\infty}^{\infty} s_2(t) f_1^*(t) \, \mathrm{d}t$$

The projection of $s_2(t)$ onto $f_1(t)$ is simply $c_{12} f_1(t)$. Therefore, the function

$$f_2'(t) = s_2(t) - c_{12} f_1(t)$$

is *orthogonal* to $f_1(t)$, i.e., $< f_1(t),\ f_2'(t) >= 0$. A normalized version of $f_2'(t)$ gives the second basis function

$$f_2(t) = \frac{f_2'(t)}{\sqrt{E_2}},$$

with

$$E_j = \int\limits_{-\infty}^{\infty} |f'_j(t)|^2 \, \mathrm{d}t, \qquad j = 2,\, 3,\, \ldots$$

3. kth *Step*

$$f'_k(t) = s_k(t) - \sum_{i=1}^{k-1} c_{ik} f_i(t)$$

with

$$c_{ik} = \int\limits_{-\infty}^{\infty} s_k(t) f_i^*(t) \, \mathrm{d}t$$

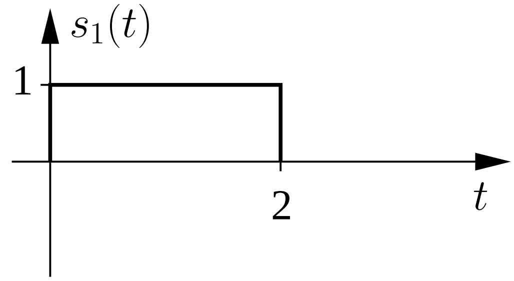
The kth basis function is obtained as

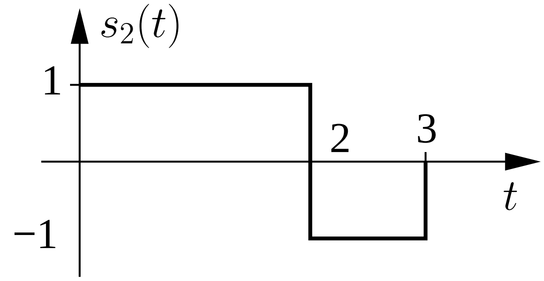
$$f_k(t) = \frac{f'_k(t)}{\sqrt{E_k}},$$

4. The orthonormalization process continues until all M functions of the set $\{s_i(t)\}$ have been processed. The result are $N \leq M$ orthonormal waveforms $\{f_i(t)\}$.

Example:

Given are the following two signals



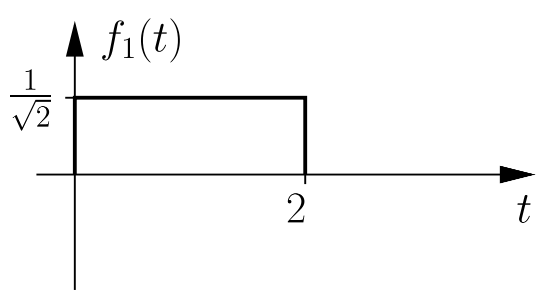


1. *First Step:*
   Signal $s_1(t)$ has energy

$$E_1 = \int_{-\infty}^{\infty} |s_1(t)|^2 \,\mathrm{d}t = 2.$$

   Therefore, the first basis function is

$$f_1(t) = \frac{s_1(t)}{\sqrt{E_1}} = \frac{s_1(t)}{\sqrt{2}}$$



2. *Second Step:*
   The inner product between the first basis function $f_1(t)$ and

the second signal $s_2(t)$ is

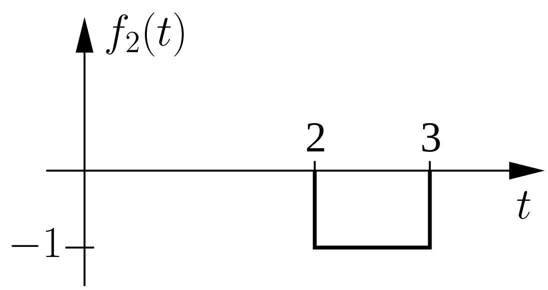
$$c_{12} = \int_{-\infty}^{\infty} s_2(t) f_1^*(t)\,\mathrm{d}t = \frac{1}{\sqrt{2}} 2.$$

Therefore, we obtain

$$\begin{aligned} f_2'(t) &= s_2(t) - c_{12} f_1(t) \\ &= s_2(t) - s_1(t) \end{aligned}$$

Since the energy of $f_2'(t)$ is $E_2 = 1$, the second basis function is

$$f_2(t) = s_2(t) - s_1(t)$$



- **Representation of $s_k(t)$ by Basis Functions $f_k(t)$**

  The original M signals $s_k(t)$ can be represented in terms of the N orthogonal basis functions found by the Gram–Schmidt procedure. In particular, we get

$$s_k(t) = \sum_{n=1}^{N} s_{kn} f_n(t), \qquad k = 1,\, 2,\, \ldots,\, M$$

  with coefficients

$$s_{kn} = \int_{-\infty}^{\infty} s_k(t) f_n^*(t)\, \mathrm{d}t, \qquad n = 1,\, 2,\, \ldots,\, N.$$

  The energy E_k of the coefficient vector $\boldsymbol{s}_k = [s_{k1}\ s_{k2}\ \ldots\ s_{kN}]^T$ of $s_k(t)$ is identical to the energy of $s_k(t)$ itself.

$$E_k = \int_{-\infty}^{\infty} |s_k(t)|^2\, \mathrm{d}t = \sum_{n=1}^{N} |s_{kn}|^2 = ||\boldsymbol{s}_k||^2$$

- **Signal Space Representation of $s_k(t)$**

  For given basis vectors $f_n(t)$, $1 \le n \le N$, the signals $s_k(t)$ are fully described by their coefficient vector $\boldsymbol{s}_k$. The N orthonormal basis functions span the N–dimensional *signal space* and vector $\boldsymbol{s}_k$ contains the coordinates of $s_k(t)$ in that signal space. Therefore, $\boldsymbol{s}_k$ is referred to as the *signal space representation* of $s_k(t)$. Using the signal space representation of signals, many operations such as correlation or energy calculation can be performed with *vectors* instead of *signals*. This has the advantage that for example tedious evaluations of integrals are avoided.

Example:

Continuation of previous example:
Since $s_1(t) = \sqrt{2} f_1(t)$, the signal space representation of $s_1(t)$ is simply

$$\boldsymbol{s}_1 = [\sqrt{2} \;\; 0]^T.$$

For s_{21} and s_{22} we get, respectively,

$$s_{21} = \int_{-\infty}^{\infty} s_2(t) f_1^*(t) \, \mathrm{d}t = \sqrt{2}$$

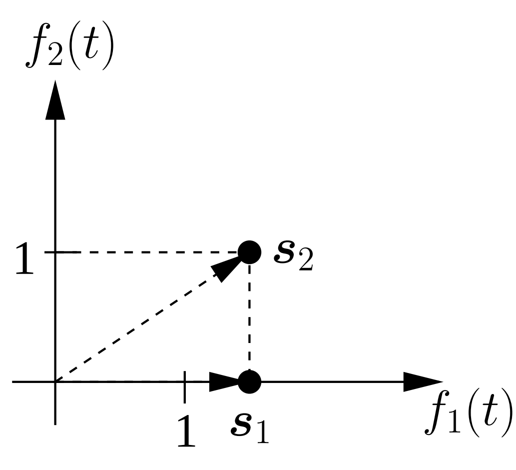
and

$$s_{22} = \int_{-\infty}^{\infty} s_2(t) f_2^*(t) \, \mathrm{d}t = 1.$$

Consequently, the signal space representation of $s_2(t)$ is

$$\boldsymbol{s}_2 = [\sqrt{2} \;\; 1]^T.$$

A graphical representation of the signal space for this example is given below.



Using the signal space representation, e.g. the inner product between $s_1(t)$ and $s_2(t)$ can be easily obtained as

$$< s_1(t),\, s_2(t) > = \boldsymbol{s}_1 \bullet \boldsymbol{s}_2 = \sqrt{2}\sqrt{2} + 0 \cdot 1 = 2$$

- **Complex Baseband Case**

The passband signals $s_m(t)$, $m = 1,\, 2,\, \ldots,\, M$, can be represented by their complex baseband equivalents $s_{bm}(t)$ as

$$s_m(t) = \sqrt{2}\mathrm{Re}\left\{s_{bm}(t)e^{j2\pi f_c t}\right\}.$$

Recall that $s_m(t)$ and $s_{bm}(t)$ have the same energy

$$E_m = \int\limits_{-\infty}^{\infty} s_m^2(t)\,\mathrm{d}t = \int\limits_{-\infty}^{\infty} |s_{bm}(t)|^2\,\mathrm{d}t.$$

Inner Product

We express the inner product between $s_m(t)$ and $s_k(t)$ in terms of the inner product between $s_{bm}(t)$ and $s_{bk}(t)$.

$$\int\limits_{-\infty}^{\infty} s_m(t)s_k^*(t)\,\mathrm{d}t = 2\int\limits_{-\infty}^{\infty} \mathrm{Re}\left\{s_{bm}(t)e^{j2\pi f_c t}\right\} \mathrm{Re}\left\{s_{bk}(t)e^{j2\pi f_c t}\right\}\,\mathrm{d}t$$

$$= \frac{2}{4}\int\limits_{-\infty}^{\infty} \left(s_{bm}(t)e^{j2\pi f_c t} + s_{bm}^*(t)e^{-j2\pi f_c t}\right)$$

$$\left(s_{bk}(t)e^{j2\pi f_c t} + s_{bk}^*(t)e^{-j2\pi f_c t}\right)\,\mathrm{d}t$$

Using a generalized form of *Parsevals Theorem*

$$\int\limits_{-\infty}^{\infty} x(t)y^*(t)\,\mathrm{d}t = \int\limits_{-\infty}^{\infty} X(f)Y^*(f)\,\mathrm{d}f$$

and the fact that both $s_m(t)$ and $s_k(t)$ are narrowband signals, the above integral can be simplified to

$$\begin{aligned}
\int_{-\infty}^{\infty} s_m(t)s_k^*(t)\,\mathrm{d}t &= \frac{1}{2}\int_{-\infty}^{\infty}[S_{bm}(f-f_c)S_{bk}^*(f-f_c)+ \\
&\qquad S_{bm}^*(-f-f_c)S_{bk}(-f-f_c)]\,\mathrm{d}f \\
&= \frac{1}{2}\int_{-\infty}^{\infty}[S_{bm}(f)S_{bk}^*(f)+S_{bm}^*(f)S_{bk}(f)]\,\mathrm{d}f \\
&= \frac{1}{2}\int_{-\infty}^{\infty}[s_{bm}(t)s_{bk}^*(t)+s_{bm}^*(t)s_{bk}(t)]\,\mathrm{d}t \\
&= \int_{-\infty}^{\infty}\mathsf{Re}\left\{s_{bm}(t)s_{bk}^*(t)\right\}\,\mathrm{d}t
\end{aligned}$$

This result shows that the inner product of the passband signals is identical to the real part of the inner product of the corresponding baseband signals.

$$< s_m(t),\, s_k(t) >= \mathsf{Re}\{< s_{bm}(t),\, s_{bk}(t) >\}$$

If we have a signal space representation of passband and baseband signals respectively, we can rewrite the above equation as

$$\boldsymbol{s}_m \bullet \boldsymbol{s}_k = \mathsf{Re}\{\boldsymbol{s}_{bm} \bullet \boldsymbol{s}_{bk}\},$$

where $\boldsymbol{s}_m$ and $\boldsymbol{s}_k$ denote the signals space representation of $s_m(t)$ and $s_k(t)$, respectively, whereas $\boldsymbol{s}_{bm}$ and $\boldsymbol{s}_{bk}$ denote those of $s_{bm}(t)$ and $s_{bk}(t)$, respectively.

Correlation ρ_{km}

In general, the (cross–)correlation between two signals allows to quantify the "similarity" of the signals. The complex correlation ρ^b_{km} of two baseband signals $s_{bk}(t)$ and $s_{bm}(t)$ is defined as

$$\begin{aligned}
\rho^b_{km} &= \frac{< s_{bk}(t),\, s_{bm}(t) >}{\sqrt{E_k E_m}} \\
&= \frac{1}{\sqrt{E_k E_m}} \int\limits_{-\infty}^{\infty} s_{bk}(t) s^*_{bm}(t)\, \mathrm{d}t \\
&= \frac{\boldsymbol{s}_{bk} \bullet \boldsymbol{s}_{bm}}{\sqrt{||\boldsymbol{s}_{bk}|| \cdot ||\boldsymbol{s}_{bm}||}}
\end{aligned}$$

Similarly, the correlation of the passband signals is defined as

$$\begin{aligned}
\rho_{km} &= \frac{< s_k(t),\, s_m(t) >}{\sqrt{E_k E_m}} \\
&= \frac{1}{\sqrt{E_k E_m}} \int\limits_{-\infty}^{\infty} s_k(t) s_m(t)\, \mathrm{d}t \\
&= \frac{\boldsymbol{s}_k \bullet \boldsymbol{s}_m}{\sqrt{||\boldsymbol{s}_k|| \cdot ||\boldsymbol{s}_m||}} \\
&= \mathsf{Re}\{\rho^b_{km}\}
\end{aligned}$$

Euclidean Distance Between Signals $s_m(t)$ and $s_k(t)$

The Euclidean distance d^e_{km} between two passband signals $s_m(t)$ and $s_k(t)$ can be calculated as

$$\begin{aligned} d^e_{km} &= \sqrt{\int\limits_{-\infty}^{\infty} [s_m(t) - s_k(t)]^2 \, \mathrm{d}t} \\ &= ||\boldsymbol{s}_m - \boldsymbol{s}_k|| = \sqrt{||\boldsymbol{s}_m||^2 + ||\boldsymbol{s}_k||^2 - 2\boldsymbol{s}_m \bullet \boldsymbol{s}_k} \\ &= \left(E_m + E_k - 2\sqrt{E_m E_k}\rho_{km}\right)^{1/2} \end{aligned}$$

For comparison, we may calculate the Euclidean distance d^{be}_{km} between the corresponding baseband signals $s_{bm}(t)$ and $s_{bk}(t)$

$$\begin{aligned} d^{be}_{km} &= \sqrt{\int\limits_{-\infty}^{\infty} |s_{bm}(t) - s_{bk}(t)|^2 \, \mathrm{d}t} \\ &= ||\boldsymbol{s}_{bm} - \boldsymbol{s}_{bk}|| = \sqrt{||\boldsymbol{s}_{bm}||^2 + ||\boldsymbol{s}_{bk}||^2 - 2\mathsf{Re}\{\boldsymbol{s}_{bm} \bullet \boldsymbol{s}_{bk}\}} \\ &= \left(E_m + E_k - 2\sqrt{E_m E_k}\mathsf{Re}\{\rho^b_{km}\}\right)^{1/2}. \end{aligned}$$

Since $\mathsf{Re}\{\rho^b_{km}\} = \rho_{km}$ is valid, we conclude that the Euclidean distance of the passband signals is identical to that of the corresponding baseband signals.

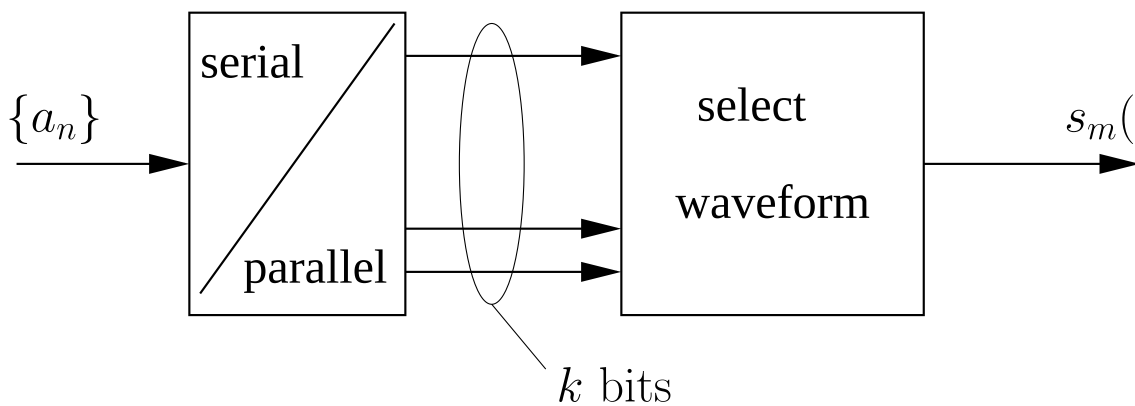
$$d^{be}_{km} = d^e_{km}$$

This important result shows once again, that the baseband representation is really *equivalent* to the passband signal. We will see later that the Euclidean distance between signals used for digital communication determines the achievable error probability, i.e., the probability that a signal different from the actually transmitted signal is detected at the receiver.

3.3 Representation of Digitally Modulated Signals

Modulation:

- We wish to transmit a sequence of binary digits $\{a_n\}$, $a_n \in \{0, 1\}$, over a given physical channel, e.g., wireless channel, cable, etc.
- The *modulator* is the interface between the source that emits the binary digits a_n and the channel. The modulator selects one of $M = 2^k$ *waveforms* $\{s_m(t),\ m = 1, 2, \ldots, M\}$ based on a block of $k = \log_2 M$ binary digits (bits) a_n.



- In the following, we distinguish between

  a) *memoryless modulation* and *modulation with memory*,

  b) *linear modulation* and *non–linear modulation*.

3.3.1 Memoryless Modulation

- In this case, the transmitted waveform $s_m(t)$ depends only on the current k bits but not on previous bits (or, equivalently, previously transmitted waveforms).
- We assume that the bits enter the modulator at a rate of R bits/s.

3.3.1.1 M-ary Pulse–Amplitude Modulation (MPAM)

- MPAM is also referred to as M–ary Amplitude–Shift Keying (MASK).
- **MPAM Waveform**
  The MPAM waveform in passband representation is given by

$$\begin{aligned} s_m(t) &= \sqrt{2}\mathsf{Re}\left\{A_m g(t) e^{j2\pi f_c t}\right\} \\ &= \sqrt{2} A_m g(t) \cos(2\pi f_c t) \qquad m = 1, 2 \ldots, M, \end{aligned}$$

  where we assume for the moment that $s_m(t) = 0$ outside the interval $t \in [0, T]$. T is referred to as the *symbol duration*. We use the following definitions:

  – $A_m = (2m - 1 - M)d$, $m = 1, 2 \ldots, M$, are the M possible amplitudes or symbols, where $2d$ is the distance between two adjacent amplitudes.

  – $g(t)$: real–valued signal pulse of duration T. **Note:** The finite duration condition will be relaxed later.

  – Bit interval (duration) $T_b = 1/R$. The symbol duration is related to the bit duration by $T = kT_b$.

  – PAM symbol rate $R_S = R/k$ symbols/s.

- **Complex Baseband Representation**
  The MPAM waveform can be represented in complex baseband as

$$s_{bm}(t) = A_m g(t).$$

- **Transmitted Waveform**
  The transmitted waveform $s(t)$ for continuous transmission is given by

$$s(t) = \sum_{k=-\infty}^{\infty} s_m(t - kT).$$

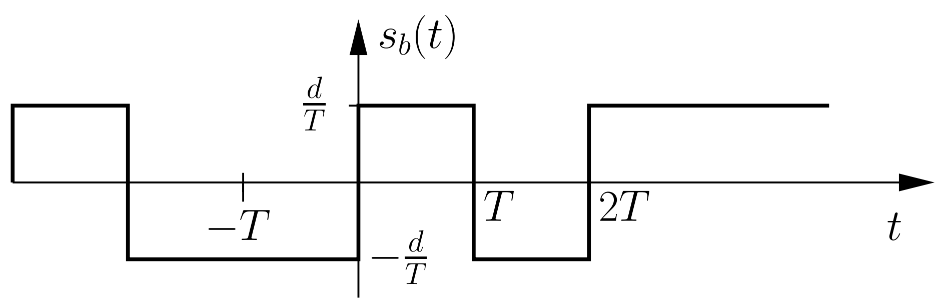
In complex baseband representation, we have

$$s_b(t) = \sum_{k=-\infty}^{\infty} s_{bm}(t - kT) = \sum_{k=-\infty}^{\infty} A_m[k] g(t - kT),$$

where the index k in $A_m[k]$ indicates that the amplitude coefficients depend on time.

Example:

$M = 2$, $g(t)$: rectangular pulse with amplitude $1/T$ and duration T.



Signal Energy

$$E_m = \int_0^T |s_{bm}(t)|^2 \,\mathrm{d}t$$

$$= A_m^2 \int_0^T |g(t)|^2 \,\mathrm{d}t = A_m^2 E_g$$

with pulse energy

$$E_g = \int_0^T |g(t)|^2 \,\mathrm{d}t$$

Signal Space Representation

$s_{bm}(t)$ can be expressed as

$$s_{bm}(t) = s_{bm} f_b(t)$$

with the unit energy waveform (basis function)

$$f_b(t) = \frac{1}{\sqrt{E_g}} g(t)$$

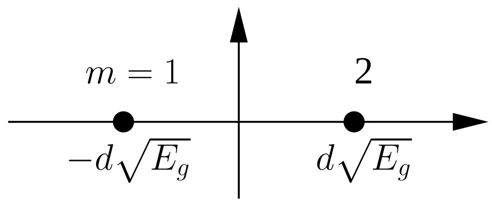
and signal space coefficient

$$s_m = \sqrt{E_g} A_m.$$

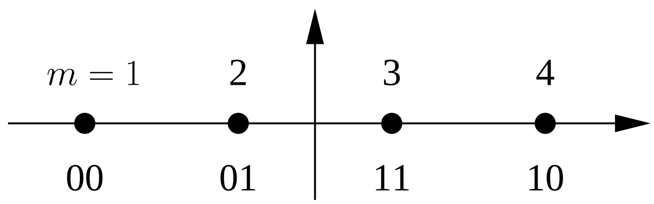
The same representation is obtained if we start from the passband signal $s_m(t) = s_m f(t)$ and use basis function $f(t) = \sqrt{\frac{2}{E_g}} g(t) \cos(2\pi f_c t)$.

Example:

1. $M = 2$



2. $M = 4$ with *Gray labeling* of signal points



- **Labeling of Signal Points**
  Each of the $M = 2^k$ possible combinations that can be generated by k bits, has to address one signal $s_m(t)$ and consequently one signal point $\mathbf{s}_m$. The assignment of bit combinations to symbols is called *labeling* or *mapping*. At first glance, this labeling seems to be arbitrary. However, we will see later on that it is advantageous if **adjacent signal points differ only in one bit**. Such a labeling is called **Gray labeling**. The advantage of Gray labeling is a comparatively low bit error probability, since after transmission over a noisy channel the most likely error event is that we select an adjacent signal point instead of the actually transmitted one. In case of Gray labeling, one such symbol error translates into exactly one bit error.

- **Euclidean Distance Between Signal Points**

For the Euclidean distance between two signals or equivalently two signal points, we get

$$\begin{aligned} d^e_{mn} &= \sqrt{(s_m - s_n)^2} \\ &= \sqrt{E_g}|A_m - A_n| \\ &= 2\sqrt{E_g}d|m-n| \end{aligned}$$

In practice, often the *minimum* Euclidean distance $d^e_{\min}$ of a modulation scheme plays an important role. For PAM signals we get,

$$\begin{aligned} d^e_{\min} &= \min_{\substack{n,m \\ m \neq n}} \{d^e_{mn}\} \\ &= 2\sqrt{E_g}d. \end{aligned}$$

- **Special Case:** $M = 2$

In this case, the signal set contains only two functions:

$$\begin{aligned} s_1(t) &= -\sqrt{2}dg(t)\cos(2\pi f_c t) \\ s_2(t) &= \sqrt{2}dg(t)\cos(2\pi f_c t). \end{aligned}$$

Obviously,

$$s_1(t) = -s_2(t)$$

is valid. The correlation in that case is

$$\rho_{12} = -1.$$

Because of the above properties, $s_1(t)$ and $s_2(t)$ are referred to as *antipodal* signals. 2PAM is an example for antipodal modulation.

3.3.1.2 M-ary Phase–Shift Keying (MPSK)

- **MPSK Waveform**

The MPSK waveform in passband representation is given by

$$\begin{aligned} s_m(t) &= \sqrt{2}\mathsf{Re}\left\{e^{j2\pi(m-1)/M} g(t) e^{j2\pi f_c t}\right\} \\ &= \sqrt{2} g(t)\mathsf{cos}(2\pi f_c t + \Theta_m) \\ &= \sqrt{2} g(t)\mathsf{cos}(\Theta_m)\mathsf{cos}(2\pi f_c t) \\ &\quad -\sqrt{2} g(t)\mathsf{sin}(\Theta_m)\mathsf{sin}(2\pi f_c t), \qquad m = 1, 2 \ldots, M, \end{aligned}$$

where we assume again that $s_m(t) = 0$ outside the interval $t \in [0, T]$ and

$$\Theta_m = 2\pi(m-1)/M, \qquad m = 1, 2 \ldots, M,$$

denotes the information conveying phase of the carrier.

- **Complex Baseband Representation**

The MPSK waveform can be represented in complex baseband as

$$s_{bm}(t) = e^{j2\pi(m-1)/M} g(t) = e^{j\Theta_m} g(t).$$

- **Signal Energy**

$$\begin{aligned} E_m &= \int_0^T |s_{bm}(t)|^2 \,\mathrm{d}t \\ &= \int_0^T |g(t)|^2 \,\mathrm{d}t = E_g \end{aligned}$$

This means that all signals of the set have the *same* energy.

■ Signal Space Representation

$s_{bm}(t)$ can be expressed as

$$s_{bm}(t) = s_{bm} f_b(t)$$

with the unit energy waveform (basis function)

$$f_b(t) = \frac{1}{\sqrt{E_g}} g(t)$$

and (complex–valued) signal space coefficient

$$\begin{aligned} s_{bm} &= \sqrt{E_g} e^{j2\pi(m-1)/M} = \sqrt{E_g} e^{j\Theta_m} \\ &= \sqrt{E_g} \mathsf{cos}(\Theta_m) + j\sqrt{E_g} \mathsf{sin}(\Theta_m). \end{aligned}$$

On the other hand, the *passband* signal can be written as

$$s_m(t) = s_{m1} f_1(t) + s_{m2} f_2(t)$$

with the orthonormal functions

$$\begin{aligned} f_1(t) &= \sqrt{\frac{2}{E_g}} g(t) \mathsf{cos}(2\pi f_c t) \\ f_2(t) &= -\sqrt{\frac{2}{E_g}} g(t) \mathsf{sin}(2\pi f_c t) \end{aligned}$$

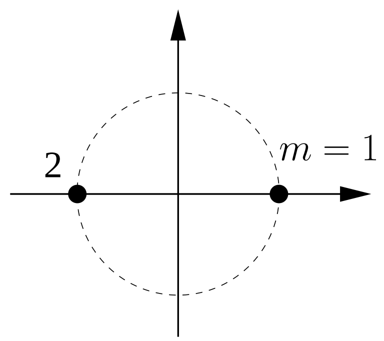
and the signal space representation

$$\boldsymbol{s}_m = [\sqrt{E_g} \mathsf{cos}(\Theta_m) \;\; \sqrt{E_g} \mathsf{sin}(\Theta_m)]^T.$$

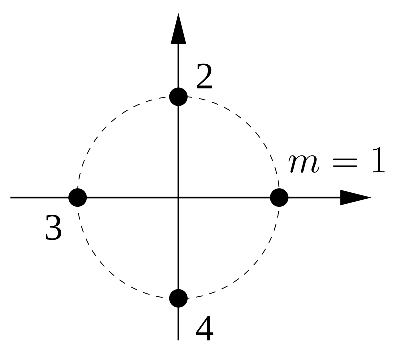
It is interesting to compare the signal space representation of the baseband and passband signals. In the complex baseband, we get one basis function but the coefficients s_{bm} are complex valued, i.e., we have an one–dimensional *complex* signal space. In the passband, we have two basis functions and the elements of $\boldsymbol{s}_m$ are real valued, i.e., we have a two–dimensional *real* signal space. Of course, both representations are equivalent and contain the same information about the original passband signal.

Example:

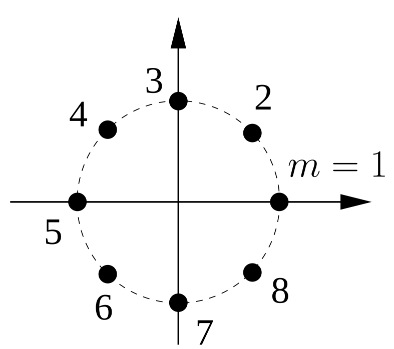
1. $M = 2$



2. $M = 4$



3. $M = 8$



- **Euclidean Distance**

$$
\begin{aligned}
d^e_{mn} &= ||s_{bm} - s_{bn}|| \\
&= \sqrt{|\sqrt{E_g}e^{j\Theta_m} - \sqrt{E_g}e^{j\Theta_n}|^2} \\
&= \sqrt{E_g(2 - 2\mathsf{Re}\{e^{j(\Theta_m - \Theta_n)}\})^2} \\
&= \sqrt{2E_g(1 - \mathsf{cos}[2\pi(m-n)/M])}
\end{aligned}
$$

The minimum Euclidean distance of MPSK is given by

$$
d^e_{\min} = \sqrt{2E_g(1 - \mathsf{cos}[2\pi/M])}
$$

3.3.1.3 M-ary Quadrature Amplitude Modulation (MQAM)

- **MQAM Waveform**

The MQAM waveform in passband representation is given by

$$
\begin{aligned}
s_m(t) &= \sqrt{2}\mathsf{Re}\left\{(A_{cm} + jA_{sm})g(t)e^{j2\pi f_c t}\right\} \\
&= \sqrt{2}\mathsf{Re}\left\{A_m g(t)e^{j2\pi f_c t}\right\} \\
&= \sqrt{2}A_{cm}g(t)\mathsf{cos}(2\pi f_c t) - \sqrt{2}A_{sm}g(t)\mathsf{sin}(2\pi f_c t),
\end{aligned}
$$

for $m = 1, 2 \ldots, M$, and we assume again that $g(t) = 0$ outside the interval $t \in [0, T]$.

- $\mathsf{cos}(2\pi f_c t)$ and $\mathsf{sin}(2\pi f_c t)$ are referred to as the *quadrature carriers*.
- $A_m = A_{cm} + jA_{sm}$ is the *complex* information carrying amplitude.
- $\frac{1}{2}\log_2 M$ bits are mapped to A_{cm} and A_{sm}, respectively.

- **Complex Baseband Representation**

  The MPSK waveform can be represented in complex baseband as

$$s_{bm}(t) = (A_{cm} + jA_{sm})g(t) = A_m g(t).$$

- **Signal Energy**

$$\begin{aligned} E_m &= \int_0^T |s_{bm}(t)|^2 \, \mathrm{d}t \\ &= |A_m|^2 \int_0^T |g(t)|^2 \, \mathrm{d}t = |A_m|^2 E_g. \end{aligned}$$

- **Signal Space Representation**

  $s_{bm}(t)$ can be expressed as

$$s_{bm}(t) = s_{bm} f_b(t)$$

  with the unit energy waveform (basis function)

$$f_b(t) = \frac{1}{\sqrt{E_g}} g(t)$$

  and (complex–valued) signal space coefficient

$$s_{bm} = \sqrt{E_g}(A_{cm} + jA_{sm}) = \sqrt{E_g} A_m$$

  Similar to the PSK case, also for QAM a one–dimensional complex signal space representation results. Also for QAM the signal space representation of the passband signal requires a two–dimensional real space.

- **Euclidean Distance**

$$\begin{aligned} d^e_{mn} &= ||s_{bm} - s_{bn}|| \\ &= \sqrt{|\sqrt{E_g}(A_m - A_n)|^2} \\ &= \sqrt{E_g}|A_m - A_n| \\ &= \sqrt{E_g}\sqrt{(A_{cm} - A_{cn})^2 + (A_{sm} - A_{sn})^2} \end{aligned}$$

- **How to Choose Amplitudes**

So far, we have not specified the amplitudes A_{cm} and A_{sm}. In principle, arbitrary sets of amplitudes A_{cm} and A_{sm} can be chosen. However, in practice, usually the spacing of the amplitudes is equidistant, i.e.,

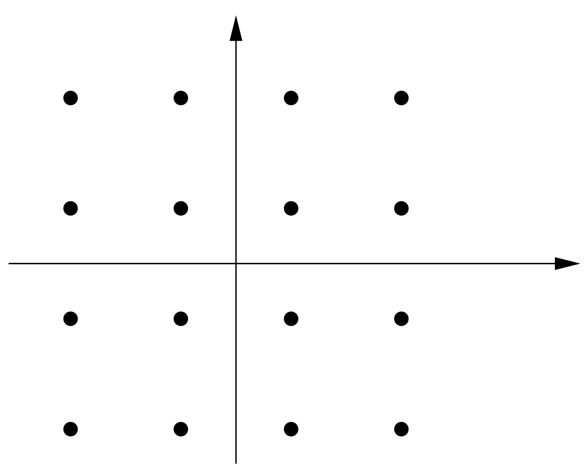
$$A_{cm}, A_{sm} \in \{\pm d, \pm 3d, \ldots, \pm(\sqrt{M} - 1)d\}.$$

In that case, the minimum distance of MQAM is

$$d^e_{\min} = 2\sqrt{E_g}d$$

Example:

$M = 16$:



3.3.1.4 Multi–Dimensional Modulation

In the previous sections, we found out that PAM has a one–dimensional (real) signal space representation, i.e., PAM is a one–dimensional modulation format. On the other hand, PSK and QAM have a two–dimensional real signal space representation (or equivalently a one–dimensional complex representation). This can be concept can be further extended to *multi–dimensional* modulation schemes. Multi–dimensional modulation schemes can be obtained by jointly modulating several symbols in the time and/or frequency domain.

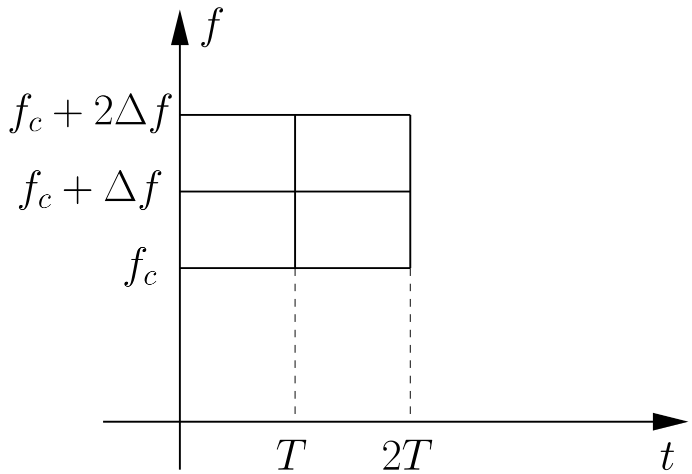
- **Time Domain Approach**
  In the time domain, we may *jointly* modulate the signals transmitted in N consecutive symbol intervals. If we use a PAM waveform in each interval, an N–dimensional modulation scheme results, whereas an $2N$–dimensional scheme results if we base the modulation on PSK or QAM waveforms.

- **Frequency Domain Approach**
  N–dimensional (or $2N$–dimensional) modulation schemes can also be constructed by jointly modulating the signals transmitted over N carriers, that are separated by frequency differences of Δf.

- **Combined Approach**
  Of course, the time and the frequency domain approaches can be combined. For example, if we jointly modulate the PAM waveforms transmitted in N_1 symbol intervals and over N_2 carriers, an $N_1 N_2$–dimensional signal results.

Example:

Assume $N_1 = 2$, $N_2 = 3$:



3.3.1.5 M-ary Frequency–Shift Keying (MFSK)

- MFSK is an example for an (orthogonal) multi–dimensional modulation scheme.
- **MFSK Waveform**
  The MFSK waveform in passband representation is given by

$$
\begin{aligned}
s_m(t) &= \sqrt{\frac{2E}{T}} \mathsf{Re}\left\{ e^{j2\pi m \Delta f t} e^{j2\pi f_c t} \right\} \\
&= \sqrt{\frac{2E}{T}} \mathsf{cos}[2\pi (f_c + m\Delta f)t], \qquad m = 1,\, 2\, \ldots,\, M
\end{aligned}
$$

  and we assume again that $s_m(t) = 0$ outside the interval $t \in [0,\, T]$. E is the energy of $s_m(t)$.

- **Complex Baseband Representation**
  The MFSK waveform can be represented in complex baseband as

$$s_{bm}(t) = \sqrt{\frac{E}{T}} e^{j2\pi m\Delta f t}.$$

- **Correlation and Orthogonality**
  The correlation ρ^b_{mn} between the baseband signals $s_{bn}(t)$ and $s_{bn}(t)$ is given by

$$\begin{aligned}
\rho^b_{mn} &= \frac{1}{\sqrt{E_m E_n}} \int\limits_{-\infty}^{\infty} s_{bm}(t) s^*_{bn}(t)\, \mathrm{d}t \\
&= \frac{1}{E}\frac{E}{T} \int\limits_{0}^{T} e^{j2\pi m\Delta f t} e^{-j2\pi n\Delta f t}\, \mathrm{d}t \\
&= \frac{1}{T} \frac{e^{j2\pi(m-n)\Delta f t}}{j2\pi(m-n)\Delta f}\Bigg|_0^T \\
&= \frac{\sin[\pi(m-n)\Delta f T]}{\pi(m-n)\Delta f T}\, e^{j\pi(m-n)\Delta f T}
\end{aligned}$$

  Using this result, we obtain for the correlation ρ_{mn} of the corresponding passband signals $s_m(t)$ and $s_n(t)$

$$\begin{aligned}
\rho_{mn} &= \mathrm{Re}\{\rho^b_{mm}\} \\
&= \frac{\sin[\pi(m-n)\Delta f T]}{\pi(m-n)\Delta f T} \cos[\pi(m-n)\Delta f T] \\
&= \frac{\sin[2\pi(m-n)\Delta f T]}{2\pi(m-n)\Delta f T}
\end{aligned}$$

  Now, it is easy to show that $\rho_{mn} = 0$ is true for $m \neq n$ if

$$\Delta f T = k/2, \qquad k \in \{\pm 1, \pm 2, \ldots\}$$

The smallest frequency separation that results in M *orthogonal* signals is $\Delta f = 1/(2T)$. In practice, both orthogonality and narrow spacing in frequency are desirable. Therefore, usually $\Delta f = 1/(2T)$ is chosen for MFSK.

It is interesting to calculate the complex correlation in that case:

$$\begin{aligned}\rho^b_{mn} &= \frac{\sin[\pi(m-n)/2]}{\pi(m-n)/2}\, e^{j\pi(m-n)/2} \\ &= \begin{cases} 0, & (m-n)\,\text{even},\ m \neq n \\ 2j/[\pi(m-n)], & (m-n)\,\text{odd} \end{cases}\end{aligned}$$

This means that for a given signal $s_{bm}(t)$ $(M/2-1)$ other signals are orthogonal in the sense that the correlation is zero, whereas the remaining $M/2$ signals are orthogonal in the sense that the correlation is purely imaginary.

- **Signal Space Representation** $(\Delta f T = 1/2)$

Since the M passband signals $s_m(t)$ are orthogonal to each other, they can be directly used as basis functions after proper normalization, i.e., the basis functions are given by

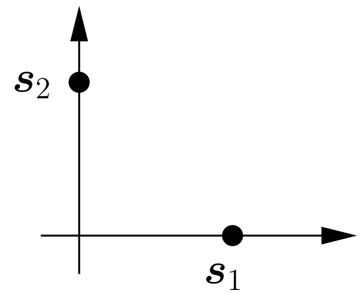
$$f_m(t) = \sqrt{\frac{2}{T}}\cos(2\pi f_c t + \pi m t/T),$$

and the resulting signal space representation is

$$\begin{aligned}\boldsymbol{s}_1 &= [\sqrt{E}\;\; 0\;\; \ldots\;\; 0]^T \\ \boldsymbol{s}_2 &= [0\;\; \sqrt{E}\;\; 0\;\; \ldots\;\; 0]^T \\ &\;\;\vdots \\ \boldsymbol{s}_M &= [0\;\; \ldots\;\; 0\;\; \sqrt{E}]^T\end{aligned}$$

Example:

$M = 2$:



- **Euclidean Distance**

$$\begin{aligned} d^e_{mn} &= ||s_m - s_n|| \\ &= \sqrt{2E}. \end{aligned}$$

This means the Euclidean distance between *any* two signal points is $\sqrt{2E}$. Therefore, the *minimum* Euclidean distance is also $d^e_{\min} = \sqrt{2E}$.

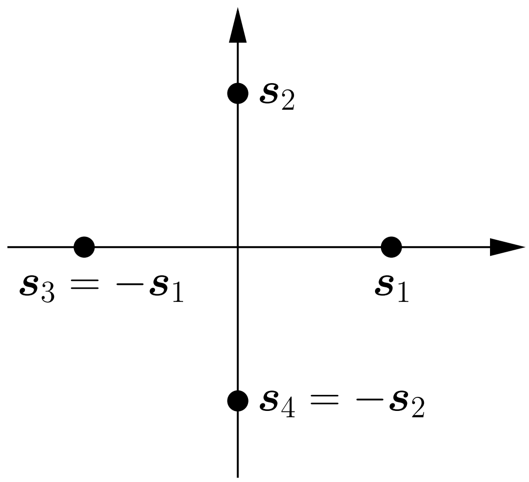
- **Biorthogonal Signals**

A set of $2M$ biorthogonal signals is derived from a set of M orthogonal signals $\{s_m(t)\}$ by also including the negative signals $\{-s_m(t)\}$.

For biorthogonal signals the Euclidean distance between pairs of signals is either $d^e_{mn} = \sqrt{2E}$ or $d^e_{mn} = 2\sqrt{E}$. The correlation is either $\rho_{mn} = -1$ or $\rho_{mn} = 0$.

Example:

$M = 2$:



■ Simplex Signals

– Consider a set of M orthogonal waveforms $s_m(t)$, $1 \le m \le M$.

– The mean of the waveforms is

$$\bar{\boldsymbol{s}} = \frac{1}{M}\sum_{m=1}^{M} \boldsymbol{s}_m = \frac{\sqrt{E}}{M}\mathbf{1}_M,$$

where $\mathbf{1}_M$ is the M–dimensional all–ones column vector.

– In practice, often zero–mean waveforms are preferred. Therefore, it is desirable to modify the orthogonal signal set to a signal set with zero mean.

– The mean can be removed by

$$\boldsymbol{s}'_m = \boldsymbol{s}_m - \bar{\boldsymbol{s}}.$$

The set $\{\boldsymbol{s}'_m\}$ has zero mean and the waveforms $\boldsymbol{s}'_m$ are called *simplex signals*.

– Energy

$$\begin{aligned}
||\boldsymbol{s}'_m||^2 &= ||\boldsymbol{s}_m - \bar{\boldsymbol{s}}||^2 = ||\boldsymbol{s}_m||^2 - 2\boldsymbol{s}_m \bullet \bar{\boldsymbol{s}} + ||\bar{\boldsymbol{s}}||^2 \\
&= E - 2\frac{1}{M}E + \frac{1}{M}E = E\left(1 - \frac{1}{M}\right)
\end{aligned}$$

We observe that simplex waveforms require less energy than orthogonal waveforms to achieve the same minimum Euclidean distance $d^e_{\min} = \sqrt{2E}$.

– Correlation

$$\rho_{mn} = \frac{\boldsymbol{s}'_m \bullet \boldsymbol{s}'_n}{||\boldsymbol{s}'_m|| \cdot ||\boldsymbol{s}'_n||} = \frac{-1/M}{1 - 1/M} = -\frac{1}{M-1}$$

Simplex waveforms are equally correlated.

3.3.2 Linear Modulation With Memory

- Waveforms transmitted in successive symbol intervals are mutually dependent. This dependence is responsible for the memory of the modulation.

- One example for a linear modulation scheme with memory is a *line code*. In that case, the memory is introduced by filtering the transmitted signal (e.g. PAM or QAM signal) with a linear filter to shape its spectrum.

- Another important example is *differentially encoded PSK* (or just differential PSK). Here, memory is introduced in order to enable noncoherent detection at the receiver, i.e., detection without knowledge of the channel phase.

3.3.2.1 M–ary Differential Phase–Shift Keying (MDPSK)

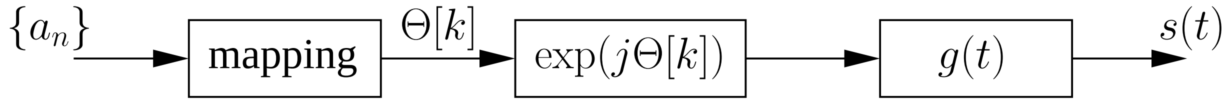
- **PSK Signal**
  Recall that the PSK waveform in complex baseband representation is given by

$$s_{bm}(t) = e^{j\Theta_m} g(t),$$

where $\Theta_m = 2\pi(m-1)/M$, $m \in \{1, 2, \dots, M\}$. For continuous–transmission the transmit signal $s(t)$ is given by

$$s(t) = \sum_{k=-\infty}^{\infty} e^{j\Theta[k]} g(t - kT),$$

where the index k in $\Theta[k]$ indicates that a new phase is chosen in every symbol interval and where we have dropped the symbol index m for simplicity.



- **DPSK**

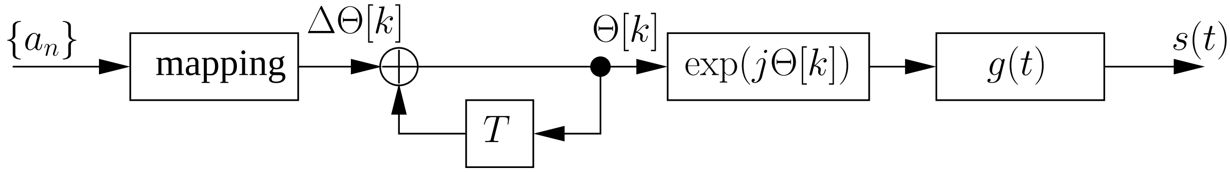
  In DPSK, the bits are mapped to the *difference* of two consecutive signal phases. If we denote the phase difference by $\Delta\Theta_m$, then in MDPSK $k = \log_2 M$ bits are mapped to

$$\Delta\Theta_m = 2\pi(m-1)/M, \qquad m \in \{1, 2\, \dots, M\}.$$

  If we drop again the symbol index m, and introduce the time index k, the absolute phase $\Theta[k]$ of the transmitted symbol is obtained as

$$\Theta[k] = \Theta[k-1] + \Delta\Theta[k].$$

  Since the information is carried in the *phase difference*, in the receiver detection can be performed based on the phase difference of the waveforms received in two consecutive symbol intervals, i.e., knowledge of the absolute phase of the received signal is not required.



3.3.3 Nonlinear Modulation With Memory

In most nonlinear modulation schemes with memory, the memory is introduced by forcing the transmitted signal to have a *continuous* phase. Here, we will discuss *continuous–phase FSK (CPFSK)* and more general *continuous–phase modulation (CPM)*.

3.3.3.1 Continuous–Phase FSK (CPFSK)

- **FSK**

  – In conventional FSK, the transmit signal $s(t)$ at time k can be generated by shifting the carrier frequency $f[k]$ by an amount of

  $$\Delta f[k] = \frac{1}{2}\Delta f\, I[k], \qquad I[k] \in \{\pm 1, \pm 3, \ldots, \pm(M-1)\},$$

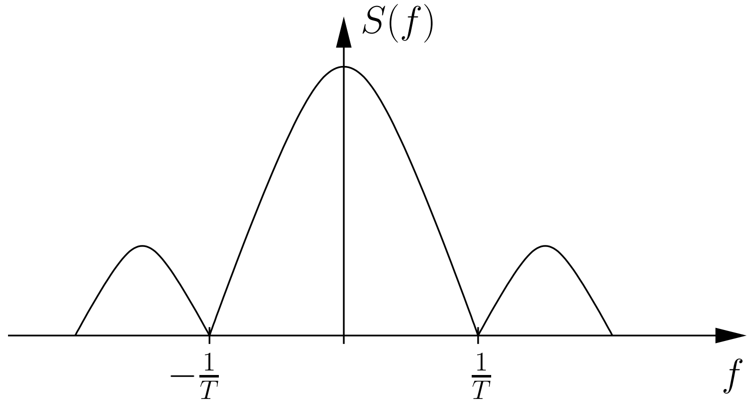
  where $I[k]$ reflects the transmitted digital information at time k (i.e., in the kth symbol interval).

  Proof. For FSK we have

  $$\begin{aligned} f[k] &= f_c + m[k]\Delta f \\ &= f_c + \frac{2m[k] - M - 1 + M + 1}{2}\Delta f \\ &= f_c + \left(\frac{1}{2}(2m[k] - M - 1) + \frac{M+1}{2}\right)\Delta f \\ &= f'_c + \frac{1}{2}\Delta f\, I[k], \end{aligned}$$

  where $f'_c = f_c + \frac{M+1}{2}\Delta f$ and $I[k] = 2m[k] - M - 1$. If we interpret f'_c as new carrier frequency, we have the desired representation. □

– FSK is memoryless and the abrupt switching from one frequency to another in successive symbol intervals results in a relatively *broad* spectrum $S(f) = \mathcal{F}\{s(t)\}$ of the transmit signal with large *side lobes*.



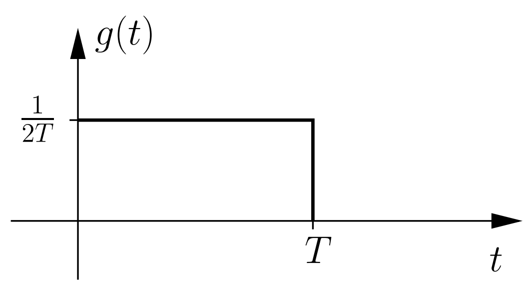
– The large side lobes can be avoided by changing the *carrier frequency* smoothly. The resulting signal has a *continuous* phase.

■ CPFSK Signal

– We first consider the PAM signal

$$d(t) = \sum_{k=-\infty}^{\infty} I[k]g(t-kT),$$

where $I[k] \in \{\pm 1, \pm 2, \ldots, \pm(M-1)\}$ and $g(t)$ is a rectangular pulse with amplitude $1/(2T)$ and duration T.



Clearly, the FSK signal in complex baseband representation can be expressed as

$$s_b(t) = \sqrt{\frac{E}{T}} \exp\left(j2\pi\Delta f T \sum_{k=-\infty}^{\infty} I[k](t-kT)g(t-kT) \right),$$

where E denotes the energy of the signal in one symbol interval. Since $t\,g(t)$ is a discontinuous function, the phase $\phi(t) = 2\pi\Delta f T \sum_{k=-\infty}^{\infty} I[k](t-kT)g(t-kT)$ of $s(t)$ will jump between symbol intervals. The main idea behind CPFSK is now to avoid this jumping of the phase by integrating over $d(t)$.

– **CPFSK Baseband Signal**

The CPFSK signal in complex baseband representation is

$$s_b(t) = \sqrt{\frac{E}{T}} \exp(j[\phi(t,\, \boldsymbol{I}) + \phi_0])$$

with information carrying *carrier phase*

$$\phi(t,\, \boldsymbol{I}) = 4\pi f_d T \int_{-\infty}^{t} d(\tau)\, \mathrm{d}\tau$$

and the definitions

f_d: peak frequency deviation
ϕ_0: initial carrier phase
$\boldsymbol{I}$: information sequence $\{I[k]\}$

– **CPFSK Passband Signal**

The CPFSK signal in passband representation is given by

$$s(t) = \sqrt{\frac{2E}{T}} \cos[2\pi f_c t + \phi(t,\, \boldsymbol{I}) + \phi_0].$$

– **Information Carrying Phase**

The information carrying phase in the interval $kT \leq t \leq (k+1)T$ can be rewritten as

$$
\begin{aligned}
\phi(t, \boldsymbol{I}) &= 4\pi f_d T \int\limits_{-\infty}^{t} d(\tau)\,\mathrm{d}\tau \\
&= 4\pi f_d T \int\limits_{-\infty}^{t} \sum_{n=-\infty}^{\infty} I[n] g(\tau - nT)\,\mathrm{d}\tau \\
&= 4\pi f_d T \sum_{n=-\infty}^{k-1} I[n] \underbrace{\int\limits_{-\infty}^{kT} g(\tau - nT)\,\mathrm{d}\tau}_{\frac{1}{2}} \\
&\quad +4\pi f_d T I[k] \underbrace{\int\limits_{kT}^{(k+1)T} g(\tau - kT)\,\mathrm{d}\tau}_{q(t-kT)=\frac{1}{2T}(t-kT)} \\
&= \underbrace{2\pi f_d T \sum_{n=-\infty}^{k-1} I[n]}_{\Theta[k]} + 4\pi f_d T I[k] q(t-kT) \\
&= \Theta[k] + 2\pi h I[k] q(t-kT),
\end{aligned}
$$

where

$h = 2f_dT$: modulation index

$\Theta[k]$: accumulated phase *memory*

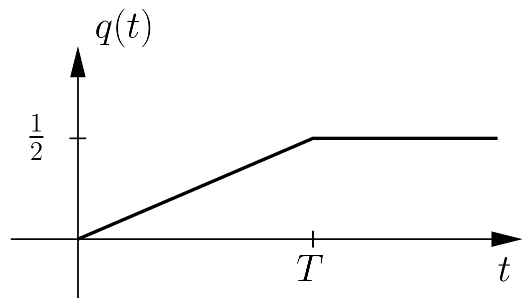
$q(t)$ is referred to as the *phase pulse shape*, and since

$$g(t) = \mathrm{d}q(t)/\mathrm{d}t$$

is valid, $g(t)$ is referred to as the *frequency pulse shape*. The above representation of the information carrying phase $\phi(t,\, \boldsymbol{I})$ clearly illustrates that CPFSK is a modulation format with *memory*.

The phase pulse shape $q(t)$ is given by

$$q(t) = \begin{cases} 0, & t < 0 \\ \frac{t}{2T}, & 0 \leq t \leq T \\ \frac{1}{2}, & t > T \end{cases}$$



3.3.3.2 Continuous Phase Modulation (CPM)

- CPM can be viewed as a generalization of CPFSK. For CPM more general frequency pulse shapes $g(t)$ and consequently more general phase pulse shapes

$$q(t) = \int\limits_{-\infty}^{t} g(\tau)\,\mathrm{d}\tau$$

  are used.

■ CPM Waveform

The CPM waveform in complex baseband representation is given by

$$s_b(t) = \sqrt{\frac{E}{T}} \exp(j[\phi(t,\, \boldsymbol{I}) + \phi_0]),$$

whereas the corresponding passband signal is

$$s(t) = \sqrt{\frac{2E}{T}} \cos[2\pi f_c t + \phi(t,\, \boldsymbol{I}) + \phi_0],$$

where the information–carrying phase in the interval $kT \leq t \leq (k+1)T$ is given by

$$\phi(t,\, \boldsymbol{I}) = 2\pi h \sum_{n=-\infty}^{k} I[n] q(t - nT).$$

h is the *modulation index*.

■ Properties of $g(t)$

– The integral over the so–called frequency pulse $g(t)$ is always 1/2.

$$\int\limits_0^\infty g(\tau)\, \mathrm{d}\tau = q(\infty) = \frac{1}{2}$$

– **Full Response CPM**

In *full response* CPM

$$g(t) = 0, \qquad t \geq T.$$

is valid.

– **Partial Response CPM**
In *partial response* CPM

$$g(t) \neq 0, \qquad t \geq T.$$

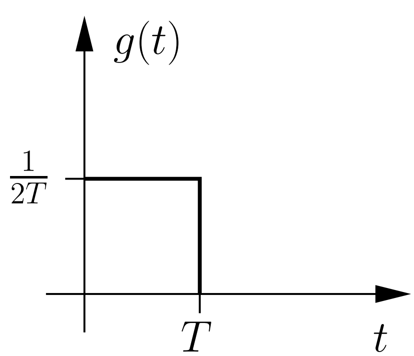
is valid.

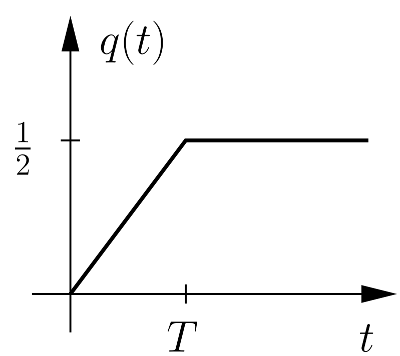
Example:

1. CPM with rectangular frequency pulse of length L (LREC)

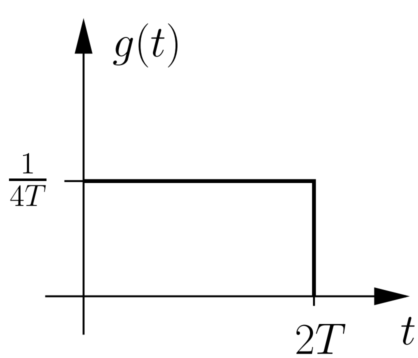
$$g(t) = \begin{cases} \frac{1}{2LT}, & 0 \leq t \leq LT \\ 0, & \text{otherwise} \end{cases}$$

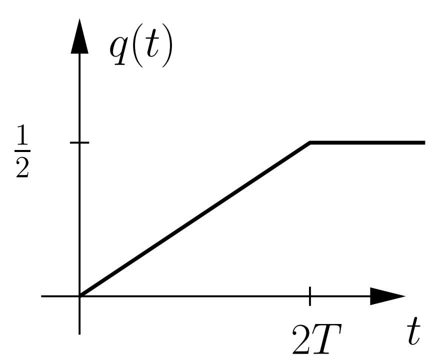
$L = 1$ (CPFSK):





$L = 2$:

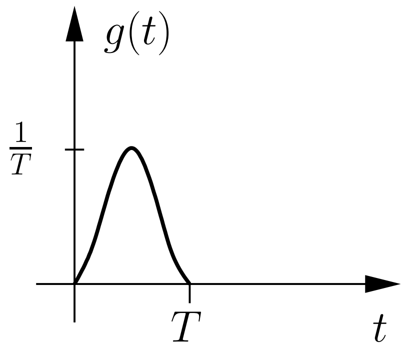


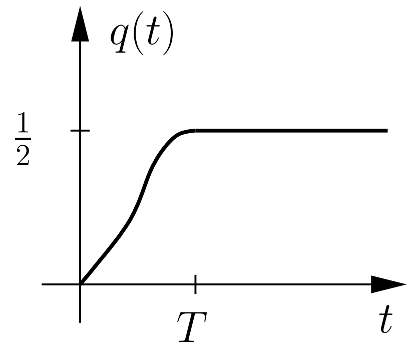


2. CPM with raised cosine frequency pulse of length L (LRC)

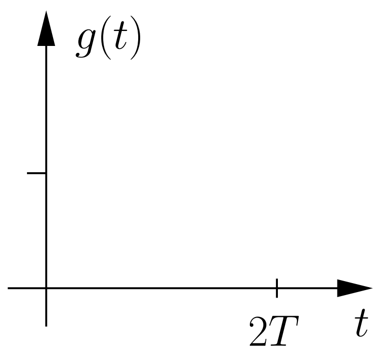
$$g(t) = \begin{cases} \frac{1}{2LT}\left(1 - \cos\left(\frac{2\pi t}{LT}\right)\right), & 0 \le t \le LT \\ 0, & \text{otherwise} \end{cases}$$

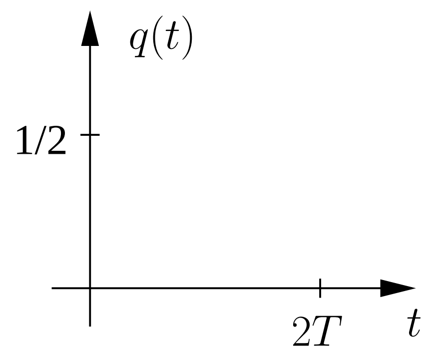
$L = 1$:





$L = 2$:





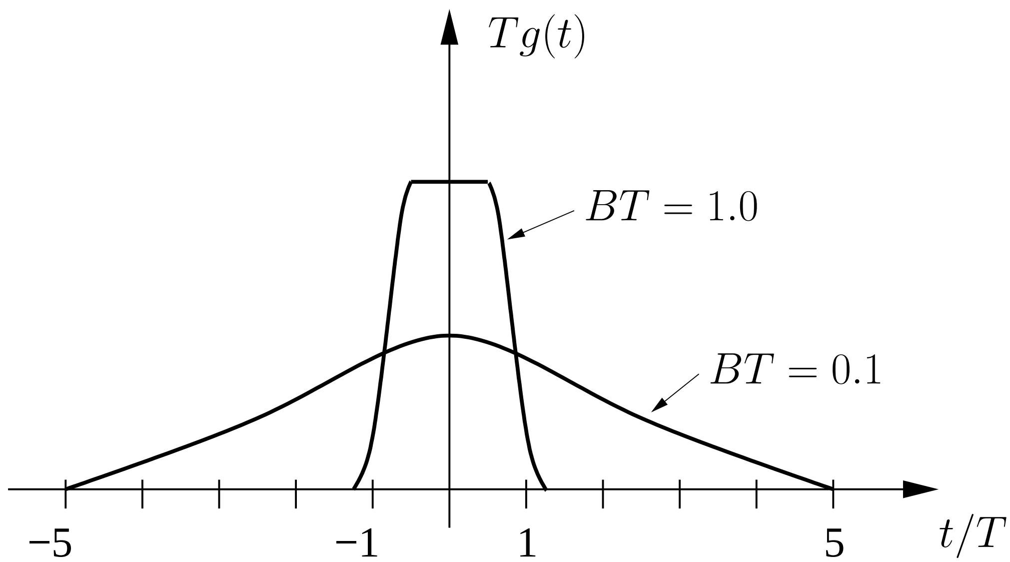
3. Gaussian Minimum–Shift Keying (GMSK)

$$g(t) = \frac{1}{T}\left(Q\left[\frac{2\pi B}{\sqrt{\ln 2}}\left(t - \frac{T}{2}\right)\right] - Q\left[\frac{2\pi B}{\sqrt{\ln 2}}\left(t + \frac{T}{2}\right)\right]\right)$$

with the Q–function

$$Q(x) = \frac{1}{\sqrt{2\pi}} \int\limits_{x}^{\infty} e^{-t^2/2}\, \mathrm{d}t$$

BT is the normalized bandwidth parameter and represents the -3 dB bandwidth of the GMSK impulse. Note that the GMSK pulse duration increases with decreasing BT.



■ Phase Tree

The *phase trajectories* $\phi(t, \boldsymbol{I})$ of all possible input sequences $\{I[k]\}$ can be represented by a *phase tree*. For construction of the tree it is assumed that the phase at a certain time t_0 (usually $t_0 = 0$) is known. Usually, $\phi(0, \boldsymbol{I}) = 0$ is assumed.

Example:

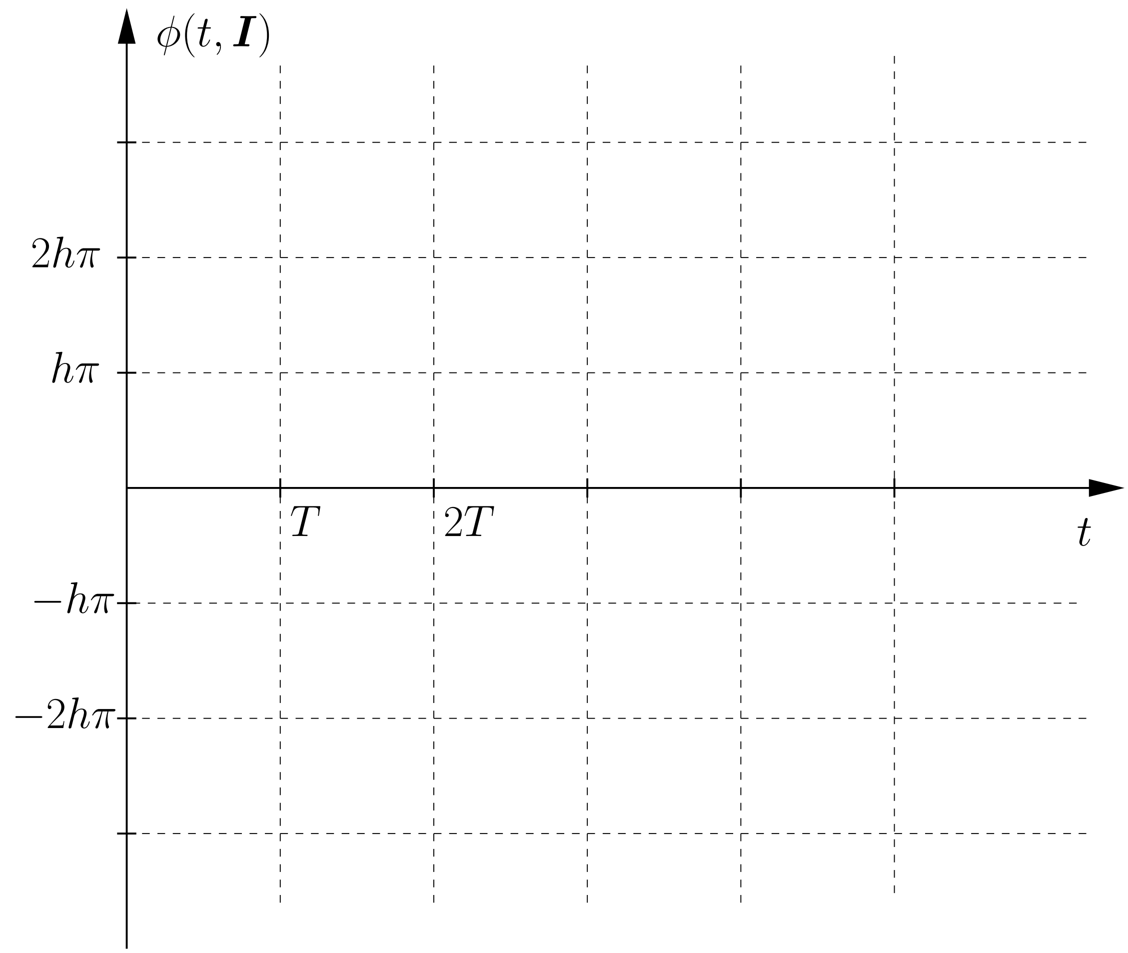
Binary CPFSK
In the interval $kT \leq t \leq (k+1)T$, we get

$$\phi(t, \boldsymbol{I}) = \pi h \sum_{n=-\infty}^{k-1} I[n] + 2\pi h I[k] q(t - kT),$$

where $I[k] \in \{\pm 1\}$ and

$$q(t) = \begin{cases} 0, & t < 0 \\ t/(2T), & 0 \leq t \leq T \\ 1/2, & t > T \end{cases}$$

If we assume $\phi(0, \boldsymbol{I}) = 0$, we get the following phase tree.



- **Phase Trellis**

  The *phase trellis* is obtained by plotting the phase trajectories modulo 2π. If h is furthermore given by

  $$h = \frac{q}{p},$$

  where q and p are relatively prime integers, it can be shown that $\phi(kT, \boldsymbol{I})$ can assume only a *finite* number of different values. These values are called the (terminal) states of the trellis.

Example:

Full response CPM with odd q
In that case, we get

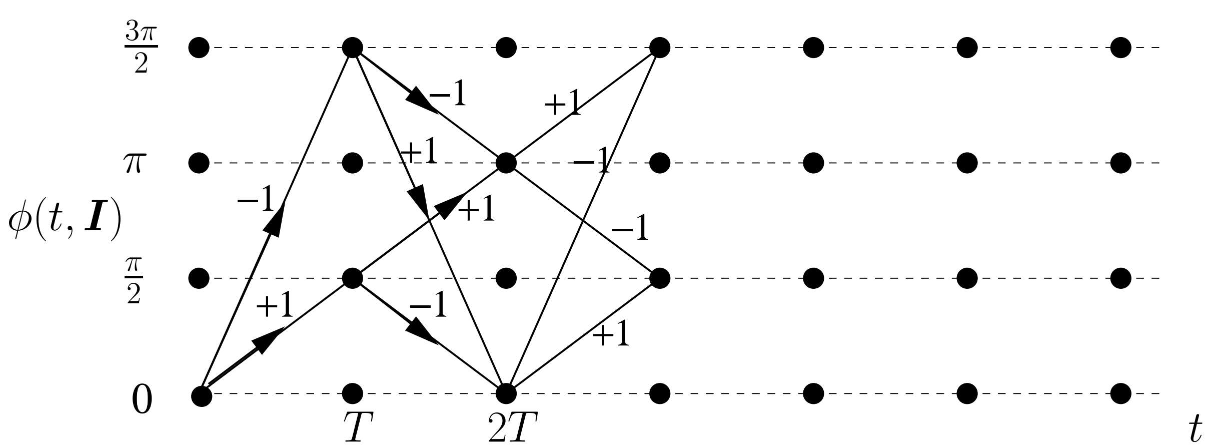
$$\begin{aligned}\phi(kT,\, \boldsymbol{I}) &= \pi h \sum_{n=-\infty}^{k} I[n] \\ &= \pi \frac{q}{p}\{0,\, \pm 1,\, \pm 2,\, \ldots\}.\end{aligned}$$

From the above equation we conclude that $\phi(kT,\, \boldsymbol{I}) \bmod 2\pi$ can assume only the values

$$\Theta_S = \left[0,\, \frac{\pi q}{p},\, 2\frac{\pi q}{p},\, \ldots,\, (2p-1)\frac{\pi q}{p}\right]$$

If we further specialize this result to CPFSK with $h = \frac{1}{2}$, which is also referred to as *minimum shift–keying (MSK)*, we get

$$\Theta_S = \left[0,\, \frac{\pi}{2},\, \pi,\, \frac{3\pi}{2}\right]$$



Number of States

More generally, it can be shown that for general M–ary CPM the number S of states is given by

$$S = \begin{cases} pM^{L-1}, & \text{even}\, q \\ 2pM^{L-1}, & \text{odd}\, q \end{cases}$$

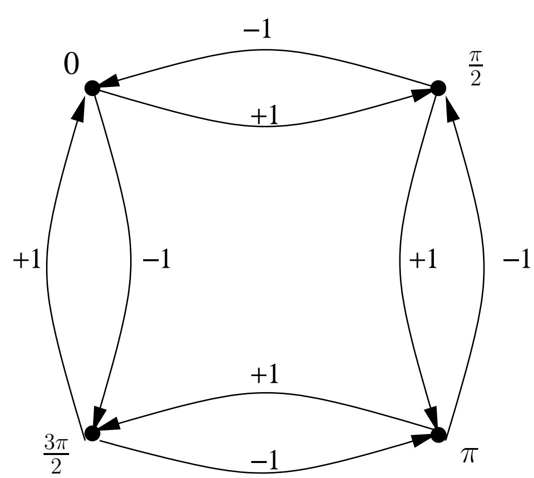
The number of states is an important indicator for the receiver complexity required for optimum detection of the corresponding CPM scheme.

- **Phase State Diagram**

  Yet another representation of the CPM phase results if we just display the terminal phase states along with the possible transitions between these states. This representation is referred to as *phase state diagram* and is more compact than the trellis representation. However, all information related to time is not contained in the phase state diagram.

Example:

Binary CPFSK with $h = \frac{1}{2}$



- **Minimum Shift Keying (MSK)**
  CPFSK with modulation index $h = \frac{1}{2}$ is also referred to as MSK. In this case, in the interval $kT \leq t \leq (k+1)T$ the signal phase can be expressed as

$$\phi(t, \boldsymbol{I}) = \Theta[k] + \frac{1}{2}\pi I[k] \left(\frac{t - kT}{T} \right)$$

with

$$\Theta[k] = \frac{1}{2}\pi \sum_{n=-\infty}^{i-1} I[n]$$

Therefore, the modulated signal is given by

$$\begin{aligned} s(t) &= \sqrt{\frac{2E}{T}} \cos\left[2\pi f_c t + \Theta[k] + \frac{1}{2}\pi I[k] \left(\frac{t - kT}{T} \right) \right] \\ &= \sqrt{\frac{2E}{T}} \cos\left[2\pi \left(f_c + \frac{1}{4T} I[k] \right) t + \Theta[k] - \frac{1}{2} k\pi I[k] \right]. \end{aligned}$$

Obviously, we can interpret $s(t)$ as a sinusoid having one of two possible frequencies. Using the definitions

$$\begin{aligned} f_1 &= f_c - \frac{1}{4T} \\ f_2 &= f_c + \frac{1}{4T} \end{aligned}$$

we can rewrite $s(t)$ as

$$s_i(t) = \sqrt{\frac{2E}{T}} \cos\left[2\pi f_i t + \Theta[k] + \frac{1}{2} k\pi (-1)^{i-1} \right], \qquad i = 1, 2.$$

Obviously, $s_1(t)$ and $s_2(t)$ are two sinusoids that are separated by $\Delta f = f_2 - f_1 = 1/(2T)$ in frequency. Since this is the minimum frequency separation required for orthogonality of $s_1(t)$ and $s_2(t)$, CPFSK with $h = \frac{1}{2}$ is referred to as MSK.

- **Offset Quadrature PSK (OQPSK)**
It can be shown that the MSK signal $s_b(t)$ in complex baseband representation can also be rewritten as

$$s_b(t) = \sqrt{\frac{E}{T}} \sum_{k=-\infty}^{\infty} [I[2k]g(t-2kT) - jI[2k+1]g(t-2kT-T)],$$

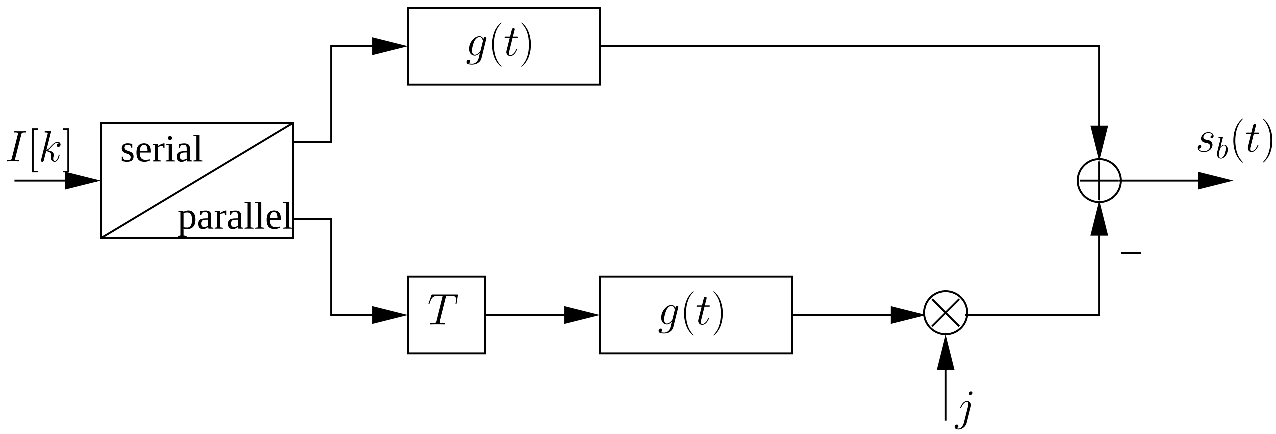
where the transmit pulse shape $g(t)$ is given by

$$g(t) = \begin{cases} \sin\left(\frac{\pi t}{2T}\right), & 0 \le t \le 2T \\ 0, & \text{otherwise} \end{cases}$$

Therefore, the passband signal can be represented as

$$\begin{aligned} s(t) &= \sqrt{\frac{2E}{T}} \left[\sum_{k=-\infty}^{\infty} I[2k]g(t-2kT)\right] \cos(2\pi f_c t) \\ &\quad + \sqrt{\frac{2E}{T}} \left[\sum_{k=-\infty}^{\infty} I[2k+1]g(t-2kT-T)\right] \sin(2\pi f_c t). \end{aligned}$$

The above representation of MSK allows the interpretation as a 4PSK signal, where the inphase and quadrature components are *staggered* by T, which corresponds to half the symbol duration of $g(t)$. Therefore, MSK is also referred to as *staggered QPSK* or *offset QPSK (OQPSK)*.



Although the equivalence between MSK and OQPSK only holds for the special $g(t)$ given above, in practice often other transmit pulse shapes are employed for OQPSK. In that case, OQPSK does not have a constant envelope but the variation of the envelope is still smaller than in case of conventional QPSK.

- **Important Properties of CPM**

  – CPM signals have a *constant envelope*, i.e., $|s_b(t)| = \text{const.}$ Therefore, efficient nonlinear amplifiers can be employed to boost the CPM transmit signal.

  – For low–to–moderate M (e.g. $M \leq 4$) CPM can be made more *power and bandwidth efficient* than simpler linear modulation schemes such as PSK or QAM.

  – *Receiver design* for CPM is more complex than for linear modulation schemes.

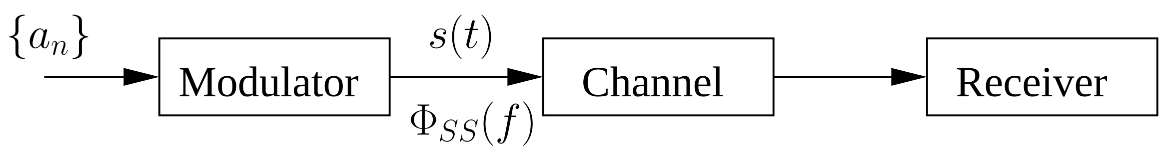
■ Alternative Representations for CPM Signal

- *Signal Space Representation*
  Because of the inherent memory, a simple description of CPM signals in the signal space is in general not possible.
- *Linear Representation*
  CPM signals can be described as a linear superposition of PAM waveforms (Laurent representation, 1986).
- *Representation as Trellis Encoder and Memoryless Mapper*
  CPM signals can also be described as a trellis encoder followed by a memoryless mapper (Rimoldi decomposition, 1988).

Both Laurent's and Rimoldi's alternative representations of CPM signals are very useful for receiver design.

3.4 Spectral Characteristics of Digitally Modulated Signals

In a practical system, the available bandwidth that can be used for transmission is *limited*. Limiting factors may be the bandlimitedness of the channel (e.g. wireline channel) or other users that use the same transmission medium in frequency multiplexing systems (e.g. wireless channel, cellular system). Therefore, it is important that we design communication schemes that use the available bandwidth as efficiently as possible. For this reason, it is necessary to know the spectral characteristics of digitally modulated signals.



3.4.1 Linearly Modulated Signals

- **Given:**
  Passband signal

$$s(t) = \sqrt{2}\mathsf{Re}\left\{s_b(t)e^{j2\pi f_c t}\right\}$$

with baseband signal

$$s_b(t) = \sum_{k=-\infty}^{\infty} I[k]g(t-kT)$$

where

  – $I[k]$: Sequence of symbols, e.g. PAM, PSK, or QAM symbols
    Note: $I[k]$ is complex valued for PSK and QAM.

- T: Symbol duration. $1/T = R/k$ symbols/s is the transmission rate, where R and k denote the bit rate and the number of bits mapped to one symbol.
- $g(t)$: Transmit pulse shape

- **Problem:** Find the frequency spectrum of $s(t)$.
- **Solution:**
  - The spectrum $\Phi_{SS}(f)$ of the passband signal $s(t)$ can be expressed as
  $$\Phi_{SS}(f) = \frac{1}{2}[\Phi_{S_bS_b}(f - f_c) + \Phi^*_{S_bS_b}(-f - f_c)],$$
  where $\Phi_{S_bS_b}(f)$ denotes the spectrum of the equivalent baseband signal $s_b(t)$. We observe that $\Phi_{SS}(f)$ can be easily determined, if $\Phi_{S_bS_b}(f)$ is known. Therefore, in the following, we concentrate on the calculation of $\Phi_{S_bS_b}(f)$.
  - $\{I[k]\}$ is a sequence of random variables, i.e., $\{I[k]\}$ is a discrete–time random process. Therefore, $s_b(t)$ is also a random process and we have to calculate the *power spectral density* of $s_b(t)$, since a spectrum in the deterministic sense does not exist.

ACF of $s_b(t)$

$$\phi_{S_bS_b}(t+\tau, t) = \mathcal{E}\{s_b^*(t)s_b(t+\tau)\}$$

$$= \mathcal{E}\left\{\left(\sum_{k=-\infty}^{\infty} I^*[k]g^*(t-kT)\right)\left(\sum_{k=-\infty}^{\infty} I[k]g(t+\tau-kT)\right)\right\}$$

$$= \sum_{n=-\infty}^{\infty}\sum_{k=-\infty}^{\infty} \mathcal{E}\{I^*[k]I[n]\}\, g^*(t-kT)g(t+\tau-nT)$$

$\{I[k]\}$ can be assumed to be wide–sense stationary with mean

$$\mu_I = \mathcal{E}\{I[k]\}$$

and ACF

$$\phi_{II}[\lambda] = \mathcal{E}\{I^*[k]I[k+\lambda]\}.$$

Therefore, the ACF of $s_b(t)$ can be simplified to

$$\begin{aligned}\phi_{S_bS_b}(t+\tau,t) &= \sum_{n=-\infty}^{\infty}\sum_{k=-\infty}^{\infty}\phi_{II}[n-k]g^*(t-kT)g(t+\tau-nT)\\ &= \sum_{\lambda=-\infty}^{\infty}\phi_{II}[\lambda]\sum_{k=-\infty}^{\infty}g^*(t-kT)g(t+\tau-kT-\lambda T)\end{aligned}$$

From the above representation we observe that

$$\phi_{S_bS_b}(t+\tau+mT,t+mT) = \phi_{S_bS_b}(t+\tau,t),$$

for $m \in \{\ldots,\, -1,\, 0,\, 1,\, \ldots\}$. In addition, the mean $\mu_{S_b}(t)$ is also periodic in T

$$\mu_{S_b}(t) = \mathcal{E}\{s_b(t)\} = \mu_I \sum_{k=-\infty}^{\infty} g(t-kT).$$

Since both mean $\mu_{S_b}(t)$ and ACF $\phi_{S_bS_b}(t+\tau,t)$ are periodic, $s_b(t)$ is a *cyclostationary* process with period T.

Average ACF of $s_b(t)$

Since we are only interested in the *average* spectrum of $s_b(t)$, we eliminate the dependence of $\phi_{S_bS_b}(t+\tau, t)$ on t by averaging over one period T.

$$\begin{aligned}
\bar{\phi}_{S_bS_b}(\tau) &= \frac{1}{T}\int\limits_{-T/2}^{T/2} \phi_{S_bS_b}(t+\tau, t)\,\mathrm{d}t \\
&= \sum_{\lambda=-\infty}^{\infty}\phi_{II}[\lambda]\sum_{k=-\infty}^{\infty}\frac{1}{T}\int\limits_{-T/2}^{T/2} g^*(t-kT)\cdot \\
&\qquad\qquad g(t+\tau-kT-\lambda T)\,\mathrm{d}t \\
&= \sum_{\lambda=-\infty}^{\infty}\phi_{II}[\lambda]\sum_{k=-\infty}^{\infty}\frac{1}{T}\int\limits_{-T/2-kT}^{T/2-kT} g^*(t')\cdot \\
&\qquad\qquad g(t'+\tau-\lambda T)\,\mathrm{d}t' \\
&= \sum_{\lambda=-\infty}^{\infty}\phi_{II}[\lambda]\frac{1}{T}\int\limits_{-\infty}^{\infty} g^*(t')g(t'+\tau-\lambda T)\,\mathrm{d}t'
\end{aligned}$$

Now, we introduce the *deterministic* ACF of $g(t)$ as

$$\phi_{gg}(\tau) = \int\limits_{-\infty}^{\infty} g^*(t)g(t+\tau)\,\mathrm{d}t,$$

and obtain

$$\boxed{\bar{\phi}_{S_bS_b}(\tau) = \frac{1}{T}\sum_{\lambda=-\infty}^{\infty}\phi_{II}[\lambda]\phi_{gg}(\tau-\lambda T)}$$

(Average) Power Spectral Density of $s_b(t)$

The (average) power spectral density $\Phi_{S_bS_b}(f)$ of $s_b(t)$ is given by

$$\begin{aligned}
\Phi_{S_bS_b}(f) &= \mathcal{F}\{\bar{\phi}_{S_bS_b}(\tau)\} \\
&= \int_{-\infty}^{\infty}\left(\frac{1}{T}\sum_{\lambda=-\infty}^{\infty}\int_{-\infty}^{\infty}\phi_{II}[\lambda]g^*(t)g(t+\tau-\lambda T)\,\mathrm{d}t\right)e^{-j2\pi f\tau}\,\mathrm{d}\tau \\
&= \frac{1}{T}\sum_{\lambda=-\infty}^{\infty}\phi_{II}[\lambda]\int_{-\infty}^{\infty}g^*(t)\int_{-\infty}^{\infty}g(t+\tau-\lambda T)e^{-j2\pi f\tau}\,\mathrm{d}\tau\,\mathrm{d}t \\
&= \frac{1}{T}\sum_{\lambda=-\infty}^{\infty}\phi_{II}[\lambda]\int_{-\infty}^{\infty}g^*(t)G(f)e^{j2\pi ft}e^{-j2\pi fT\lambda}\,\mathrm{d}t \\
&= \frac{1}{T}\sum_{\lambda=-\infty}^{\infty}\phi_{II}[\lambda]e^{-j2\pi fT\lambda}|G(f)|^2
\end{aligned}$$

Using the fact that the *discrete–time Fourier transform* of the ACF $\phi_{II}[\lambda]$ is given by

$$\Phi_{II}(f) = \sum_{\lambda=-\infty}^{\infty}\phi_{II}[\lambda]e^{-j2\pi fT\lambda},$$

we obtain for the average power spectral density of $s_b(t)$ the elegant expression

$$\boxed{\Phi_{S_bS_b}(f) = \frac{1}{T}|G(f)|^2\Phi_{II}(f)}$$

– Observe that $G(f) = \mathcal{F}\{g(t)\}$ directly influences the spectrum $\Phi_{II}(f)$. Therefore, the transmit pulse shape $g(t)$ can be used to *shape* the spectrum of $s_b(t)$, and consequently that of the passband signal $s(t)$.

– Notice that $\Phi_{II}(f)$ is periodic with period $1/T$.

Important Special Case

In practice, the information symbols $\{I[k]\}$ are usually mutually uncorrelated and have variance σ_I^2. The corresponding ACF $\phi_{II}[\lambda]$ is given by

$$\phi_{II}[\lambda] = \begin{cases} \sigma_I^2 + |\mu_I|^2, & \lambda = 0 \\ |\mu_I|^2, & \lambda \neq 0 \end{cases}$$

In this case, the spectrum of $I[k]$ can be rewritten as

$$\Phi_{II}(f) = \sigma_I^2 + |\mu_I|^2 \sum_{\lambda=-\infty}^{\infty} e^{-j2\pi fT\lambda}$$

Since $\sum_{\lambda=-\infty}^{\infty} e^{-j2\pi fT\lambda}$ can be interpreted as a *Fourier series* representation of $\frac{1}{T}\sum_{\lambda=-\infty}^{\infty} \delta(f-\lambda/T)$, we get

$$\Phi_{II}(f) = \sigma_I^2 + \frac{|\mu_I|^2}{T} \sum_{\lambda=-\infty}^{\infty} \delta\left(f - \frac{\lambda}{T}\right),$$

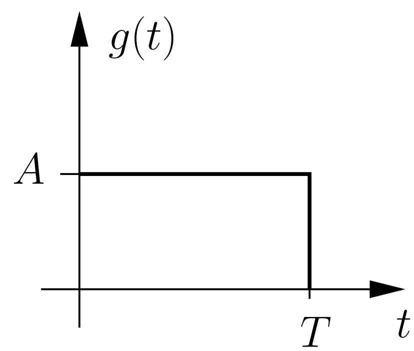
we obtain for $\Phi_{S_bS_b}(f)$ the expression

$$\boxed{\Phi_{S_bS_b}(f) = \frac{\sigma_I^2}{T}|G(f)|^2 + \frac{|\mu_I|^2}{T^2} \sum_{\lambda=-\infty}^{\infty} \left|G\left(\frac{\lambda}{T}\right)\right|^2 \delta\left(f - \frac{\lambda}{T}\right)}$$

This representation of $\Phi_{S_bS_b}(f)$ shows that a nonzero mean $\mu_I \neq 0$ leads to Dirac–Impulses in the spectrum of $s_b(t)$, which is in general not desirable. Therefore, in practice, nonzero mean information sequences are preferred.

Example:

1. Rectangular Pulse

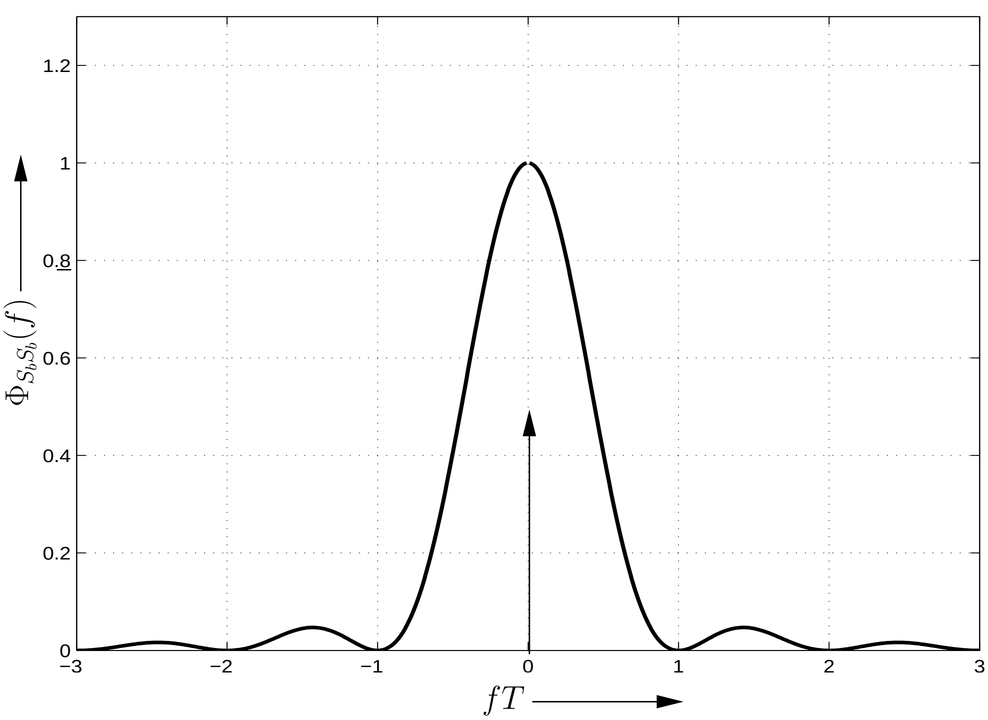


The frequency response of $g(t)$ is given by

$$G(f) = AT\,\frac{\sin(\pi f T)}{\pi f T}\,e^{-j\pi f T}$$

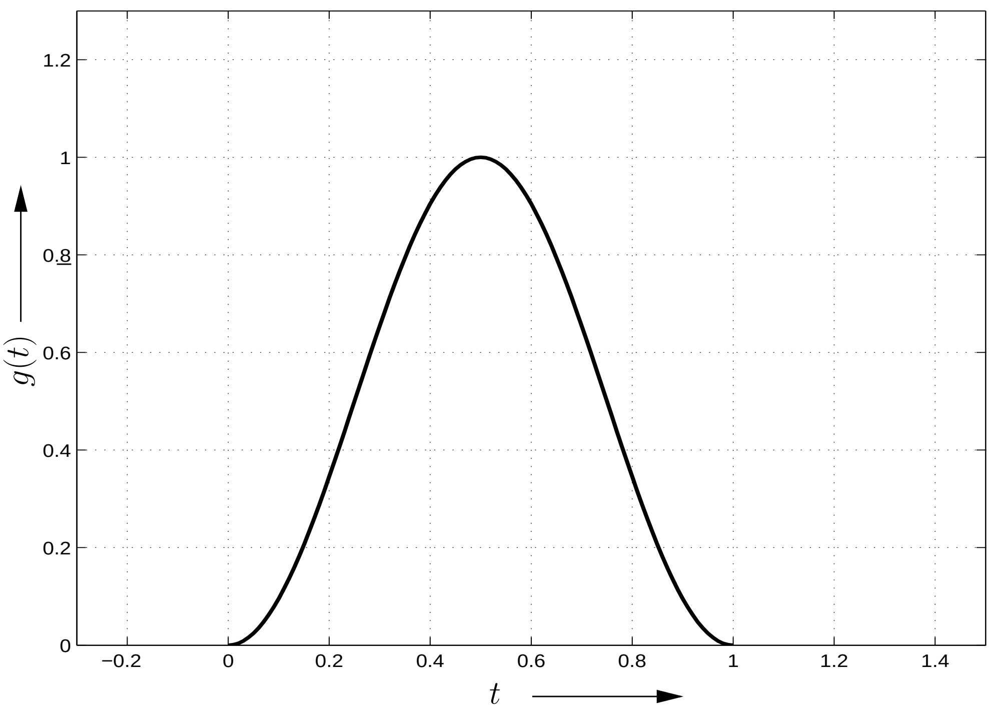
Thus, the spectrum of $s_b(t)$ is

$$\Phi_{S_bS_b}(f) = \sigma_I^2 A^2 T \left(\frac{\sin(\pi f T)}{\pi f T}\right)^2 + A^2|\mu_I|^2\delta(f)$$



2. Raised Cosine Pulse The raised cosine pulse is given by

$$g(t) = \frac{A}{2}\left(1 + \cos\left[\frac{2\pi}{T}\left(t - \frac{T}{2}\right)\right]\right), \quad 0 \leq T \leq T$$



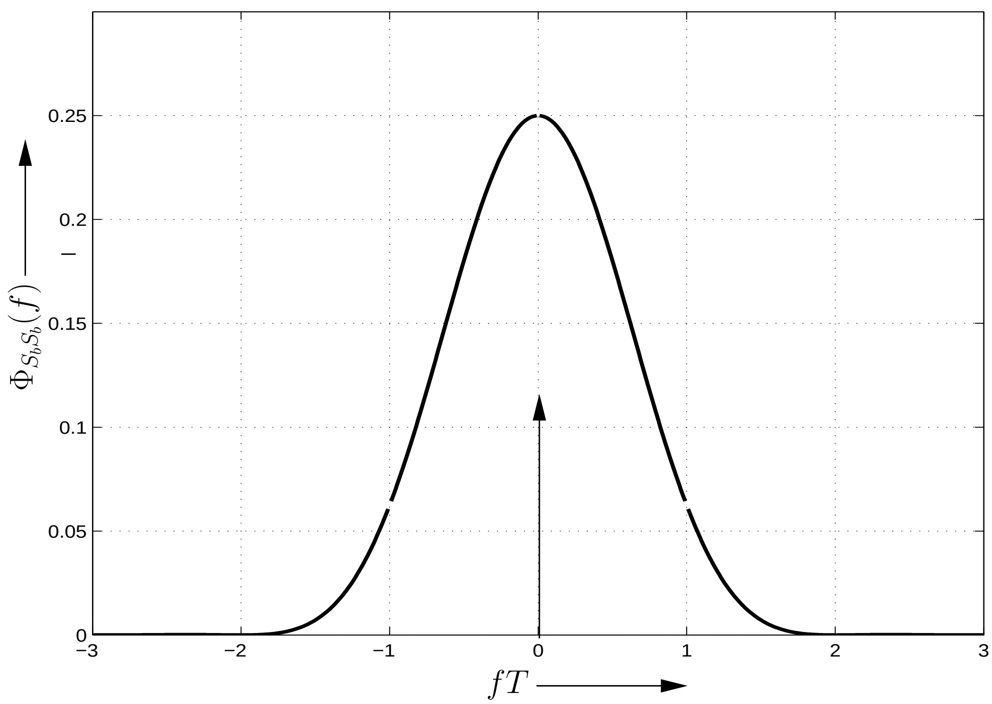
The frequency response of the raised cosine pulse is

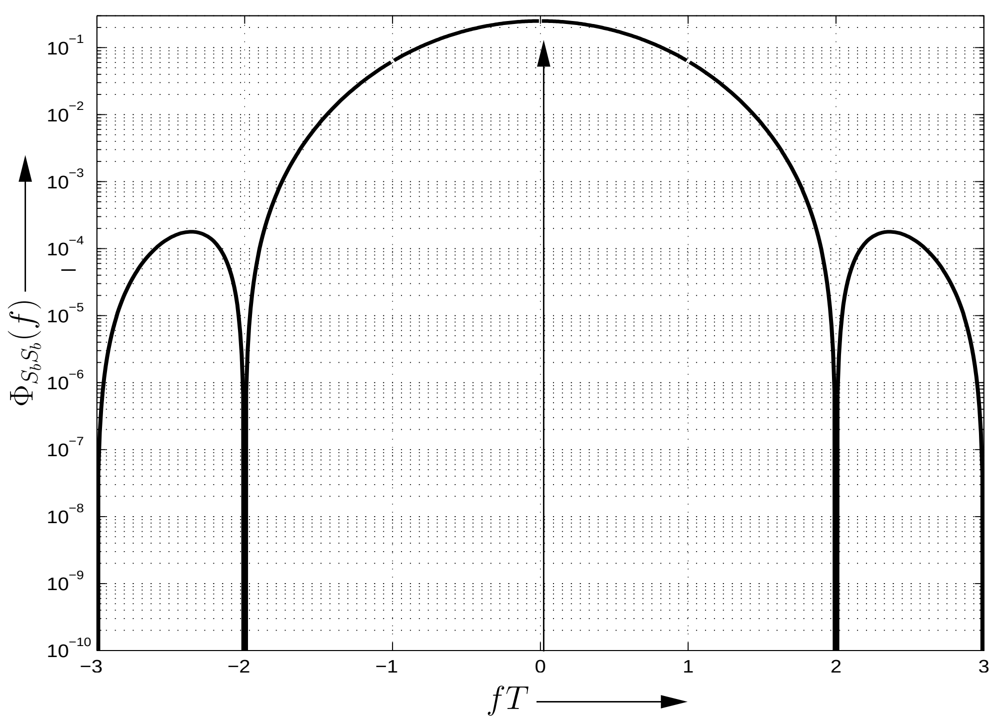
$$G(f) = \frac{AT}{2}\,\frac{\sin(\pi f T}{\pi f T(1 - f^2T^2)}\,e^{-j\pi f T}$$

The spectrum of $s_b(t)$ can be obtained as

$$\Phi_{S_bS_b}(f) = \frac{\sigma_I^2 A^2 T^2}{4}\,\left(\frac{\sin(\pi f T}{\pi f T(1 - f^2T^2)}\right)^2 + \frac{|\mu_I|^2 A^2}{4}\,\delta(f)$$

The continuous part of the spectrum decays much more rapidly than for the rectangular pulse.


0.25
0.2
0.15
0.1
0.05
0
−3
−2
−1
0
1
2
3
$\Phi_{S_bS_b}(f)$
fT


10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^{-5}
10^{-6}
10^{-7}
10^{-8}
10^{-9}
10^{-10}
−3
−2
−1
0
1
2
3
$\Phi_{S_bS_b}(f)$
fT

3.4.2 CPFSK and CPM

- In general, calculation of the power spectral density of nonlinear modulation formats is very involved, cf. text book pp. 207–221.

Example:

Text book Figs. 4.4-3 – 4.4-10.

4 Optimum Reception in Additive White Gaussian Noise (AWGN)

In this chapter, we derive the optimum receiver structures for the modulation schemes introduced in Chapter 3 and analyze their performance.

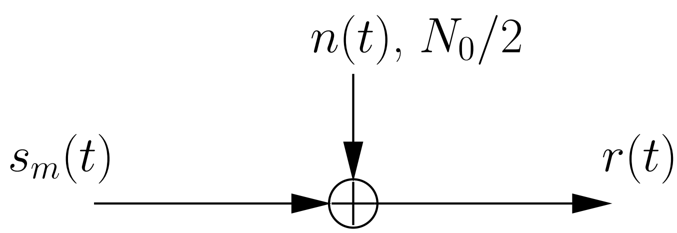
4.1 Optimum Receivers for Signals Corrupted by AWGN

■ Problem Formulation

– We first consider memoryless linear modulation formats. In symbol interval $0 \le t \le T$, information is transmitted using one of M possible waveforms $s_m(t)$, $1 \le m \le M$.

– The received passband signal $r(t)$ is corrupted by real–valued AWGN $n(t)$:

$$r(t) = s_m(t) + n(t), \qquad 0 \le t \le T.$$



– The AWGN, $n(t)$, has *power spectral density*

$$\Phi_{NN}(f) = \frac{N_0}{2} \;\left[\frac{\mathrm{W}}{\mathrm{Hz}}\right]$$

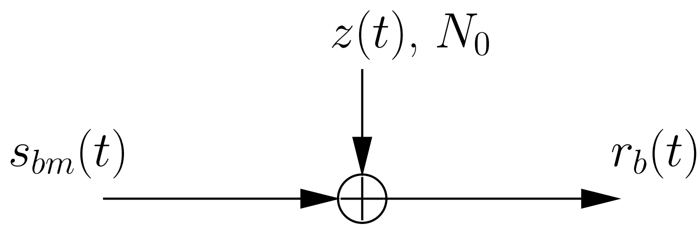
– At the receiver, we observe $r(t)$ and the question we ask is: **What is the best decision rule for determining $s_m(t)$?**

– This problem can be equivalently formulated in the complex baseband. The received baseband signal $r_b(t)$ is

$$r_b(t) = s_{bm}(t) + z(t)$$

where $z(t)$ is complex AWGN, whose real and imaginary parts are independent. $z(t)$ has a power spectral density of

$$\Phi_{ZZ}(f) = N_0 \left[\frac{\mathrm{W}}{\mathrm{Hz}}\right]$$



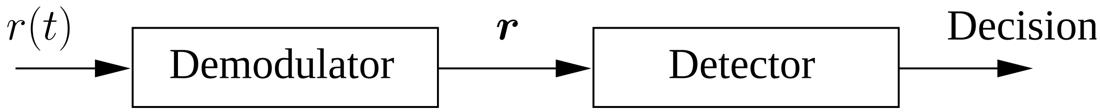
■ **Strategy:** We divide the problem into two parts:

1. First we transform the received continuous–time signal $r(t)$ (or equivalently $r_b(t)$) into an N–dimensional vector

$$\boldsymbol{r} = [r_1 \;\; r_2 \;\; \ldots \;\; r_N]^T$$

(or $\boldsymbol{r}_b$), which forms a *sufficient statistic* for the detection of $s_m(t)$ ($s_{bm}(t)$). This transformation is referred to as *demodulation*.

2. Subsequently, we determine an estimate for $s_m(t)$ (or $s_{bm}(t)$) based on vector $\boldsymbol{r}$ (or $\boldsymbol{r}_b$). This process is referred to as *detection*.



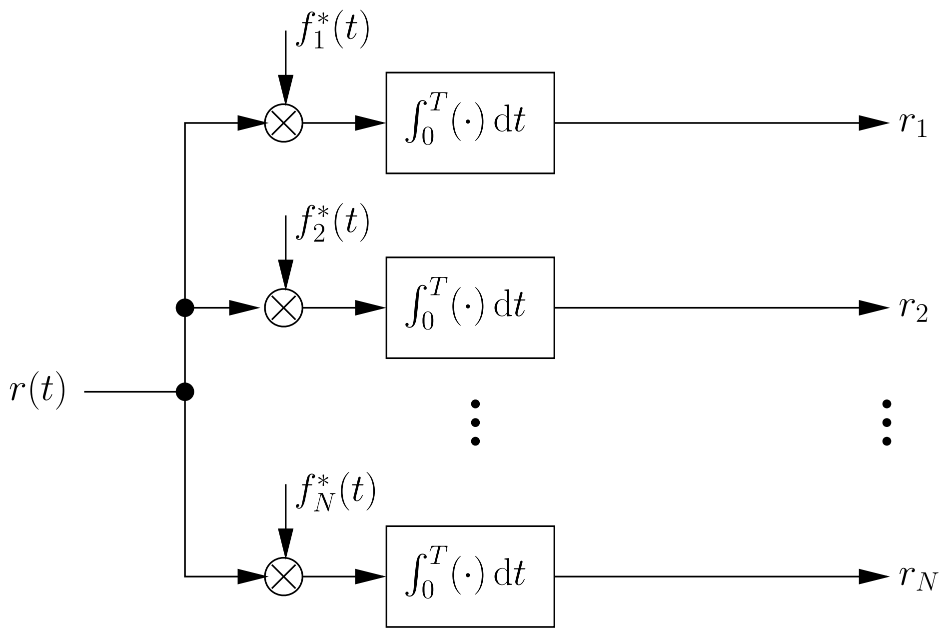
4.1.1 Demodulation

The demodulator extracts the information required for optimal detection of $s_m(t)$ and eliminates those parts of the received signal $r(t)$ that are irrelevant for the detection process.

4.1.1.1 Correlation Demodulation

- Recall that the transmit waveforms $\{s_m(t)\}$ can be represented by a set of N orthogonal basis functions $f_k(t)$, $1 \leq k \leq N$.
- For a complete representation of the noise $n(t)$, $0 \leq t \leq T$, an infinite number of basis functions are required. But fortunately, only the noise components that lie in the signal space spanned by $f_k(t)$, $1 \leq k \leq N$, are relevant for detection of $s_m(t)$.
- We obtain vector $\boldsymbol{r}$ by correlating $r(t)$ with $f_k(t)$, $1 \leq k \leq N$

$$
\begin{aligned}
r_k &= \int_0^T r(t) f_k^*(t)\,\mathrm{d}t = \int_0^T [s_m(t) + n(t)] f_k^*(t)\,\mathrm{d}t \\
&= \underbrace{\int_0^T s_m(t) f_k^*(t)\,\mathrm{d}t}_{s_{mk}} + \underbrace{\int_0^T n(t) f_k^*(t)\,\mathrm{d}t}_{n_k} \\
&= s_{mk} + n_k, \qquad 1 \leq k \leq N
\end{aligned}
$$



- $r(t)$ can be represented by

$$
\begin{aligned}
r(t) &= \sum_{k=1}^{N} s_{mk} f_k(t) + \sum_{k=1}^{N} n_k f_k(t) + n'(t) \\
&= \sum_{k=1}^{N} r_k f_k(t) + n'(t),
\end{aligned}
$$

where noise $n'(t)$ is given by

$$
n'(t) = n(t) - \sum_{k=1}^{N} n_k f_k(t)
$$

Since $n'(t)$ does not lie in the signal space spanned by the basis functions of $s_m(t)$, it is irrelevant for detection of $s_m(t)$. Therefore, without loss of optimality, we can estimate the transmitted waveform $s_m(t)$ from $\boldsymbol{r}$ instead of $r(t)$.

■ Properties of n_k

– n_k is a Gaussian random variable (RV), since $n(t)$ is Gaussian.

– Mean:

$$\begin{aligned}
\mathcal{E}\{n_k\} &= \mathcal{E}\left\{\int_0^T n(t) f_k^*(t)\,\mathrm{d}t\right\} \\
&= \int_0^T \mathcal{E}\{n(t)\} f_k^*(t)\,\mathrm{d}t \\
&= 0
\end{aligned}$$

– Covariance:

$$\begin{aligned}
\mathcal{E}\{n_k n_m^*\} &= \mathcal{E}\left\{\left(\int_0^T n(t) f_k^*(t)\,\mathrm{d}t\right)\left(\int_0^T n(t) f_m^*(t)\,\mathrm{d}t\right)^*\right\} \\
&= \int_0^T \int_0^T \underbrace{\mathcal{E}\{n(t) n^*(\tau)\}}_{\frac{N_0}{2}\delta(t-\tau)} f_k^*(t) f_m(\tau)\,\mathrm{d}t\,\mathrm{d}\tau \\
&= \frac{N_0}{2}\int_0^T f_m(t) f_k^*(t)\mathrm{d}t \\
&= \frac{N_0}{2}\,\delta[k-m]
\end{aligned}$$

where $\delta[k]$ denotes the Kronecker function

$$\delta[k] = \begin{cases} 1, & k = 0 \\ 0, & k \neq 0 \end{cases}$$

We conclude that the N noise components are zero–mean, mutually uncorrelated Gaussian RVs.

■ Conditional pdf of $\boldsymbol{r}$

$\boldsymbol{r}$ can be expressed as

$$\boldsymbol{r} = \boldsymbol{s}_m + \boldsymbol{n}$$

with

$$\begin{aligned} \boldsymbol{s}_m &= [s_{m1} \; s_{m2} \; \ldots s_{mN}]^T, \\ \boldsymbol{n} &= [n_1 \; n_2 \; \ldots n_N]^T. \end{aligned}$$

Therefore, conditioned on $\boldsymbol{s}_m$ vector $\boldsymbol{r}$ is Gaussian distributed and we obtain

$$\begin{aligned} p(\boldsymbol{r}|\boldsymbol{s}_m) &= p_{\boldsymbol{n}}(\boldsymbol{r} - \boldsymbol{s}_m) \\ &= \prod_{k=1}^{N} p_n(r_k - s_{mk}), \end{aligned}$$

where $p_{\boldsymbol{n}}(\boldsymbol{n})$ and $p_n(n_k)$ denote the pdfs of the Gaussian noise vector $\boldsymbol{n}$ and the components n_k of $\boldsymbol{n}$, respectively. $p_n(n_k)$ is given by

$$p_n(n_k) = \frac{1}{\sqrt{\pi N_0}} \exp\left(-\frac{n_k^2}{N_0}\right)$$

since n_k is a real–valued Gaussian RV with variance $\sigma_n^2 = \frac{N_0}{2}$. Therefore, $p(\boldsymbol{r}|\boldsymbol{s}_m)$ can be expressed as

$$\begin{aligned} p(\boldsymbol{r}|\boldsymbol{s}_m) &= \prod_{k=1}^{N} \frac{1}{\sqrt{\pi N_0}} \exp\left(-\frac{(r_k - s_{mk})^2}{N_0}\right) \\ &= \frac{1}{(\pi N_0)^{N/2}} \exp\left(-\frac{\sum\limits_{k=1}^{N}(r_k - s_{mk})^2}{N_0}\right), \qquad 1 \le m \le M \end{aligned}$$

$p(\boldsymbol{r}|\boldsymbol{s}_m)$ will be used later to find the optimum estimate for $\boldsymbol{s}_m$ (or equivalently $s_m(t)$).

■ **Role of** $n'(t)$

We consider the correlation between r_k and $n'(t)$:

$$\begin{aligned}
\mathcal{E}\{n'(t)r_k^*\} &= \mathcal{E}\{n'(t)(s_{mk}+n_k)^*\} \\
&= \underbrace{\mathcal{E}\{n'(t)\}}_{=0}\, s_{mk}^* + \mathcal{E}\{n'(t)n_k^*\} \\
&= \mathcal{E}\left\{\left(n(t) - \sum_{j=1}^{N} n_j f_j(t)\right) n_k^*\right\} \\
&= \int_0^T \underbrace{\mathcal{E}\{n(t)n^*(\tau)\}}_{\frac{N_0}{2}\delta(t-\tau)} f_k(\tau)\,\mathrm{d}\tau - \sum_{j=1}^{N} \underbrace{\mathcal{E}\{n_j n_k^*\}}_{\frac{N_0}{2}\delta[j-k]} f_j(t) \\
&= \frac{N_0}{2} f_k(t) - \frac{N_0}{2} f_k(t) \\
&= 0
\end{aligned}$$

We observe that $\boldsymbol{r}$ and $n'(t)$ are uncorrelated. Since $\boldsymbol{r}$ and $n'(t)$ are Gaussian distributed, they are also *statistically independent*. Therefore, $n'(t)$ cannot provide any useful information that is relevant for the decision, and consequently, $\boldsymbol{r}$ forms a *sufficient statistic* for detection of $s_m(t)$.

4.1.1.2 Matched–Filter Demodulation

- Instead of generating the $\{r_k\}$ using a bank of N correlators, we may use N linear filters instead.
- We define the N filter impulse responses $h_k(t)$ as

$$h_k(t) = f_k^*(T - t), \qquad 0 \leq t \leq T$$

where f_k, $1 \leq k \leq N$, are the N basis functions.

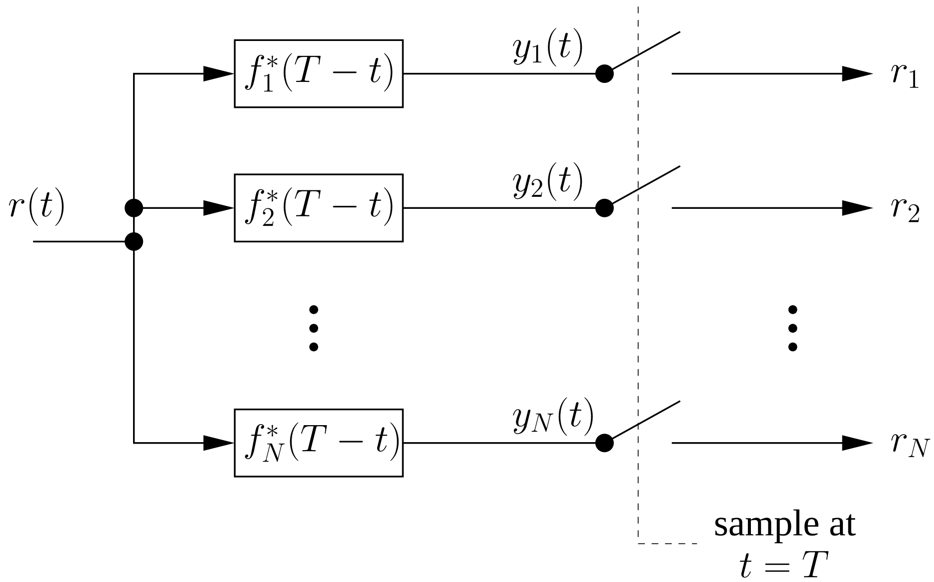
- The output of filter $h_k(t)$ with input $r(t)$ is

$$\begin{aligned} y_k(t) &= \int\limits_0^t r(\tau) h_k(t - \tau)\, \mathrm{d}\tau \\ &= \int\limits_0^t r(\tau) f_k^*(T - t + \tau)\, \mathrm{d}\tau \end{aligned}$$

- By sampling $y_k(t)$ at time $t = T$, we obtain

$$\begin{aligned} y_k(T) &= \int\limits_0^T r(\tau) f_k^*(\tau)\, \mathrm{d}\tau \\ &= r_k, \qquad 1 \leq k \leq N \end{aligned}$$

This means the sampled output of $h_k(t)$ is r_k.



■ General Properties of Matched Filters MFs

– In general, we call a filter of the form

$$h(t) = s^*(T-t)$$

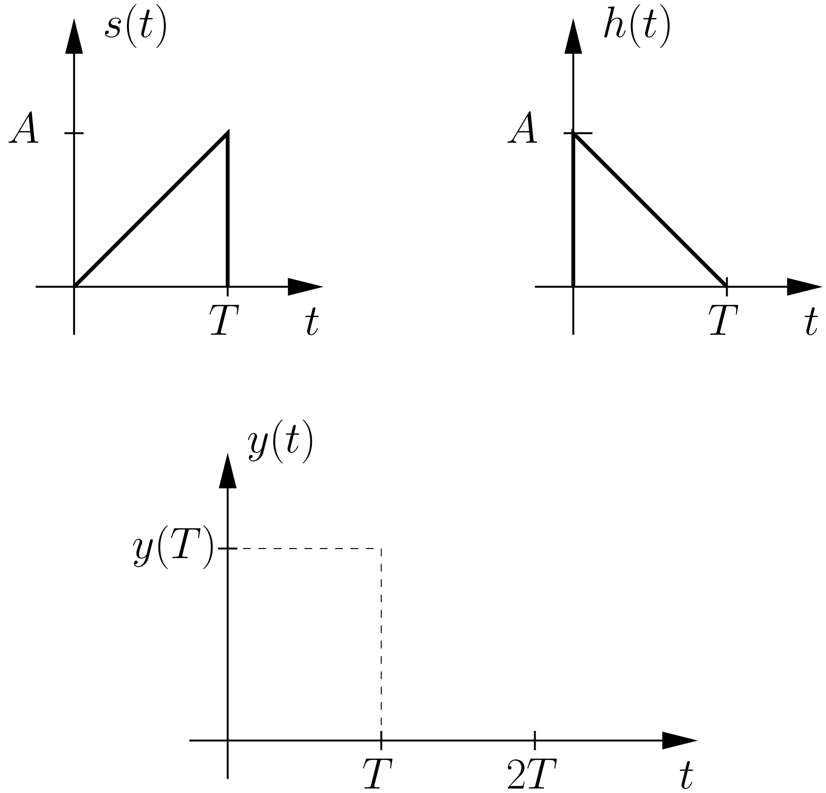
a *matched filter* for $s(t)$.

– The output

$$y(t) = \int_0^t s(\tau)s^*(T-t+\tau)\,\mathrm{d}\tau$$

is the time–shifted time–autocorrelation of $s(t)$.

Example:



– **MFs Maximize the SNR**

∗ Consider the signal

$$r(t) = s(t) + n(t), \qquad 0 \le t \le T,$$

where $s(t)$ is some known signal with energy

$$E = \int_0^T |s(t)|^2 \, \mathrm{d}t$$

and $n(t)$ is AWGN with power spectral density

$$\Phi_{NN}(f) = \frac{N_0}{2}$$

* **Problem:** Which filter $h(t)$ maximizes the SNR of

$$y(T) = h(t) * r(t)\Big|_{t=T}$$

* **Answer:** The *matched filter* $h(t) = s^*(T-t)$!

Proof. The filter output sampled at time $t = T$ is given by

$$\begin{aligned} y(T) &= \int_0^T r(\tau)h(T-\tau)\,\mathrm{d}\tau \\ &= \underbrace{\int_0^T s(\tau)h(T-\tau)\,\mathrm{d}\tau}_{y_S(T)} + \underbrace{\int_0^T n(\tau)h(T-\tau)\,\mathrm{d}\tau}_{y_N(T)} \end{aligned}$$

Now, the SNR at the filter output can be defined as

$$\mathrm{SNR} = \frac{|y_S(T)|^2}{\mathcal{E}\{|y_N(T)|^2\}}$$

The noise power in the denominator can be calculated as

$$\begin{aligned} \mathcal{E}\left\{|y_N(T)|^2\right\} &= \mathcal{E}\left\{\left(\int_0^T n(\tau)h(T-\tau)\,\mathrm{d}\tau\right)\left(\int_0^T n(\tau)h(T-\tau)\,\mathrm{d}\tau\right)^*\right\} \\ &= \int_0^T\int_0^T \underbrace{\mathcal{E}\{n(\tau)n^*(t)\}}_{\frac{N_0}{2}\delta(\tau-t)} h(T-\tau)h^*(T-t)\,\mathrm{d}\tau\,\mathrm{d}t \\ &= \frac{N_0}{2}\int_0^T |h(T-\tau)|^2\,\mathrm{d}\tau \end{aligned}$$

Therefore, the SNR can be expressed as

$$\mathrm{SNR} = \frac{\left| \int\limits_0^T s(\tau) h(T-\tau)\,\mathrm{d}\tau \right|^2}{\frac{N_0}{2} \int\limits_0^T |h(T-\tau)|^2\,\mathrm{d}\tau}.$$

From the *Cauchy–Schwartz inequality* we know

$$\left| \int\limits_0^T s(\tau) h(T-\tau)\,\mathrm{d}\tau \right|^2 \le \int\limits_0^T |s(\tau)|^2\,\mathrm{d}\tau \cdot \int\limits_0^T |h(T-\tau)|^2\,\mathrm{d}\tau,$$

where equality holds if and only if

$$h(t) = C s^*(T-t).$$

(C is an arbitrary non–zero constant). Therefore, the maximum output SNR is

$$\begin{aligned} \mathrm{SNR} &= \frac{\left| \int\limits_0^T |s(\tau)|^2\,\mathrm{d}\tau \right|^2}{\frac{N_0}{2} \int\limits_0^T |s(\tau)|^2\,\mathrm{d}\tau} \\ &= \frac{2}{N_0} \int\limits_0^T |s(\tau)|^2\,\mathrm{d}\tau \\ &= 2\frac{E}{N_0}, \end{aligned}$$

which is achieved by the MF $h(t) = s^*(T-t)$. $\square$

– **Frequency Domain Interpretation**

The frequency response of the MF is given by

$$\begin{aligned} H(f) &= \mathcal{F}\{h(t)\} \\ &= \int\limits_{-\infty}^{\infty} s^*(T-t)\, e^{-j2\pi f t}\, \mathrm{d}t \\ &= \int\limits_{-\infty}^{\infty} s^*(\tau)\, e^{j2\pi f \tau} e^{-j2\pi f T}\, \mathrm{d}\tau \\ &= e^{-j2\pi f T} \left(\int\limits_{-\infty}^{\infty} s(\tau)\, e^{-j2\pi f \tau}\, \mathrm{d}\tau \right)^* \\ &= e^{-j2\pi f T} S^*(f) \end{aligned}$$

Observe that $H(f)$ has the same magnitude as $S(f)$

$$|H(f)| = |S(f)|.$$

The factor $e^{-j2\pi f T}$ in the frequency response accounts for the time shift of $s^*(-t)$ by T.

4.1.2 Optimal Detection

Problem Formulation:

- The output $\boldsymbol{r}$ of the demodulator forms a sufficient statistic for detection of $s_m(t)$ ($\boldsymbol{s}_m$).
- We consider linear modulation formats without memory.
- *What is the optimal decision rule?*
- *Optimality criterion:* Probability for correct detection shall be maximized, i.e., probability of error shall be minimized.

Solution:

- The probability of error is minimized if we choose that $\boldsymbol{s}_{\tilde{m}}$ which maximizes the *posteriori probability*

$$P(\boldsymbol{s}_{\tilde{m}}|\boldsymbol{r}), \qquad \tilde{m} = 1, 2, \ldots, M,$$

  where the "tilde" indicates that $\boldsymbol{s}_{\tilde{m}}$ is not the transmitted symbol but a *trial symbol*.

Maximum a Posteriori (MAP) Decision Rule

The resulting decision rule can be formulated as

$$\boxed{\hat{m} = \underset{\tilde{m}}{\operatorname{argmax}} \, \{P(\boldsymbol{s}_{\tilde{m}}|\boldsymbol{r})\}}$$

where $\hat{m}$ denotes the *estimated* signal number. The above decision rule is called *maximum a posteriori (MAP)* decision rule.

- **Simplifications**

  Using Bayes rule, we can rewrite $P(\boldsymbol{s}_{\tilde{m}}|\boldsymbol{r})$ as

$$P(\boldsymbol{s}_{\tilde{m}}|\boldsymbol{r}) = \frac{p(\boldsymbol{r}|\boldsymbol{s}_{\tilde{m}})P(\boldsymbol{s}_{\tilde{m}})}{p(\boldsymbol{r})},$$

  with

  – $p(\boldsymbol{r}|\boldsymbol{s}_{\tilde{m}})$: Conditional pdf of observed vector $\boldsymbol{r}$ given $\boldsymbol{s}_{\tilde{m}}$.

  – $P(\boldsymbol{s}_{\tilde{m}})$: *A priori probability* of transmitted symbols. Normally, we have

$$P(\boldsymbol{s}_{\tilde{m}}) = \frac{1}{M}, \qquad 1 \leq \tilde{m} \leq M,$$

  i.e., all signals of the set are transmitted with equal probability.

  – $p(\boldsymbol{r})$: Probability density function of vector $\boldsymbol{r}$

$$p(\boldsymbol{r}) = \sum_{m=1}^{M} p(\boldsymbol{r}|\boldsymbol{s}_m)P(\boldsymbol{s}_m).$$

  Since $p(\boldsymbol{r})$ is obviously independent of $\boldsymbol{s}_{\tilde{m}}$, we can simplify the MAP decision rule to

$$\boxed{\hat{m} = \underset{\tilde{m}}{\operatorname{argmax}} \, \{p(\boldsymbol{r}|\boldsymbol{s}_{\tilde{m}})P(\boldsymbol{s}_{\tilde{m}})\}}$$

Maximum–Likelihood (ML) Decision Rule

- The MAP rule requires knowledge of both $p(\boldsymbol{r}|\boldsymbol{s}_{\tilde{m}})$ and $P(\boldsymbol{s}_{\tilde{m}})$.
- In some applications $P(\boldsymbol{s}_{\tilde{m}})$ is unknown at the receiver.
- If we neglect the influence of $P(\boldsymbol{s}_{\tilde{m}})$, we get the ML decision rule

$$\boxed{\hat{m} = \underset{\tilde{m}}{\operatorname{argmax}} \left\{ p(\boldsymbol{r}|\boldsymbol{s}_{\tilde{m}}) \right\}}$$

- Note that if all $\boldsymbol{s}_m$ are equally probable, i.e., $P(\boldsymbol{s}_{\tilde{m}}) = 1/M$, $1 \leq \tilde{m} \leq M$, the MAP and the ML decision rules are *identical*.

The above MAP and ML decision rules are very general. They can be applied to any channel as long as we are able to find an expression for $p(\boldsymbol{r}|\boldsymbol{s}_{\tilde{m}})$.

ML Decision Rule for AWGN Channel

- For the AWGN channel we have

$$p(\boldsymbol{r}|\boldsymbol{s}_{\tilde{m}}) = \frac{1}{(\pi N_0)^{N/2}} \exp\left(-\frac{\sum_{k=1}^{N} |r_k - s_{\tilde{m}k}|^2}{N_0} \right), \quad 1 \le \tilde{m} \le M$$

- We note that the ML decision does not change if we maximize $\ln(p(\boldsymbol{r}|\boldsymbol{s}_{\tilde{m}}))$ instead of $p(\boldsymbol{r}|\boldsymbol{s}_{\tilde{m}})$ itself, since $\ln(\cdot)$ is a *monotonic* function.

- Therefore, the ML decision rule can be simplified as

$$\begin{aligned}
\hat{m} &= \underset{\tilde{m}}{\operatorname{argmax}} \left\{ p(\boldsymbol{r}|\boldsymbol{s}_{\tilde{m}}) \right\} \\
&= \underset{\tilde{m}}{\operatorname{argmax}} \left\{ \ln(p(\boldsymbol{r}|\boldsymbol{s}_{\tilde{m}})) \right\} \\
&= \underset{\tilde{m}}{\operatorname{argmax}} \left\{ -\frac{N}{2} \ln(\pi N_0) - \frac{1}{N_0} \sum_{k=1}^{N} |r_k - s_{\tilde{m}k}|^2 \right\} \\
&= \underset{\tilde{m}}{\operatorname{argmin}} \left\{ \sum_{k=1}^{N} |r_k - s_{\tilde{m}k}|^2 \right\} \\
&= \underset{\tilde{m}}{\operatorname{argmin}} \left\{ ||\boldsymbol{r} - \boldsymbol{s}_{\tilde{m}}||^2 \right\}
\end{aligned}$$

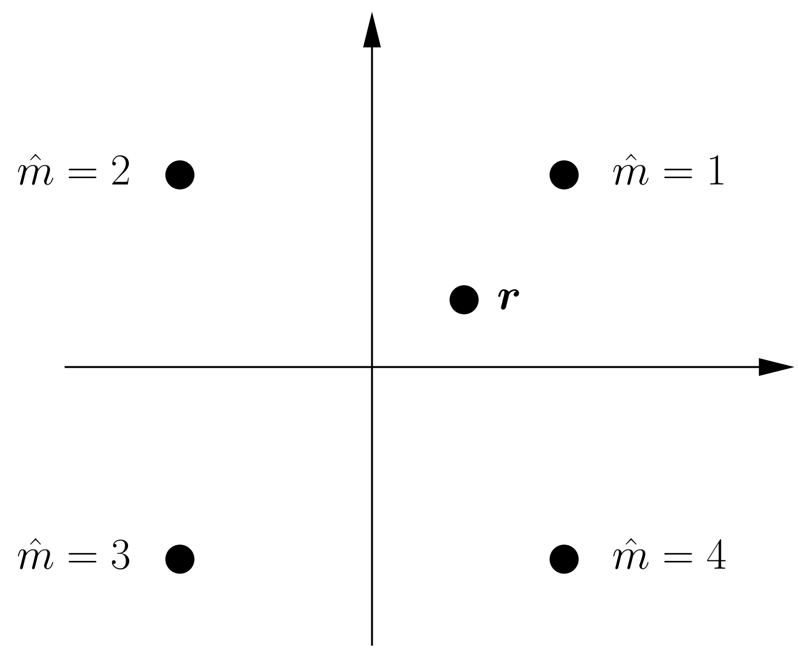
- **Interpretation:**
  We select that vector $\boldsymbol{s}_{\tilde{m}}$ which has the minimum *Euclidean distance*

$$\boxed{D(\boldsymbol{r}, \boldsymbol{s}_{\tilde{m}}) = ||\boldsymbol{r} - \boldsymbol{s}_{\tilde{m}}||}$$

from the received vector $\boldsymbol{r}$. Therefore, we can interpret the above ML decision rule graphically by dividing the signal space in *decision regions*.

Example:

4QAM



- **Alternative Representation:**
  Using the expansion

$$||\boldsymbol{r} - \boldsymbol{s}_{\tilde{m}}||^2 \;=\; ||\boldsymbol{r}||^2 - 2\mathsf{Re}\{\boldsymbol{r} \bullet \boldsymbol{s}_{\tilde{m}}\} + ||\boldsymbol{s}_{\tilde{m}}||^2,$$

we observe that $||\boldsymbol{r}||^2$ is independent of $\boldsymbol{s}_{\tilde{m}}$. Therefore, we can further simplify the ML decision rule

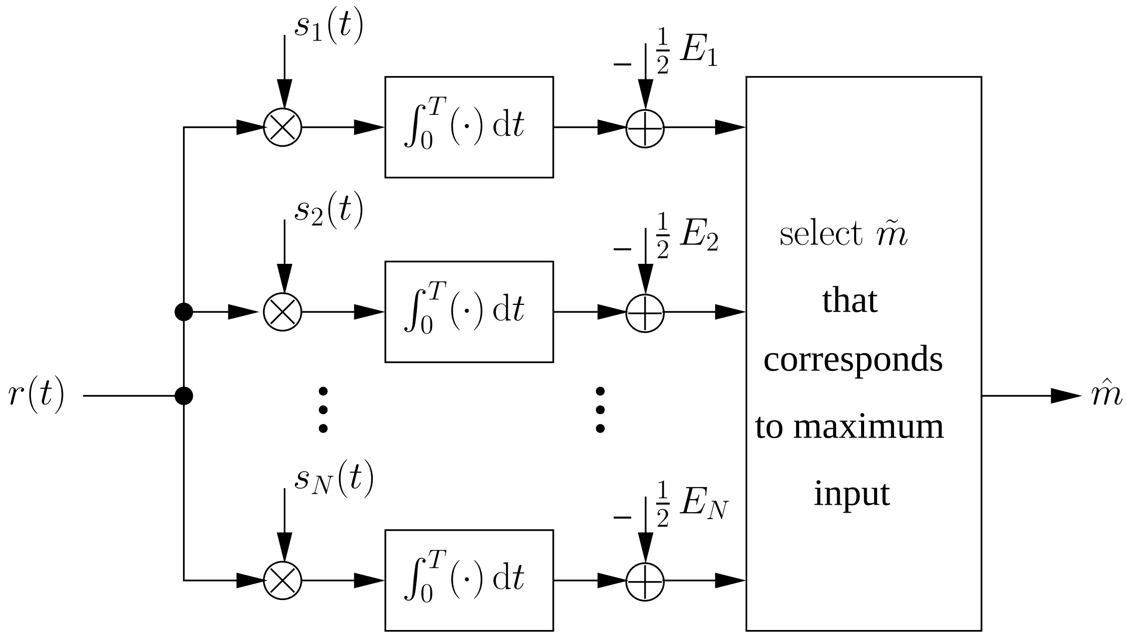
$$\begin{aligned}
\hat{m} &= \underset{\tilde{m}}{\mathrm{argmin}} \left\{ ||\boldsymbol{r} - \boldsymbol{s}_{\tilde{m}}||^2 \right\} \\
&= \underset{\tilde{m}}{\mathrm{argmin}} \left\{ -2\mathsf{Re}\{\boldsymbol{r} \bullet \boldsymbol{s}_{\tilde{m}}\} + ||\boldsymbol{s}_{\tilde{m}}||^2 \right\} \\
&= \underset{\tilde{m}}{\mathrm{argmax}} \left\{ 2\mathsf{Re}\{\boldsymbol{r} \bullet \boldsymbol{s}_{\tilde{m}}\} - ||\boldsymbol{s}_{\tilde{m}}||^2 \right\} \\
&= \underset{\tilde{m}}{\mathrm{argmax}} \left\{ \mathsf{Re}\left\{ \int_0^T r(t) s^*_{\tilde{m}}(t)\, \mathrm{d}t \right\} - \frac{1}{2} E_{\tilde{m}} \right\},
\end{aligned}$$

with

$$E_{\tilde{m}} = \int_0^T |s_{\tilde{m}}(t)|^2 \, \mathrm{d}t.$$

If we are dealing with passband signals both $r(t)$ and $s^*_{\tilde{m}}(t)$ are real–valued, and we obtain

$$\boxed{\hat{m} = \underset{\tilde{m}}{\mathrm{argmax}} \left\{ \int_0^T r(t) s_{\tilde{m}}(t)\, \mathrm{d}t - \frac{1}{2} E_{\tilde{m}} \right\}}$$



Example:

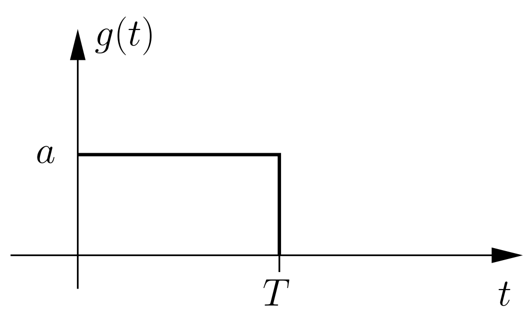
M–ary PAM transmission (baseband case)

The transmitted signals are given by

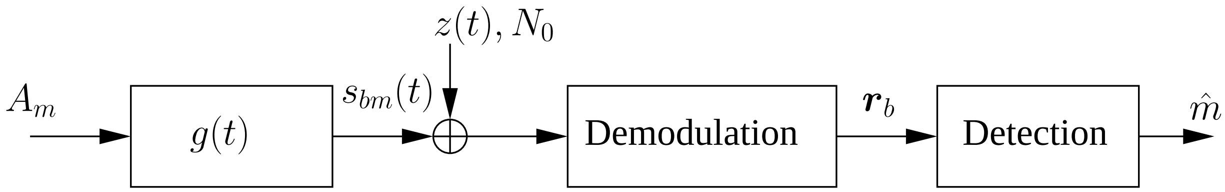
$$s_{bm}(t) = A_m g(t),$$

with $A_m = (2m-1-M)d$, $m = 1, 2, \ldots, M$, $2d$: distance between adjacent signal points.

We assume the transmit pulse $g(t)$ is as shown below.



In the interval $0 \leq t \leq T$, the transmission scheme is modeled as



1. **Demodulator**

   – Energy of transmit pulse

$$E_g = \int\limits_0^T |g(t)|^2 \, \mathrm{d}t = a^2 T$$

   – Basis function $f(t)$

$$\begin{aligned} f(t) &= \frac{1}{\sqrt{E_g}} g(t) \\ &= \begin{cases} \frac{1}{\sqrt{T}}, & 0 \leq t \leq T \\ 0, & \text{otherwise} \end{cases} \end{aligned}$$

– Correlation demodulator

$$
\begin{aligned}
\boldsymbol{r}_b &= \int_0^T r_b(t) f^*(t)\,\mathrm{d}t \\
&= \frac{1}{\sqrt{T}} \int_0^T r_b(t)\,\mathrm{d}t \\
&= \underbrace{\frac{1}{\sqrt{T}} \int_0^T s_{bm}(t)\,\mathrm{d}t}_{s_{bm}} + \underbrace{\frac{1}{\sqrt{T}} \int_0^T z(t)\,\mathrm{d}t}_{z} \\
&= s_{bm} + z
\end{aligned}
$$

s_{bm} is given by

$$
\begin{aligned}
s_{bm} &= \frac{1}{\sqrt{T}} \int_0^T A_m g(t)\,\mathrm{d}t \\
&= \frac{1}{\sqrt{T}} \int_0^T A_m a\,\mathrm{d}t \\
&= a\sqrt{T} A_m = \sqrt{E_g} A_m.
\end{aligned}
$$

On the other hand, the noise variance is

$$
\begin{aligned}
\sigma_z^2 &= \mathcal{E}\{|z|^2\} \\
&= \frac{1}{T} \int_0^T \int_0^T \underbrace{\mathcal{E}\{z(t) z^*(\tau)\}}_{N_0 \delta(t-\tau)}\,\mathrm{d}t\,\mathrm{d}\tau \\
&= N_0
\end{aligned}
$$

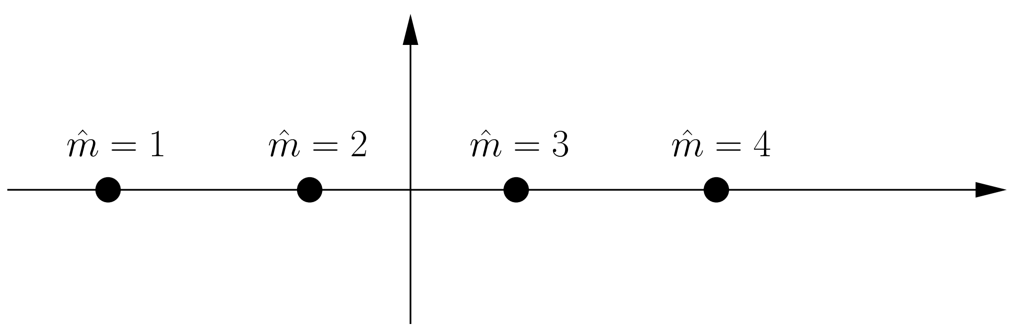
– pdf $p(r_b|A_m)$:

$$p(r_b|A_m) = \frac{1}{\pi N_0} \exp\left(-\frac{|r_b - \sqrt{E_g}A_m|^2}{N_0}\right)$$

2. **Optimum Detector**
   The ML decision rule is given by

$$\begin{aligned}\hat{m} &= \underset{\tilde{m}}{\operatorname{argmax}}\{\ln(p(r_b|A_{\tilde{m}}))\} \\ &= \underset{\tilde{m}}{\operatorname{argmax}}\{-|r_b - \sqrt{E_g}A_{\tilde{m}}|^2\} \\ &= \underset{\tilde{m}}{\operatorname{argmin}}\{|r_b - \sqrt{E_g}A_{\tilde{m}}|^2\}\end{aligned}$$

Illustration in the Signal Space



4.2 Performance of Optimum Receivers

In this section, we evaluate the performance of the optimum receivers introduced in the previous section. We assume again memoryless modulation. We adopt the *symbol error probability (SEP)* (also referred to as symbol error rate (SER)) and the *bit error probability (BEP)* (also referred to as bit error rate (BER)) as performance criteria.

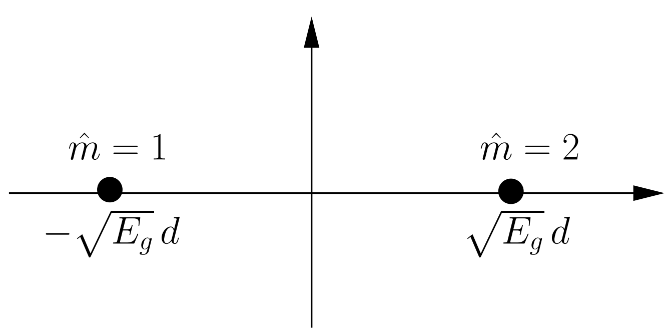
4.2.1 Binary Modulation

1. **Binary PAM** ($M = 2$)

   - From the example in the previous section we know that the detector input signal in this case is

     $$r_b = \sqrt{E_g} A_m + z, \qquad m = 1,\, 2,$$

     where the noise variance of the complex baseband noise is $\sigma_z^2 = N_0$.
   - Decision Regions



- Assuming s_1 has been transmitted, the received signal is

$$r_b = -\sqrt{E_g}d + z$$

and a correct decision is made if

$$r_R < 0,$$

whereas an error is made if

$$r_R > 0,$$

where $r_R = \mathsf{Re}\{r_b\}$ denotes the real part of r. r_R is given by

$$r_R = -\sqrt{E_g}d + z_R$$

where $z_R = \mathsf{Re}\{z\}$ is real Gaussian noise with variance $\sigma_{z_R}^2 = N_0/2$.

- Consequently, the (conditional) error probability is

$$P(e|s_1) = \int_0^\infty p_{r_R}(r_R|s_1)\,\mathrm{d}r_R.$$

Therefore, we get

$$\begin{aligned} P(e|s_1) &= \int_0^\infty \frac{1}{\sqrt{\pi N_0}} \exp\left(-\frac{(r_R - (-\sqrt{E_g}d))^2}{N_0}\right) \mathrm{d}r_R \\ &= \frac{1}{\sqrt{2\pi}} \int_{\sqrt{\frac{2E_g}{N_0}}d}^\infty \exp\left(-\frac{x^2}{2}\right) \mathrm{d}x \\ &= Q\left(\sqrt{\frac{2E_g}{N_0}}d\right) \end{aligned}$$

where we have used the substitution $x = \sqrt{2}(r_R + \sqrt{E_g}d)/\sqrt{N_0}$ and the Q–function is defined as

$$\boxed{Q(x) = \frac{1}{\sqrt{2\pi}} \int_x^\infty \exp\left(-\frac{t^2}{2}\right) \, \mathrm{d}t}$$

- The BEP, which is equal to the SEP for binary modulation, is given by

$$P_b = P(s_1)P(e|s_1) + P(s_2)P(e|s_2)$$

For the usual case, $P(s_1) = P(s_2) = \frac{1}{2}$, we get

$$\begin{aligned} P_b &= \frac{1}{2}P(e|s_1) + \frac{1}{2}P(e|s_2) \\ &= P(e|s_1), \end{aligned}$$

since $P(e|s_1) = P(e|s_2)$ is true because of the symmetry of the signal constellation.

- In general, the BEP is expressed as a function of the *received energy per bit* E_b. Here, E_b is given by

$$\begin{aligned} E_b &= \mathcal{E}\{|\sqrt{E_g}A_m|^2\} \\ &= E_g\left(\frac{1}{2}(-d)^2 + \frac{1}{2}(d)^2\right) \\ &= E_g d^2. \end{aligned}$$

Therefore, the BEP can be expressed as

$$\boxed{P_b = Q\left(\sqrt{2\frac{E_b}{N_0}}\right)}$$

- Note that binary PSK (BPSK) yields the same BEP as 2PAM.

2. **Binary Orthogonal Modulation**

- For binary orthogonal modulation, the transmitted signals can be represented as

$$\boldsymbol{s}_1 = \begin{bmatrix} \sqrt{E_b} \\ 0 \end{bmatrix}$$
$$\boldsymbol{s}_2 = \begin{bmatrix} 0 \\ \sqrt{E_b} \end{bmatrix}$$

- The demodulated received signal is given by

$$\boldsymbol{r} = \begin{bmatrix} \sqrt{E_b} + n_1 \\ n_2 \end{bmatrix}$$

and

$$\boldsymbol{r} = \begin{bmatrix} n_1 \\ \sqrt{E_b} + n_2 \end{bmatrix}$$

if $\boldsymbol{s}_1$ and $\boldsymbol{s}_2$ were sent, respectively. The noise variances are given by

$$\sigma_{n_1}^2 = \mathcal{E}\{n_1^2\} = \sigma_{n_2}^2 = \mathcal{E}\{n_2^2\} = \frac{N_0}{2},$$

and n_1 and n_2 are mutually independent Gaussian RVs.

- **Decision Rule**

  The ML decision rule is given by

$$\begin{aligned}\hat{m} &= \operatorname*{argmax}_{\tilde{m}} \left\{ 2\boldsymbol{r} \bullet \boldsymbol{s}_{\tilde{m}} - ||\boldsymbol{s}_{\tilde{m}}||^2 \right\} \\ &= \operatorname*{argmax}_{\tilde{m}} \left\{ \boldsymbol{r} \bullet \boldsymbol{s}_{\tilde{m}} \right\},\end{aligned}$$

  where we have used the fact that $||\boldsymbol{s}_{\tilde{m}}||^2 = E_b$ is independent of $\tilde{m}$.

- **Error Probability**
  - Let us assume that $m = 1$ has been transmitted.
  - From the above decision rule we conclude that an error is made if

$$\boldsymbol{r} \bullet \boldsymbol{s}_1 < \boldsymbol{r} \bullet \boldsymbol{s}_2$$

  - Therefore, the conditional BEP is given by

$$\begin{aligned}P(e|\boldsymbol{s}_1) &= P(\boldsymbol{r} \bullet \boldsymbol{s}_2 > \boldsymbol{r} \bullet \boldsymbol{s}_1) \\ &= P(\sqrt{E_b} n_2 > E_b + \sqrt{E_b} n_1) \\ &= P(\underbrace{n_2 - n_1}_{X} > \sqrt{E_b})\end{aligned}$$

    Note that X is a Gaussian RV with variance

$$\begin{aligned}\sigma_X^2 &= \mathcal{E}\left\{|n_2 - n_1|^2\right\} \\ &= \mathcal{E}\left\{|n_2|^2\right\} - 2\mathcal{E}\left\{n_1 n_2\right\} + \mathcal{E}\left\{|n_1|^2\right\} \\ &= N_0\end{aligned}$$

Therefore, $P(e|\boldsymbol{s}_1)$ can be calculated to

$$
\begin{aligned}
P(e|\boldsymbol{s}_1) &= \frac{1}{\sqrt{2\pi N_0}} \int\limits_{\sqrt{E_b}}^{\infty} \exp\left(-\frac{x^2}{2N_0}\right) \mathrm{d}x \\
&= \frac{1}{\sqrt{2\pi}} \int\limits_{\sqrt{\frac{E_b}{N_0}}}^{\infty} \exp\left(-\frac{u^2}{2}\right) \mathrm{d}u \\
&= Q\left(\sqrt{\frac{E_b}{N_0}}\right)
\end{aligned}
$$

Finally, because of the symmetry of the signal constellation we obtain $P_b = P(e|\boldsymbol{s}_1) = P(e|\boldsymbol{s}_2)$ or

$$
\boxed{P_b = Q\left(\sqrt{2\frac{E_b}{N_0}}\right)}
$$

■ Comparison of 2PAM and Binary Orthogonal Modulation

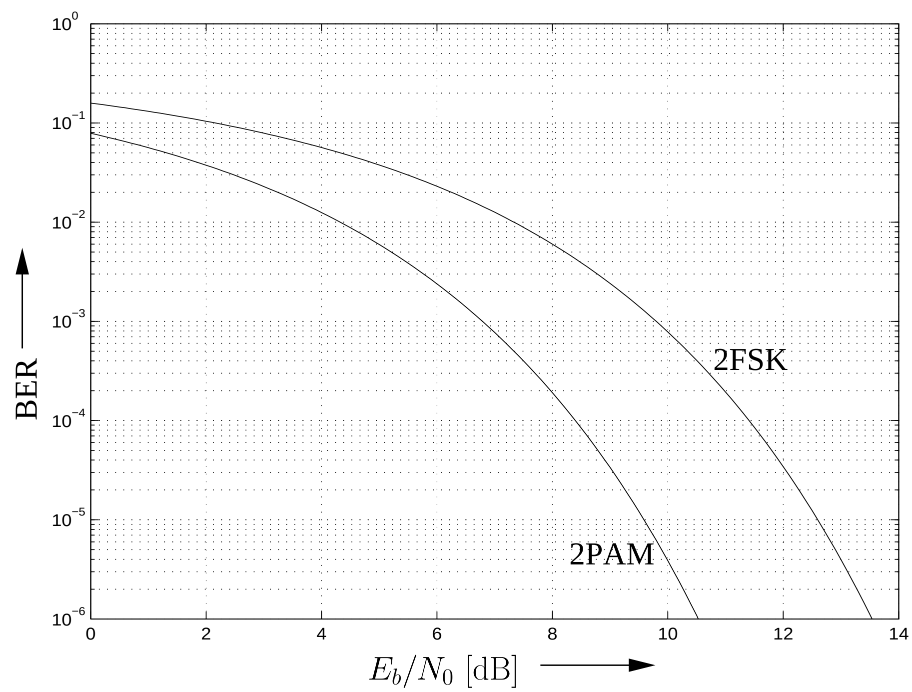
– PAM:

$$P_b = Q\left(\sqrt{2\frac{E_b}{N_0}}\right)$$

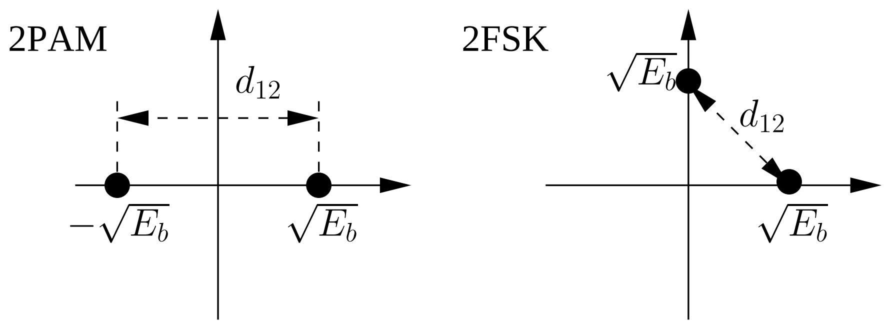
– Orthogonal Signaling (e.g. FSK)

$$P_b = Q\left(\sqrt{\frac{E_b}{N_0}}\right)$$

– We observe that in order to achieve the same BEP the E_b–to–N_0 ratio (SNR) has to be 3 dB higher for orthogonal signaling than for PAM. Therefore, orthogonal signaling (FSK) is less *power efficient* than antipodal signaling (PAM).



– Signal Space



We observe that the (minimum) Euclidean distance between signal points is given by

$$d_{12}^{\text{PAM}} = 2\sqrt{E_b}$$

and

$$d_{12}^{\text{FSK}} = \sqrt{2E_b}$$

for 2PAM and 2FSK, respectively. The ratio of the squared Euclidean distances is given by

$$\left(\frac{d_{12}^{\text{PAM}}}{d_{12}^{\text{FSK}}}\right)^2 = 2.$$

Since the average energy of the signal points is identical for both constellations, the higher power efficiency of 2PAM can be directly deduced from the higher minimum Euclidean distance of the signal points in the signal space. Note that the BEP for both 2PAM and 2FSK can also be expressed as

$$\boxed{P_b = Q\left(\sqrt{\frac{d_{12}^2}{2N_0}}\right)}$$

– **Rule of Thumb:**
In general, for a given average energy of the signal points, the BEP of a linear modulation scheme is larger if the minimum Euclidean distance of the signals in the signal space is smaller.

4.2.2 M–ary PAM

- The transmitted signal points are given by

$$s_{bm} = \sqrt{E_g} A_m, \qquad 1 \le m \le M$$

with pulse energy E_g and amplitude

$$A_m = (2m - 1 - M)d, \qquad 1 \le m \le M.$$

- **Average Energy of Signal Points**

$$\begin{aligned}
E_S &= \frac{1}{M} \sum_{m=1}^{M} E_m \\
&= \frac{1}{M} E_g d^2 \sum_{m=1}^{M} (2m - 1 - M)^2 \\
&= \frac{E_g d^2}{M} \left[4 \underbrace{\sum_{m=1}^{M} m^2}_{\frac{1}{6}M(M+1)(2M+1)} - 4(M+1) \underbrace{\sum_{m=1}^{M} m}_{\frac{1}{2}M(M+1)} + \underbrace{\sum_{m=1}^{M} (M+1)^2}_{M(M+1)^2} \right] \\
&= \frac{M^2 - 1}{3} d^2 E_g
\end{aligned}$$

- **Received Baseband Signal**

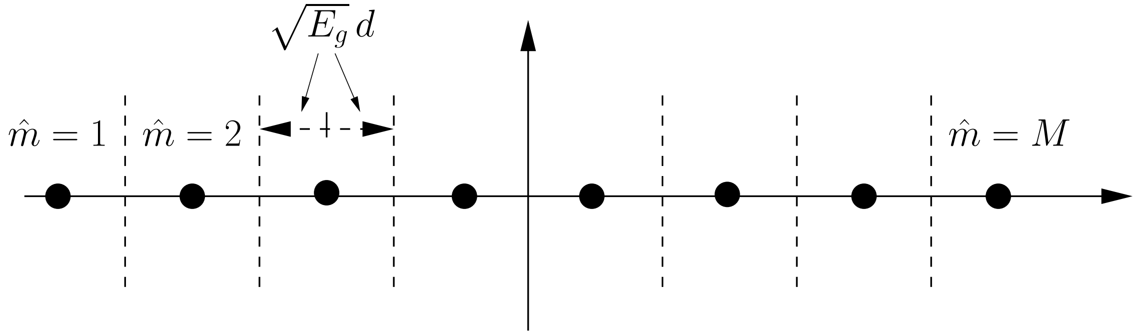
$$r_b = s_{bm} + z,$$

with $\sigma_z^2 = \mathcal{E}\{|z|^2\} = N_0$. Again only the real part of the received signal is relevant for detection and we get

$$r_R = s_{bm} + z_R,$$

with noise variance $\sigma_{z_R}^2 = \mathcal{E}\{z_R^2\} = N_0/2$.

- **Decision Regions for ML Detection**



- We observe that there are two different types of signal points:

  1. Outer Signal Points
     We refer to the signal points with $\hat{m} = 1$ and $\hat{m} = M$ as *outer signal points* since they have only one neighboring signal point. In this case, we make on average 1/2 symbol errors if

     $$|r_R - s_{bm}| > d\sqrt{E_g}$$

  2. Inner Signal Points
     Signal points with $2 \leq \hat{m} \leq M - 1$ are referred to as *inner signal points* since they have two neighbors. Here, we make on average 1 symbol error if $|r_R - s_{bm}| > d\sqrt{E_g}$.

- **Symbol Error Probability (SEP)**
  The SEP can be calculated to

$$
\begin{aligned}
P_M &= \frac{1}{M}\left[(M-2)+\frac{1}{2}\cdot 2\right] P\left(|r_R - s_{bm}| > d\sqrt{E_g}\right) \\
&= \frac{M-1}{M} P\left([r_R - s_{bm} > d\sqrt{E_g}] \vee [r_R - s_{bm} < -d\sqrt{E_g}]\right) \\
&= \frac{M-1}{M}\left(P\left(r_R - s_{bm} > d\sqrt{E_g}\right) + P\left(r_R - s_{bm} < -d\sqrt{E_g}\right)\right) \\
&= \frac{M-1}{M} 2P\left(r_R - s_{bm} > d\sqrt{E_g}\right) \\
&= 2\,\frac{M-1}{M}\,\frac{1}{\sqrt{\pi N_0}} \int\limits_{d\sqrt{E_g}}^{\infty} \exp\left(-\frac{x^2}{N_0}\right)\, \mathrm{d}x \\
&= 2\,\frac{M-1}{M}\,\frac{1}{\sqrt{2\pi}} \int\limits_{\sqrt{2d^2\frac{E_g}{N_0}}}^{\infty} \exp\left(-\frac{y^2}{2}\right)\, \mathrm{d}y \\
&= 2\,\frac{M-1}{M}\, Q\left(\sqrt{2d^2\frac{E_g}{N_0}}\right)
\end{aligned}
$$

- Using the identity

$$d^2 E_g = 3\frac{E_S}{M^2-1},$$

  we obtain

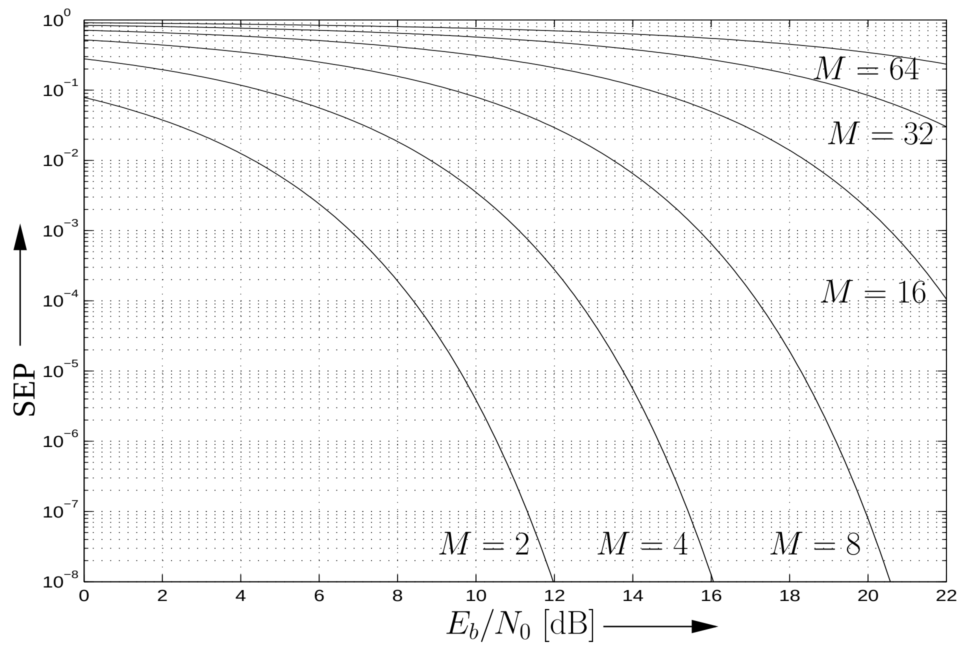
$$\boxed{P_M = 2\,\frac{M-1}{M}\, Q\left(\sqrt{\frac{6E_S}{(M^2-1)N_0}}\right).}$$

We make the following observations

1. For constant E_S the error probability increases with increasing M.
2. For a given SEP the required E_S/N_0 increases as
$$10\log_{10}(M^2-1) \approx 20\log_{10} M.$$
This means if we double the number of signal points, i.e., $M = 2^k$ is increased to $M = 2^{k+1}$, the required E_S/N_0 increases (approximately) as
$$20\log_{10}\left(2^{k+1}/2^k\right) = 20\log_{10} 2 \approx 6\,\mathrm{dB}.$$

- Alternatively, we may express P_M as a function of the average energy per bit E_b, which is given by
$$E_b = \frac{E_S}{k} = \frac{E_S}{\log_2 M}$$
Therefore, the resulting expression for P_M is
$$\boxed{P_M = 2\,\frac{M-1}{M}\,Q\left(\sqrt{\frac{6\log_2(M)E_b}{(M^2-1)N_0}}\right).}$$

- An exact expression for the *bit error probability (BEP)* is more difficult to derive than the expression for the SEP. However, for high E_S/N_0 ratios most errors only involve neighboring signal points. Therefore, if we use *Gray labeling* we make approximately one bit error per symbol error. Since there are $\log_2 M$ bits per symbol, the PEP can be approximated by
$$\boxed{P_b \approx \frac{1}{\log_2 M}\,P_M.}$$



4.2.3 M–ary PSK

- For 2PSK the same SEP as for 2PAM results.

$$P_2 = Q\left(\sqrt{\frac{2E_b}{N_0}}\right).$$

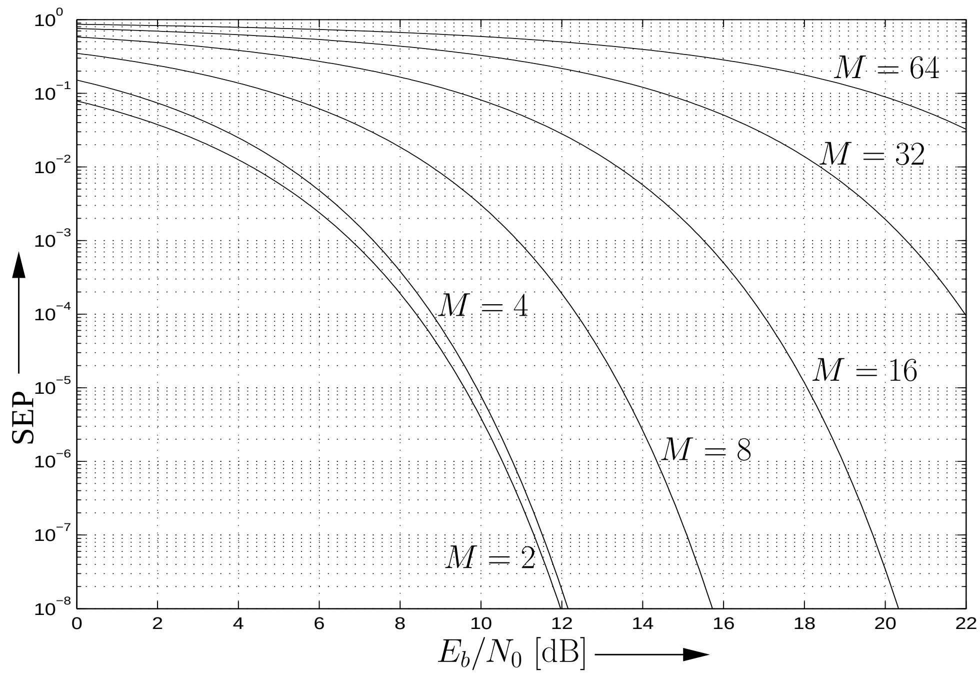
- For 4PSK the SEP is given by

$$P_4 = 2\,Q\left(\sqrt{\frac{2E_b}{N_0}}\right)\left[1 - \frac{1}{2}Q\left(\sqrt{\frac{2E_b}{N_0}}\right)\right].$$

- For optimum detection of M–ary PSK the SEP can be tightly approximated as

$$P_M \approx 2\,Q\left(\sqrt{\frac{2\log_2(M)E_b}{N_0}}\sin\frac{\pi}{M}\right).$$

- The approximate SEP is illustrated below for several values of M. For $M = 2$ and $M = 4$ the exact SEP is shown.

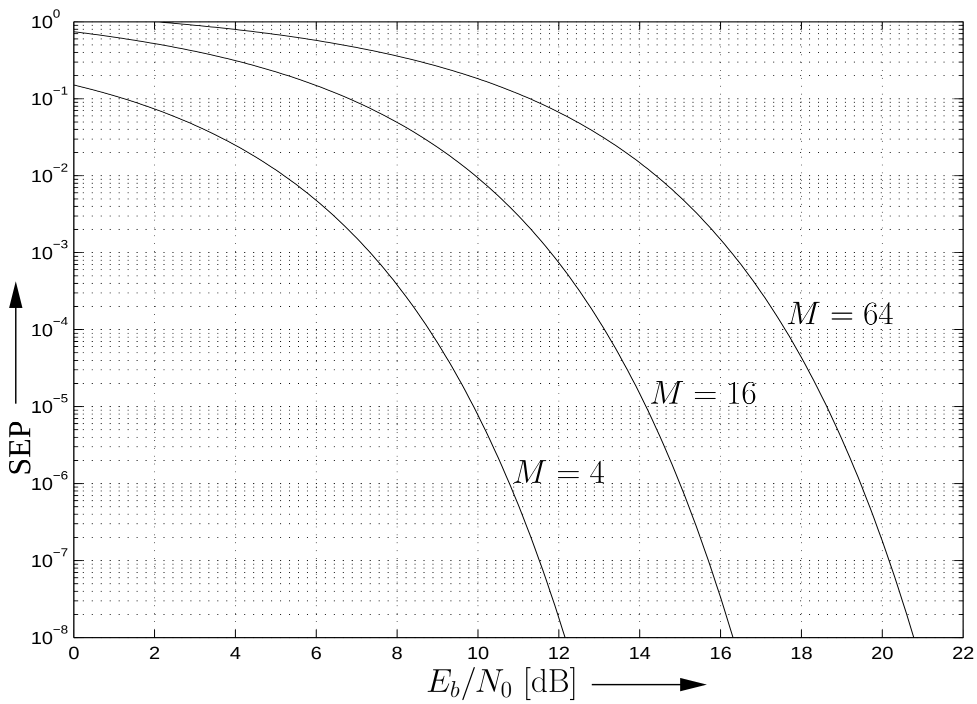


4.2.4 M–ary QAM

- For $M = 4$ the SEP of QAM is identical to that of PSK.
- In general, the SEP can be tightly upper bounded by

$$P_M \leq 4Q\left(\sqrt{\frac{3\log_2(M)E_b}{(M-1)N_0}}\right).$$

- The bound on SEP is shown below. For $M = 4$ the exact SEP is shown.



4.2.5 Upper Bound for Arbitrary Linear Modulation Schemes

Although exact (and complicated) expressions for the SEP and BEP of most regular linear modulation formats exist, it is sometimes more convenient to employ simple bounds and approximation. In this sections, we derive the union upper bound valid for arbitrary signal constellations and a related approximation for the SEP.

- We consider an M–ary modulation scheme with M signal points $\boldsymbol{s}_m$, $1 \leq m \leq M$, in the signal space.
- We denote the *pairwise* error probability of two signal points $\boldsymbol{s}_\mu$ and $\boldsymbol{s}_\nu$, $\mu \neq \nu$ by

$$\mathrm{PEP}(\boldsymbol{s}_\mu \rightarrow \boldsymbol{s}_\nu) = P(\boldsymbol{s}_\mu \, \text{transmitted}, \, \boldsymbol{s}_\nu, \, \text{detected})$$

- The union bound for the SEP can be expressed as

$$\boxed{P_M \le \frac{1}{M} \sum_{\mu=1}^{M} \sum_{\substack{\nu=1 \\ \nu \neq \mu}}^{M} \mathrm{PEP}(\boldsymbol{s}_\mu \to \boldsymbol{s}_\nu)}$$

which is an upper bound since some regions of the signal space may be included in multiple PEPs.

- The advantage of the union bound is that the PEP can be usually easily obtained. Assuming Gaussian noise and an Euclidean distance of $d_{\mu\nu} = ||\boldsymbol{s}_\mu - \boldsymbol{s}_\nu||$ between signal points $\boldsymbol{s}_\mu$ and $\boldsymbol{s}_\nu$, we obtain

$$\boxed{P_M \le \frac{1}{M} \sum_{\mu=1}^{M} \sum_{\substack{\nu=1 \\ \nu \neq \mu}}^{M} Q\left(\sqrt{\frac{d_{\mu\nu}^2}{2N_0}}\right)}$$

- Assuming that each signal point has on average C_M nearest neighbor signal points with minimum distance $d_{\min} = \min_{\mu \neq \nu}\{d_{\mu\nu}\}$ and exploiting the fact that for high SNR the minimum distance terms will dominate the union bound, we obtain the approximation

$$\boxed{P_M \approx C_M Q\left(\sqrt{\frac{d_{\min}^2}{2N_0}}\right)}$$

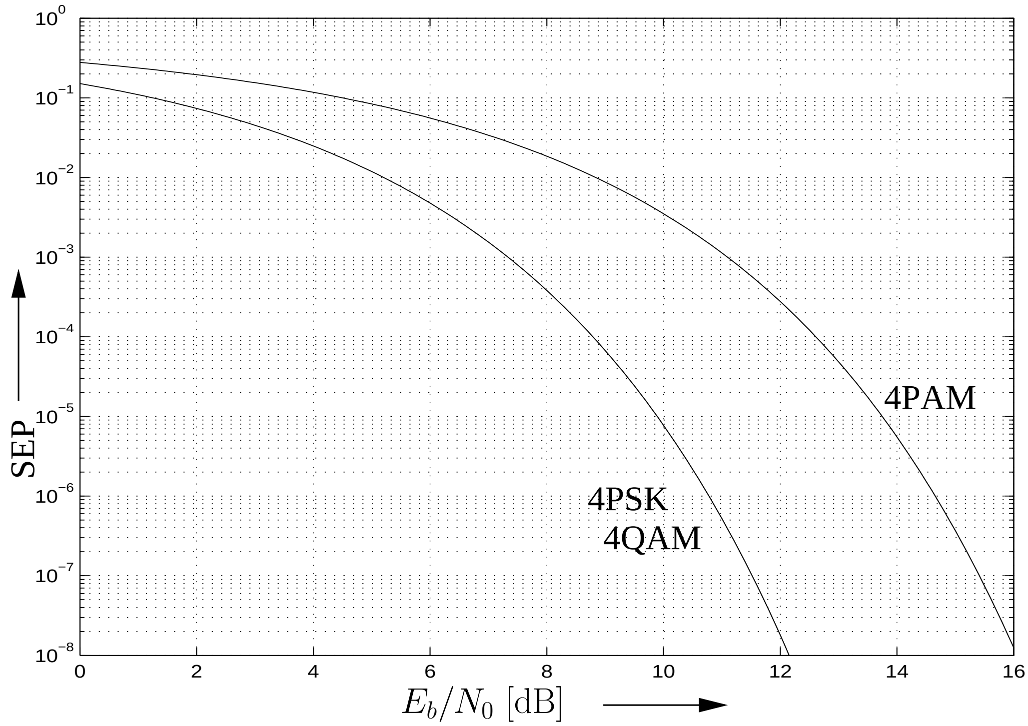
Note: The SEP approximation given above for MPSK can be obtained with this approximation.

- For Gray labeling, approximations for the BEP are obtained from the above equations with $P_b \approx P_M / \log_2(M)$

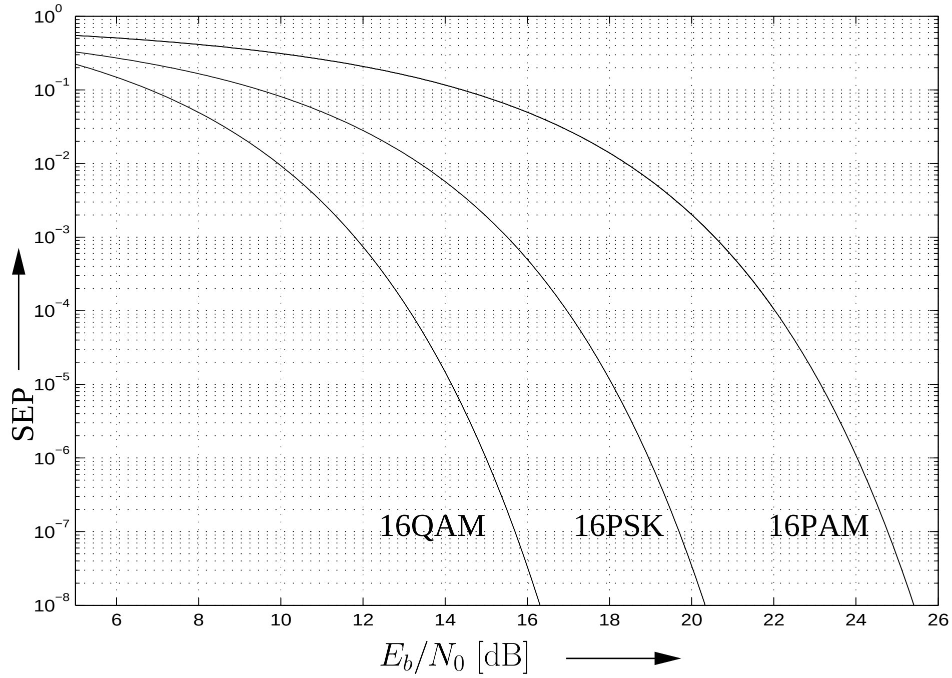
4.2.6 Comparison of Different Linear Modulations

We compare PAM, PSK, and QAM for different M.

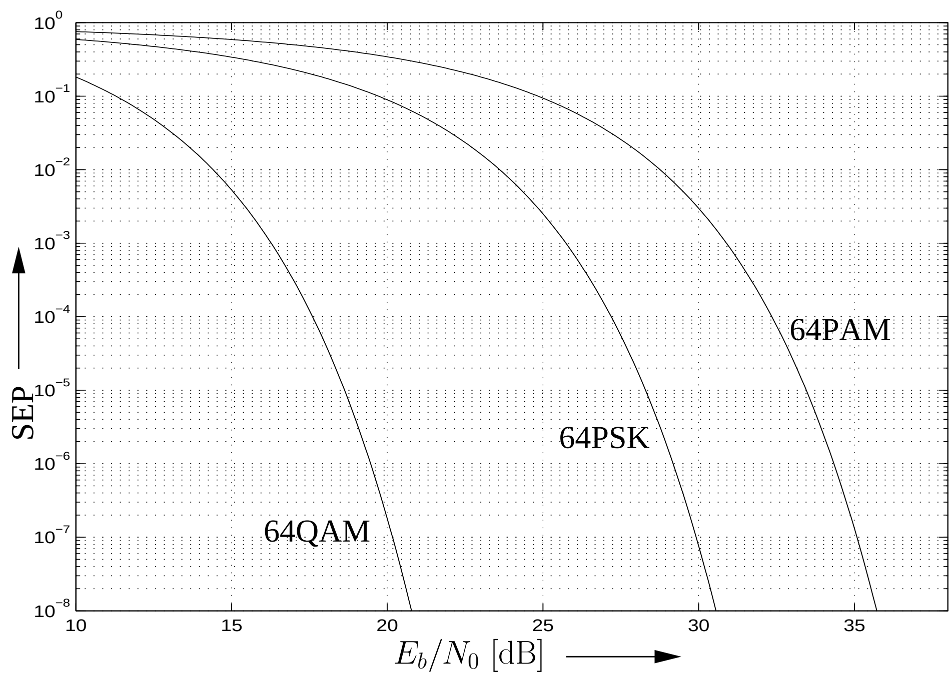
- $M = 4$



- $M = 16$



- $M = 64$



- Obviously, as M increases PAM and PSK become less favorable and the gap to QAM increases. The reason for this behavior is the smaller *minimum Euclidean distance* $d_{\min}$ of PAM and PSK. For a given transmit energy $d_{\min}$ of PAM and PSK is smaller since the signal points are confined to a line and a circle, respectively. For QAM on the other hand, the signal points are on a rectangular grid, which guarantees a comparatively large $d_{\min}$.

4.3 Receivers for Signals with Random Phase in AWGN

4.3.1 Channel Model

- **Passband Signal**

  We assume that the received passband signal can be modeled as

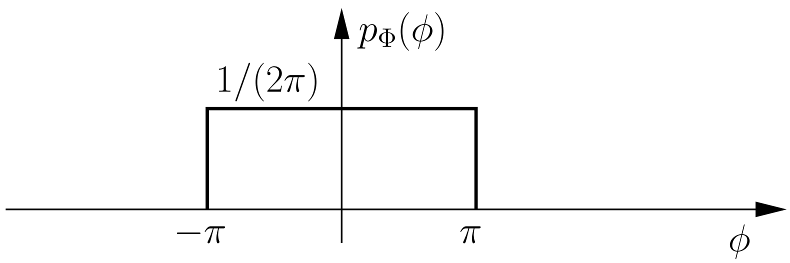
$$r(t) = \sqrt{2}\mathrm{Re}\left\{ \left(e^{j\phi} s_{bm}(t) + z(t) \right) e^{j2\pi f_c t} \right\},$$

  where $z(t)$ is complex AWGN with power spectral density $\Phi_{zz}(f) = N_0$, and ϕ is an unknown, random but constant phase.

  – ϕ may originate from the local oscillator or the transmission channel.

  – ϕ is often modeled as uniformly distributed, i.e.,

$$p_\Phi(\phi) = \begin{cases} \frac{1}{2\pi}, & -\pi \le \phi < \pi \\ 0, & \text{otherwise} \end{cases}$$



- **Baseband Signal**

  The received baseband signal is given by

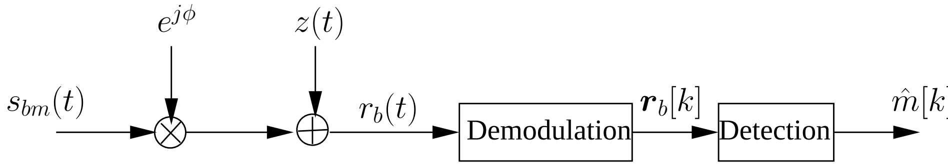
$$r_b(t) = e^{j\phi} s_{bm}(t) + z(t).$$

- **Optimal Demodulation**

  Since the unknown phase results just in the multiplication of the transmitted baseband waveform $s_{bm}(t)$ by a constant factor $e^{j\phi}$, demodulators that are optimum for $\phi = 0$ are also optimum for $\phi \neq 0$. Therefore, both correlation demodulation and matched–filter demodulation are also optimum if the channel phase is unknown. The demodulated signal in interval $kT \leq t \leq (k+1)T$ can be written as

$$\boldsymbol{r}_b[k] = e^{j\phi} \boldsymbol{s}_{bm}[k] + \boldsymbol{z}[k],$$

  where the components of the AWGN vector $\boldsymbol{z}[k]$ are mutually independent, zero–mean complex Gaussian processes with variance $\sigma_z^2 = N_0$.

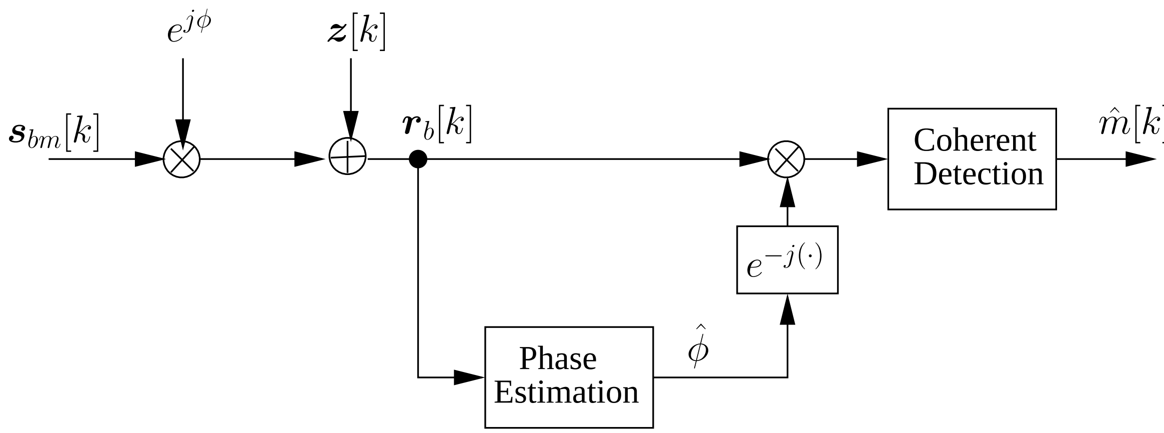


4.3.2 Noncoherent Detectors

In general, we distinguish between *coherent* and *noncoherent* detection.

1. **Coherent Detection**

   - Coherent detectors first estimate the unknown phase ϕ using e.g. known pilot symbols introduced into the transmitted signal stream.
   - For detection it is assumed that the phase estimate $\hat{\phi}$ is *perfect*, i.e., $\hat{\phi} = \phi$, and the same detectors as for the pure AWGN channel can be used.



   - The performance of ideal coherent detection with $\hat{\phi} = \phi$ constitutes an *upper* bound for any realizable non–ideal coherent or noncoherent detection scheme.
   - *Disadvantages*
     - In practice, ideal coherent detection is not possible and the *ad hoc* separation of phase estimation and detection is *suboptimum.*

– Phase estimation is often complicated and may require pilot symbols.

2. **Noncoherent Detection**

- In noncoherent detectors no attempt is made to explicitly estimate the phase ϕ.
- *Advantages*
  - For many modulation schemes simple noncoherent receiver structures exist.
  - More complex *optimal* noncoherent receivers can be derived.

4.3.2.1 A Simple Noncoherent Detector for PSK with Differential Encoding (DPSK)

- As an example for a simple, suboptimum noncoherent detector we derive the so–called *differential detector* for DPSK.
- **Transmit Signal**

  The transmitted complex baseband waveform in the interval $kT \leq t \leq (k+1)T$ is given by

  $$s_{bm}(t) = b[k]g(t - kT),$$

  where $g(t)$ denotes the transmit pulse of length T and $b[k]$ is the transmitted PSK signal which is given by

  $$b[k] = e^{j\Theta[k]}.$$

  The PSK symbols are generated from the differential PSK (DPSK) symbols $a[k]$ as

  $$b[k] = a[k]b[k-1],$$

where $a[k]$ is given by

$$a[k] = e^{j\Delta\Theta[k]}, \qquad \Delta\Theta[k] = 2\pi(m-1)/M, \; m \in \{1, 2, \ldots, M\}.$$

For simplicity, we have dropped the symbol index m in $\Delta\Theta[k]$ and $a[k]$. Note that the *absolute phase* $\Theta[k]$ is related to the *differential phase* $\Delta\Theta[k]$ by

$$\Theta[k] = \Theta[k-1] + \Delta\Theta[k].$$

- For demodulation, we use the basis function

$$f(t) = \frac{1}{\sqrt{E_g}}\, g(t)$$

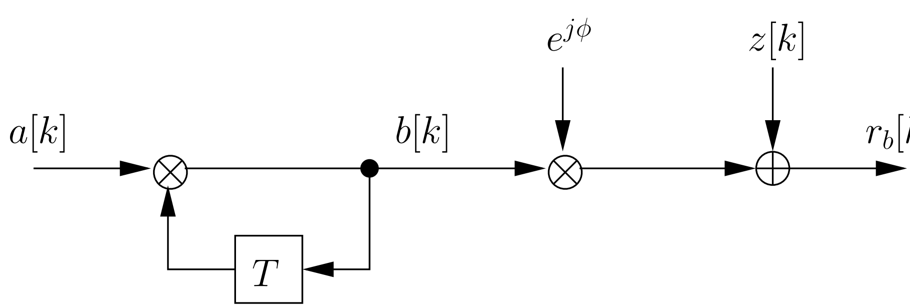
- The demodulated signal in the kth interval can be represented as

$$r_b'[k] = e^{j\phi}\sqrt{E_g}\, b[k] + z'[k],$$

where $z'[k]$ is an AWGN process with variance $\sigma_{z'}^2 = N_0$. It is convenient to define the new signal

$$\begin{aligned} r_b[k] &= \frac{1}{\sqrt{E_g}}\, r_b'[k] \\ &= e^{j\phi}\, b[k] + z[k], \end{aligned}$$

where $z[k]$ has variance $\sigma_z^2 = N_0/E_g$.



■ Differential Detection (DD)

– The received signal in the kth and the $(k-1)$st symbol intervals are given by

$$r_b[k] = e^{j\phi}a[k]b[k-1] + z[k]$$

and

$$r_b[k-1] = e^{j\phi}b[k-1] + z[k-1],$$

respectively.

– If we assume $z[k] \approx 0$ and $z[k-1] \approx 0$, the variable

$$d[k] = r_b[k]r_b^*[k-1]$$

can be simplified to

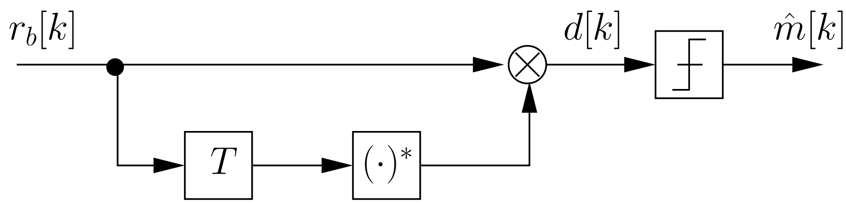
$$\begin{aligned} d[k] &= r_b[k]r_b^*[k-1] \\ &\approx e^{j\phi}a[k]b[k-1](e^{j\phi}a[k]b[k-1])^* \\ &= a[k]\,|b[k]|^2 \\ &= a[k]. \end{aligned}$$

This means $d[k]$ is independent of the phase ϕ and is suitable for detecting $a[k]$.

– A detector based on $d[k]$ is referred to as *(conventional) differential detector*. The resulting decision rule is

$$\hat{m}[k] = \underset{\tilde{m}}{\operatorname{argmin}} \left\{ |d[k] - a_{\tilde{m}}[k]|^2 \right\},$$

where $a_{\tilde{m}}[k] = e^{j2\pi(\tilde{m}-1)/M}$, $1 \leq \tilde{m} \leq M$. Alternatively, we may base our decision on the location of $d[k]$ in the signal space, as usual.



■ Comparison with Coherent Detection

– *Coherent Detection (CD)*

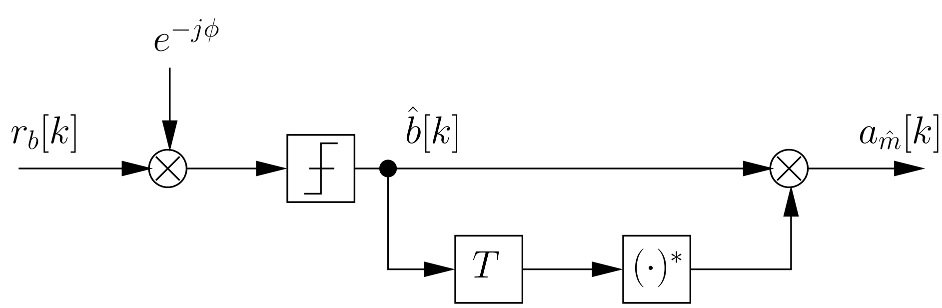
An upper bound on the performance of DD can be obtained by CD with ideal knowledge of ϕ. In that case, we can use the decision variable $r_b[k]$ and directly make a decision on the absolute phase symbols $b[k] = e^{j\Theta[k]} \in \{e^{j2\pi(m-1)/M} | m = 1, 2, \ldots, M\}$.

$$\hat{b}[k] = \underset{\tilde{b}[k]}{\operatorname{argmin}} \left\{ |\tilde{b}[k] - e^{-j\phi} r_b[k]|^2 \right\},$$

and obtain an estimate for the differential symbol from

$$a_{\hat{m}}[k] = \hat{b}[k] \cdot \hat{b}^*[k-1].$$

$\hat{b}[k]$ and $\tilde{b}[k] = e^{j\tilde{\Theta}[k]} \in \{e^{j2\pi(m-1)/M} | m = 1, 2, \ldots, M\}$ denote the estimated transmitted symbol and a trial symbol, respectively. The above decision rule is identical to that for PSK except for the inversion of the differential encoding operation.



Since two successive absolute symbols $\hat{b}[k]$ are necessary for estimation of the differential symbol $a_{\hat{m}}[k]$ (or equivalently $\hat{m}$), isolated single errors in the absolute phase symbols will lead to *two* symbol errors in the differential phase symbols, i.e., if all absolute phase symbols but that at time k_0 are correct, then all differential phase symbols but the ones at times k_0 and $k_0 + 1$ will be correct. Since at high SNRs single errors in the absolute phase symbols dominate, the SEP $\mathrm{SEP}_{\mathrm{DPSK}}^{\mathrm{CD}}$ of DPSK with CD is approximately by a factor of two higher than that of PSK with CD, which we refer to as $\mathrm{SEP}_{\mathrm{PSK}}^{\mathrm{CD}}$.

$$\mathrm{SEP}_{\mathrm{DPSK}}^{\mathrm{CD}} \approx 2\mathrm{SEP}_{\mathrm{PSK}}^{\mathrm{CD}}$$

Note that at high SNRs this factor of two difference in SEP corresponds to a negligible difference in required E_b/N_0 ratio to achieve a certain BER, since the SEP decays approximately exponentially.

– *Differential Detection (DD)*

The decision variable $d[k]$ can be rewritten as

$$\begin{aligned} d[k] &= r_b[k]r_b^*[k-1] \\ &= (e^{j\phi}a[k]b[k-1] + z[k])(e^{j\phi}b[k-1] + z[k-1])^* \\ &= a[k] + \underbrace{e^{-j\phi}b^*[k-1]z[k] + e^{j\phi}b[k]z^*[k-1] + z[k]z^*[k-1]}_{z_{\mathrm{eff}}[k]}, \end{aligned}$$

where $z_{\mathrm{eff}}[k]$ denotes the *effective noise* in the decision variable $d[k]$. It can be shown that $z_{\mathrm{eff}}[k]$ is a white process with variance

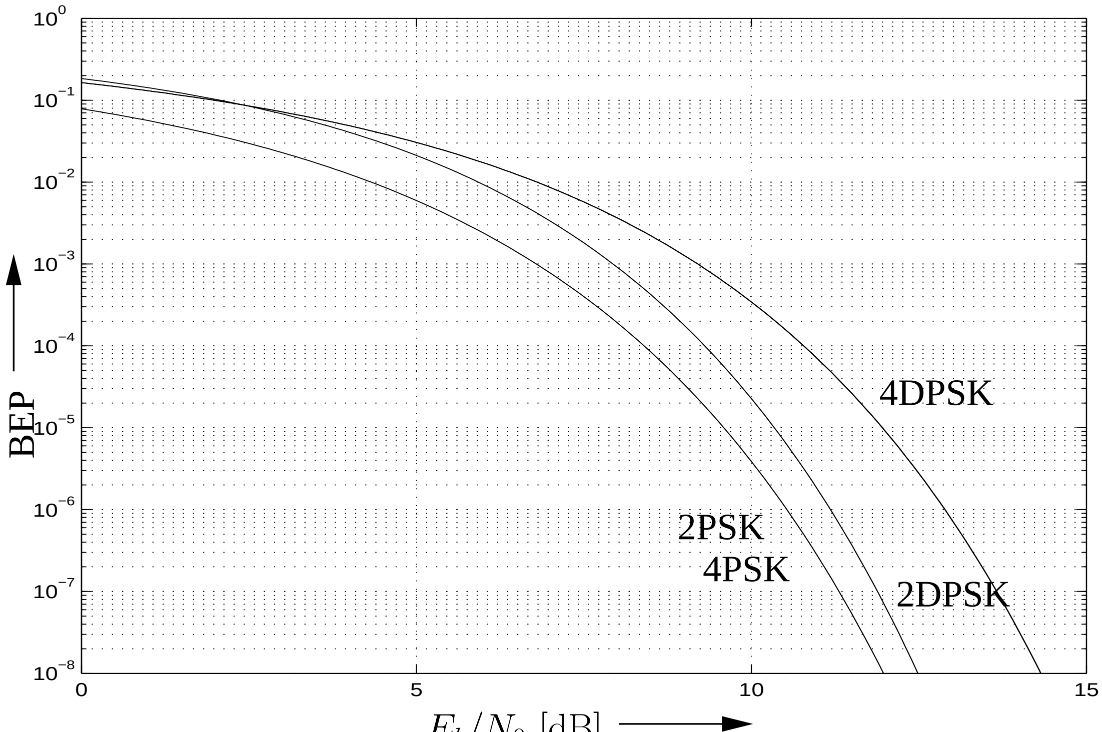
$$\begin{aligned}\sigma^2_{z_{\text{eff}}} &= 2\sigma^2_z + \sigma^4_z \\ &= 2\frac{N_0}{E_g} + \left(\frac{N_0}{E_g}\right)^2\end{aligned}$$

For high SNRs we can approximate $\sigma^2_{z_{\text{eff}}}$ as

$$\begin{aligned}\sigma^2_{z_{\text{eff}}} &\approx 2\sigma^2_z \\ &= 2\frac{N_0}{E_g}.\end{aligned}$$

– *Comparison*

We observe that the variance $\sigma^2_{z_{\text{eff}}}$ of the effective noise in the decision variable for DD is twice as high as that for CD. However, for small M the distribution of $z_{\text{eff}}[k]$ is significantly different from a Gaussian distribution. Therefore, for small M a direct comparison of DD and CD is difficult and requires a detailed BEP or SEP analysis. On the other hand, for large $M \geq 8$ the distribution of $z_{\text{eff}}[k]$ can be well approximated as Gaussian. Therefore, we expect that at high SNRs DPSK with DD requires approximately a 3 dB higher E_b/N_0 ratio to achieve the same BEP as CD. For $M = 2$ and $M = 4$ this difference is smaller. At a BEP of 10^{-5} the loss of DD compared to CD is only about 0.8 dB and 2.3 dB for $M = 2$ and $M = 4$, respectively.



4.3.2.2 Optimum Noncoherent Detection

- The demodulated received baseband signal is given by

$$\boldsymbol{r}_b = e^{j\phi}\,\boldsymbol{s}_{bm} + \boldsymbol{z}.$$

- We assume that all possible symbols $\boldsymbol{s}_{bm}$ are transmitted with equal probability, i.e., ML detection is optimum.

- We already know that the ML decision rule is given by

$$\hat{m} = \underset{\tilde{m}}{\operatorname{argmax}}\,\{p(\boldsymbol{r}|\boldsymbol{s}_{b\tilde{m}})\}.$$

  Thus, we only have to find an analytic expression for $p(\boldsymbol{r}|\boldsymbol{s}_{b\tilde{m}})$. The problem we encounter here is that, in contrast to coherent detection, there is an additional random variable, namely the unknown phase ϕ, involved.

- We know the pdf of $\boldsymbol{r}_b$ conditioned on both ϕ and $\boldsymbol{s}_{b\tilde{m}}$. It is given by

$$p(\boldsymbol{r}_b|\boldsymbol{s}_{b\tilde{m}},\,\phi) = \frac{1}{(\pi N_0)^N}\exp\left(-\frac{||\boldsymbol{r}_b - e^{j\phi}\,\boldsymbol{s}_{b\tilde{m}}||^2}{N_0}\right)$$

- We obtain $p(\boldsymbol{r}_b|\boldsymbol{s}_{b\tilde{m}})$ from $p(\boldsymbol{r}_b|\boldsymbol{s}_{b\tilde{m}},\,\phi)$ as

$$p(\boldsymbol{r}_b|\boldsymbol{s}_{b\tilde{m}}) = \int\limits_{-\infty}^{\infty} p(\boldsymbol{r}_b|\boldsymbol{s}_{b\tilde{m}},\,\phi)\,p_\Phi(\phi)\,\mathrm{d}\phi.$$

Since we assume for the distribution of ϕ

$$p_\Phi(\phi) = \begin{cases} \frac{1}{2\pi}, & -\pi \le \phi < \pi \\ 0, & \text{otherwise} \end{cases}$$

we get

$$\begin{aligned} p(\boldsymbol{r}_b|\boldsymbol{s}_{b\tilde{m}}) &= \frac{1}{2\pi}\int\limits_{-\pi}^{\pi} p(\boldsymbol{r}_b|\boldsymbol{s}_{b\tilde{m}},\,\phi)\,\mathrm{d}\phi \\ &= \frac{1}{(\pi N_0)^N}\,\frac{1}{2\pi}\int\limits_{-\pi}^{\pi}\exp\left(-\frac{||\boldsymbol{r}_b - e^{j\phi}\,\boldsymbol{s}_{b\tilde{m}}||^2}{N_0}\right)\,\mathrm{d}\phi \\ &= \frac{1}{(\pi N_0)^N}\,\frac{1}{2\pi}\int\limits_{-\pi}^{\pi}\exp\left(-\frac{||\boldsymbol{r}_b||^2 + ||\boldsymbol{s}_{b\tilde{m}}||^2 - 2\mathsf{Re}\{\boldsymbol{r}_b \bullet e^{j\phi}\boldsymbol{s}_{b\tilde{m}}\}}{N_0}\right)\,\mathrm{d}\phi \end{aligned}$$

Now, we use the relations

$$\begin{aligned} \mathsf{Re}\{\boldsymbol{r}_b \bullet e^{j\phi}\boldsymbol{s}_{b\tilde{m}}\} &= \mathsf{Re}\{|\boldsymbol{r}_b \bullet \boldsymbol{s}_{b\tilde{m}}|\,e^{j\alpha}e^{j\phi}\} \\ &= |\boldsymbol{r}_b \bullet \boldsymbol{s}_{b\tilde{m}}|\cos(\phi+\alpha), \end{aligned}$$

where α is the argument of $\boldsymbol{r}_b \bullet \boldsymbol{s}_{b\tilde{m}}$. With this simplification $p(\boldsymbol{r}_b|\boldsymbol{s}_{b\tilde{m}})$ can be rewritten as

$$\begin{aligned} p(\boldsymbol{r}_b|\boldsymbol{s}_{b\tilde{m}}) \;=\; & \frac{1}{(\pi N_0)^N} \exp\left(-\frac{||\boldsymbol{r}_b||^2 + ||\boldsymbol{s}_{b\tilde{m}}||^2}{N_0}\right) \\ & \cdot \frac{1}{2\pi} \int\limits_{-\pi}^{\pi} \exp\left(\frac{2}{N_0} |\boldsymbol{r}_b \bullet \boldsymbol{s}_{b\tilde{m}}| \mathsf{cos}(\phi + \alpha)\right) \,\mathrm{d}\phi \end{aligned}$$

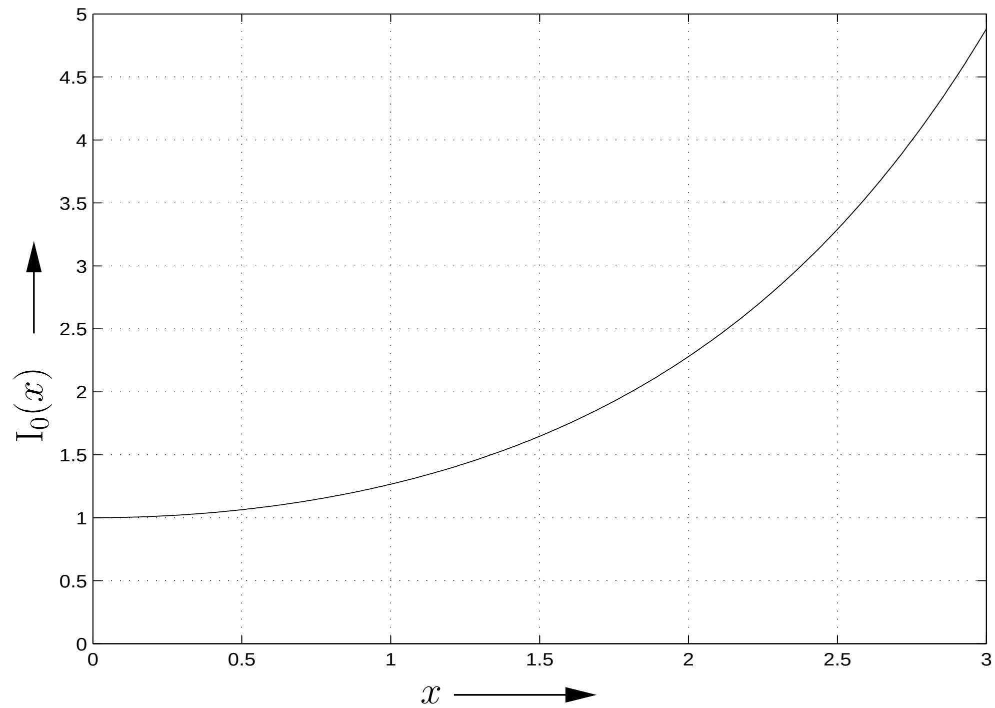
Note that the above integral is *independent* of α since α is independent of ϕ and we integrate over an entire period of the cosine function. Therefore, using the definition

$$\boxed{\mathrm{I}_0(x) = \frac{1}{2\pi} \int\limits_{-\pi}^{\pi} \exp(x \mathsf{cos}\phi) \,\mathrm{d}\phi}$$

we finally obtain

$$\boxed{p(\boldsymbol{r}_b|\boldsymbol{s}_{b\tilde{m}}) = \frac{1}{(\pi N_0)^N} \exp\left(-\frac{||\boldsymbol{r}_b||^2 + ||\boldsymbol{s}_{b\tilde{m}}||^2}{N_0}\right) \cdot \mathrm{I}_0\left(\frac{2}{N_0} |\boldsymbol{r}_b \bullet \boldsymbol{s}_{b\tilde{m}}|\right)}$$

Note that $\mathrm{I}_0(x)$ is the *zeroth order modified Bessel function of the first kind.*



- **ML Detection**

  With the above expression for $p(\boldsymbol{r}_b|\boldsymbol{s}_{b\tilde{m}})$ the noncoherent ML decision rule becomes

$$\begin{aligned}\hat{m} &= \underset{\tilde{m}}{\operatorname{argmax}}\{p(\boldsymbol{r}|\boldsymbol{s}_{b\tilde{m}})\}.\\ &= \underset{\tilde{m}}{\operatorname{argmax}}\{\ln[p(\boldsymbol{r}|\boldsymbol{s}_{b\tilde{m}})]\}.\\ &= \underset{\tilde{m}}{\operatorname{argmax}}\left\{-\frac{||\boldsymbol{s}_{b\tilde{m}}||^2}{N_0}+\ln\left[\mathrm{I}_0\left(\frac{2}{N_0}|\boldsymbol{r}_b \bullet \boldsymbol{s}_{b\tilde{m}}|\right)\right]\right\}\end{aligned}$$

- **Simplification**

  The above ML decision rule depends on N_0, which may not be desirable in practice, since N_0 (or equivalently the SNR) has to be estimated. However, for $x \gg 1$ the approximation

  $$\ln[\mathrm{I}_0(x)] \approx x$$

  holds. Therefore, at high SNR (or small N_0) the above ML metric can be simplified to

  $$\hat{m} \;=\; \underset{\tilde{m}}{\mathrm{argmax}} \left\{ -||\boldsymbol{s}_{b\tilde{m}}||^2 + 2|\boldsymbol{r}_b \bullet \boldsymbol{s}_{b\tilde{m}}| \right\},$$

  which is independent of N_0. In practice, the above simplification has a negligible impact on performance even for small arguments (corresponding to low SNRs) of the Bessel function.

4.3.2.3 Optimum Noncoherent Detection of Binary Orthogonal Modulation

- **Transmitted Waveform**

  We assume that the signal space representation of the *complex baseband* transmit waveforms is

  $$\begin{aligned} \boldsymbol{s}_{b1} &= [\sqrt{E_b} \qquad 0]^T \\ \boldsymbol{s}_{b2} &= [0 \qquad \sqrt{E_b}]^T \end{aligned}$$

- **Received Signal**

  The corresponding demodulated received signal is

  $$\boldsymbol{r}_b = [e^{j\phi}\sqrt{E_b} + z_1 \qquad z_2]^T$$

  and

  $$\boldsymbol{r}_b = [z_1 \qquad e^{j\phi}\sqrt{E_b} + z_2]^T$$

if $\boldsymbol{s}_{b1}$ and $\boldsymbol{s}_{b2}$ have been transmitted, respectively. z_1 and z_2 are mutually independent complex Gaussian noise processes with identical variances $\sigma_z^2 = N_0$.

- **ML Detection**
  For the simplification of the general ML decision rule, we exploit the fact that for binary orthogonal modulation the relation

$$||\boldsymbol{s}_{b1}||^2 = ||\boldsymbol{s}_{b2}||^2 = E_b$$

holds. The ML decision rule can be simplified as

$$\begin{aligned}
\hat{m} &= \underset{\tilde{m}}{\operatorname{argmax}} \left\{ -\frac{||\boldsymbol{s}_{b\tilde{m}}||^2}{N_0} + \ln\left[\mathrm{I}_0\left(\frac{2}{N_0}|\boldsymbol{r}_b \bullet \boldsymbol{s}_{b\tilde{m}}|\right)\right]\right\} \\
&= \underset{\tilde{m}}{\operatorname{argmax}} \left\{ -\frac{E_b}{N_0} + \ln\left[\mathrm{I}_0\left(\frac{2}{N_0}|\boldsymbol{r}_b \bullet \boldsymbol{s}_{b\tilde{m}}|\right)\right]\right\} \\
&= \underset{\tilde{m}}{\operatorname{argmax}} \left\{ \ln\left[\mathrm{I}_0\left(\frac{2}{N_0}|\boldsymbol{r}_b \bullet \boldsymbol{s}_{b\tilde{m}}|\right)\right]\right\} \\
&= \underset{\tilde{m}}{\operatorname{argmax}} \left\{ |\boldsymbol{r}_b \bullet \boldsymbol{s}_{b\tilde{m}}| \right\} \\
&= \underset{\tilde{m}}{\operatorname{argmax}} \left\{ |\boldsymbol{r}_b \bullet \boldsymbol{s}_{b\tilde{m}}|^2 \right\}
\end{aligned}$$

We decide in favor of that signal point which has a larger correlation with the received signal.

- We decide for $\hat{m} = 1$ if

$$|\boldsymbol{r}_b \bullet \boldsymbol{s}_{b1}| > |\boldsymbol{r}_b \bullet \boldsymbol{s}_{b2}|.$$

Using the definition

$$\boldsymbol{r}_b = [r_{b1} \qquad r_{b2}]^T,$$

we obtain

$$\left|\begin{bmatrix} r_{b1} \\ r_{b2} \end{bmatrix} \bullet \begin{bmatrix} \sqrt{E_b} \\ 0 \end{bmatrix}\right|^2 > \left|\begin{bmatrix} r_{b1} \\ r_{b2} \end{bmatrix} \bullet \begin{bmatrix} 0 \\ \sqrt{E_b} \end{bmatrix}\right|^2$$
$$|r_1|^2 > |r_2|^2.$$

In other words, the ML decision rule can be simplified to

$$\begin{array}{lll} \hat{m} = 1 & \text{if} & |r_1|^2 > |r_2|^2 \\ \hat{m} = 2 & \text{if} & |r_1|^2 < |r_2|^2 \end{array}$$

Example:

Binary FSK

– In this case, in the interval $0 \le t \le T$ we have

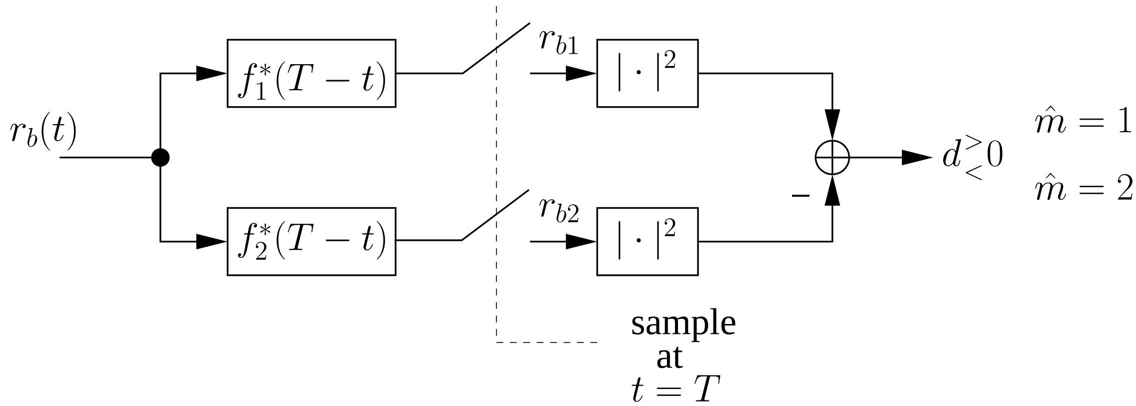
$$s_{b1}(t) = \sqrt{\frac{E_b}{T}}$$
$$s_{b2}(t) = \sqrt{\frac{E_b}{T}} e^{j2\pi\Delta f t}$$

– If $s_{b1}(t)$ and $s_{b2}(t)$ are orthogonal, the basis functions are

$$f_1(t) = \begin{cases} \frac{1}{\sqrt{T}}, & 0 \le t \le T \\ 0, & \text{otherwise} \end{cases}$$

$$f_2(t) = \begin{cases} \frac{1}{\sqrt{T}} e^{j2\pi\Delta f t}, & 0 \le t \le T \\ 0, & \text{otherwise} \end{cases}$$

– **Receiver Structure**



– If $s_{b2}(t)$ was transmitted, we get for r_{b1}

$$
\begin{aligned}
r_{b1} &= \int\limits_0^T r_b(t) f_1^*(t)\,\mathrm{d}t \\
&= \int\limits_0^T e^{j\phi} s_{b2}(t) f_1^*(t)\,\mathrm{d}t + z_1 \\
&= e^{j\phi}\frac{1}{\sqrt{T}} \int\limits_0^T s_{b2}(t)\,\mathrm{d}t + z_1 \\
&= \sqrt{E_b} e^{j\phi}\frac{1}{T} \int\limits_0^T e^{j2\pi\Delta f t}\,\mathrm{d}t + z_1 \\
&= \sqrt{E_b} e^{j\phi}\frac{1}{j2\pi\Delta f T}\, e^{j2\pi\Delta f t}\Big|_0^T + z_1 \\
&= \sqrt{E_b} e^{j\phi}\frac{1}{j2\pi\Delta f T}\, e^{j\pi\Delta f T}\left(e^{j\pi\Delta f T} - e^{-j\pi\Delta f T}\right) + z_1 \\
&= \sqrt{E_b} e^{j(\pi\Delta f T+\phi)}\,\mathrm{sinc}(\pi\Delta f T) + z_1
\end{aligned}
$$

For the proposed detector to be optimum we require orthogonality, i.e., $r_{b1} = z_1$ has to be valid. Therefore,

$$\mathrm{sinc}(\pi \Delta f T) = 0$$

is necessary, which implies

$$\Delta f T = \frac{1}{T} + \frac{k}{T}, \qquad k \in \{\ldots, -1, 0, 1, \ldots\}.$$

This means that for noncoherent detection of FSK signals the minimum required frequency separation for orthogonality is

$$\Delta f = \frac{1}{T}.$$

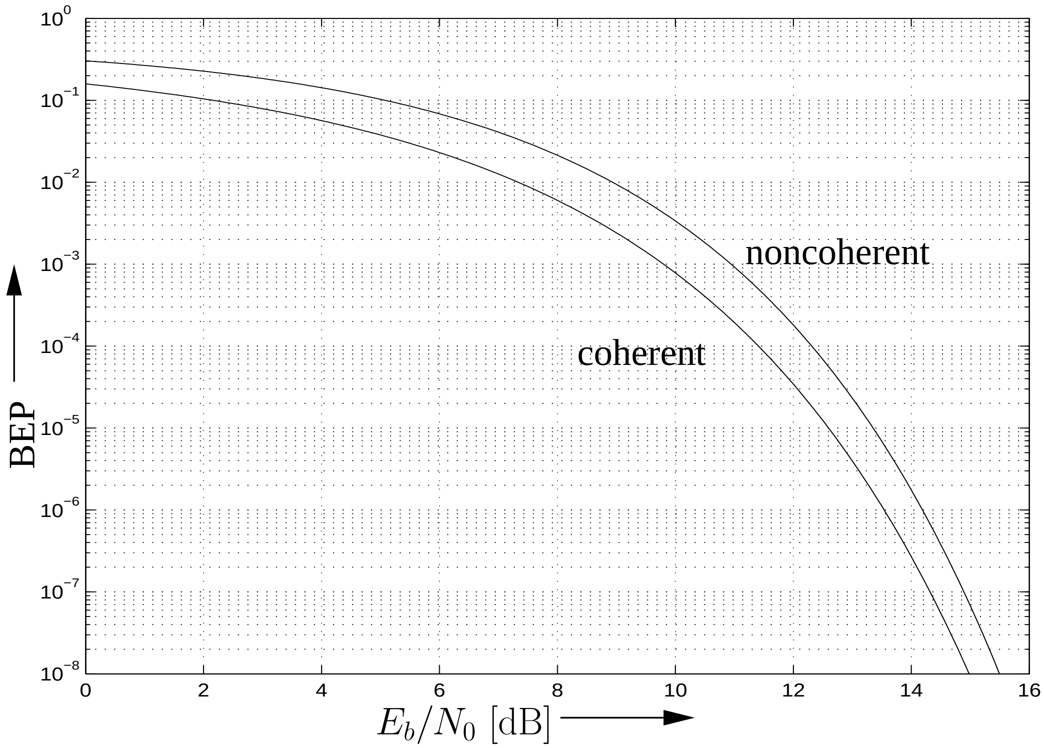
Recall that the transmitted passband signals are orthogonal for

$$\Delta f = \frac{1}{2T},$$

which is also the minimum required separation for *coherent detection* of FSK.

– The BEP (or SEP) for binary FSK with noncoherent detection can be calculated to

$$\boxed{P_b = \frac{1}{2} \exp\left(-\frac{E_b}{2N_0}\right)}$$



4.3.2.4 Optimum Noncoherent Detection of On–Off Keying

- On–off keying (OOK) is a binary modulation format, and the transmitted signal points in complex baseband representation are given by

$$\begin{aligned} s_{b1} &= \sqrt{2E_b} \\ s_{b2} &= 0 \end{aligned}$$

- The demodulated baseband signal is given by

$$r_b = \begin{cases} e^{j\phi}\sqrt{2E_b} + z & \text{if} \quad m = 1 \\ z & \text{if} \quad m = 2 \end{cases}$$

where z is complex Gaussian noise with variance $\sigma_z^2 = N_0$.

- **ML Detection**

  The ML decision rule is

$$\hat{m} \;=\; \underset{\tilde{m}}{\operatorname{argmax}} \left\{ -\frac{|s_{b\tilde{m}}|^2}{N_0} + \ln\left[\mathrm{I}_0\left(\frac{2}{N_0} |r_b s^*_{b\tilde{m}}| \right) \right] \right\}$$

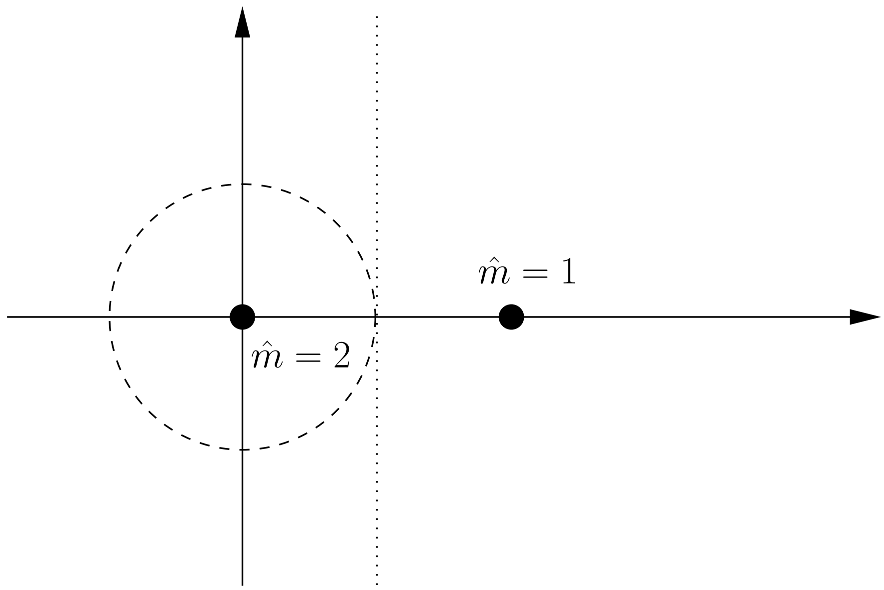
- We decide for $\hat{m} = 1$ if

$$\begin{aligned} -\frac{2E_b}{N_0} + \ln\left[\mathrm{I}_0\left(\frac{2}{N_0} |\sqrt{2E_b} r_b| \right) \right] \;&>\; \ln(\mathrm{I}_0(0)) \\ \underbrace{\ln\left[\mathrm{I}_0\left(\frac{2}{N_0} \sqrt{2E_b} |r_b| \right) \right]}_{\approx \frac{2}{N_0}\sqrt{2E_b}|r_b|} \;&>\; \frac{2E_b}{N_0} \\ |r_b| \;&>\; \sqrt{\frac{E_b}{2}} \end{aligned}$$

  Therefore, the (approximate) ML decision rule can be simplified to

$$\begin{aligned} \hat{m} = 1 \qquad \text{if} \qquad |r_b| > \sqrt{\tfrac{E_b}{2}} \\ \hat{m} = 2 \qquad \text{if} \qquad |r_b| < \sqrt{\tfrac{E_b}{2}} \end{aligned}$$

- **Signal Space Decision Regions**



4.3.2.5 Multiple–Symbol Differential Detection (MSDD) of DPSK

- Noncoherent detection of *PSK* is not possible, since for PSK the information is represented in the *absolute* phase of the transmitted signal.
- In order to enable noncoherent detection *differential encoding* is necessary. The resulting scheme is referred to as *differential* PSK (DPSK).

- The (normalized) demodulated DPSK signal in symbol interval k is given by

$$r_b[k] = e^{j\phi} b[k] + z[k]$$

with $b[k] = a[k]b[k-1]$ and noise variance $\sigma_z^2 = N_0/E_S$, where E_S denotes the received energy per DPSK symbol.

- Since the differential encoding operation introduces memory, it is advantageous to detect the transmitted symbols in a block–wise manner.

- In the following, we consider vectors (blocks) of length N

$$\boldsymbol{r}_b = e^{j\phi}\boldsymbol{b} + \boldsymbol{z},$$

where

$$\begin{aligned}
\boldsymbol{r}_b &= [r_b[k] \;\; r_b[k-1] \;\; \ldots \;\; r_b[k-(N-1)]^T \\
\boldsymbol{b} &= [b[k] \;\; b[k-1] \;\; \ldots \;\; b[k-(N-1)]^T \\
\boldsymbol{z} &= [z[k] \;\; z[k-1] \;\; \ldots \;\; z[k-(N-1)]^T
\end{aligned}$$

- Since $\boldsymbol{z}$ is a Gaussian noise vector, whose components are mutually independent and have variances of $\sigma_z^2 = N_0$, respectively, the general optimum noncoherent ML decision rule can also be applied in this case.

- In the following, we interpret $\boldsymbol{b}$ as the transmitted baseband signal points, i.e., $\boldsymbol{s}_b = \boldsymbol{b}$, and introduce the vector of $N-1$ differential *information bearing* symbols

$$\boldsymbol{a} = [a[k] \;\; a[k-1] \;\; \ldots \;\; a[k-(N-2)]]^T.$$

Note that the vector $\boldsymbol{b}$ of absolute phase symbols corresponds to just the $N-1$ differential symbol contained in $\boldsymbol{a}$.

- The estimate for $\boldsymbol{a}$ is denoted by $\hat{\boldsymbol{a}}$, and we also introduce the trial vectors $\tilde{\boldsymbol{a}}$ and $\tilde{\boldsymbol{b}}$. With these definitions the ML decision rule becomes

$$\begin{aligned}\hat{\boldsymbol{a}} &= \underset{\tilde{\boldsymbol{a}}}{\operatorname{argmax}} \left\{ -\frac{||\tilde{\boldsymbol{b}}||^2}{N_0} + \ln\left[\mathrm{I}_0\left(\frac{2}{N_0} |\boldsymbol{r}_b \bullet \tilde{\boldsymbol{b}}| \right) \right] \right\} \\ &= \underset{\tilde{\boldsymbol{a}}}{\operatorname{argmax}} \left\{ -\frac{N}{N_0} + \ln\left[\mathrm{I}_0\left(\frac{2}{N_0} |\boldsymbol{r}_b \bullet \tilde{\boldsymbol{b}}| \right) \right] \right\} \\ &= \underset{\tilde{\boldsymbol{a}}}{\operatorname{argmax}} \left\{ |\boldsymbol{r}_b \bullet \tilde{\boldsymbol{b}}| \right\} \\ &= \underset{\tilde{\boldsymbol{a}}}{\operatorname{argmax}} \left\{ \left| \sum_{\nu=0}^{N-1} r_b[k-\nu] \tilde{b}^*[k-\nu] \right| \right\}\end{aligned}$$

In order to further simplify this result we make use of the relation

$$\tilde{b}[k-\nu] = \prod_{\mu=\nu}^{N-2} \tilde{a}[k-\mu] \tilde{b}[k-(N-1)]$$

and obtain the final ML decision rule

$$\begin{aligned}\hat{\boldsymbol{a}} &= \underset{\tilde{\boldsymbol{a}}}{\operatorname{argmax}} \left\{ \left| \sum_{\nu=0}^{N-1} r_b[k-\nu] \prod_{\mu=\nu}^{N-2} \tilde{a}^*[k-\mu] \tilde{b}^*[k-(N-1)] \right| \right\} \\ &= \underset{\tilde{\boldsymbol{a}}}{\operatorname{argmax}} \left\{ \left| \sum_{\nu=0}^{N-1} r_b[k-\nu] \prod_{\mu=\nu}^{N-2} \tilde{a}^*[k-\mu] \right| \right\}\end{aligned}$$

Note that this final result is independent of $\tilde{b}[k-(N-1)]$.

- We make a *joint* decision on $N-1$ differential symbols $a[k-\nu]$, $0 \leq \nu \leq N-2$, based on the observation of N received signal points. N is also referred to as the *observation window size*.

- The above ML decision rule is known as *multiple symbol differential detection (MSDD)* and was reported first by Divsalar & Simon in 1990.

- In general, the performance of MSDD increases with increasing N. For $N \to \infty$ the performance of ideal coherent detection is approached.

- $N = 2$
  For the special case $N = 2$, we obtain

$$
\begin{aligned}
\hat{a}[k] &= \underset{\tilde{a}[k]}{\operatorname{argmax}} \left\{ \left| \sum_{\nu=0}^{1} r_b[k-\nu] \prod_{\mu=\nu}^{0} \tilde{a}^*[k-\mu] \right| \right\} \\
&= \underset{\tilde{a}[k]}{\operatorname{argmax}} \left\{ |r_b[k]\tilde{a}^*[k] + r_b[k-1]| \right\} \\
&= \underset{\tilde{a}[k]}{\operatorname{argmax}} \left\{ |r_b[k]\tilde{a}^*[k] + r_b[k-1]|^2 \right\} \\
&= \underset{\tilde{a}[k]}{\operatorname{argmax}} \left\{ |r_b[k]|^2 + |r_b[k-1]|^2 + 2\mathrm{Re}\{r_b[k]\tilde{a}^*[k]r_b^*[k-1]\} \right\} \\
&= \underset{\tilde{a}[k]}{\operatorname{argmin}} \left\{ -2\mathrm{Re}\{r_b[k]\tilde{a}^*[k]r_b^*[k-1]\} \right\} \\
&= \underset{\tilde{a}[k]}{\operatorname{argmin}} \left\{ |r_b[k]r_b^*[k-1]|^2 + |\tilde{a}[k]|^2 - 2\mathrm{Re}\{r_b[k]\tilde{a}^*[k]r_b^*[k-1]\} \right\} \\
&= \underset{\tilde{a}[k]}{\operatorname{argmin}} \left\{ |d[k] - \tilde{a}[k]|^2 \right\}
\end{aligned}
$$

  with

$$
d[k] = r_b[k]r_b^*[k-1].
$$

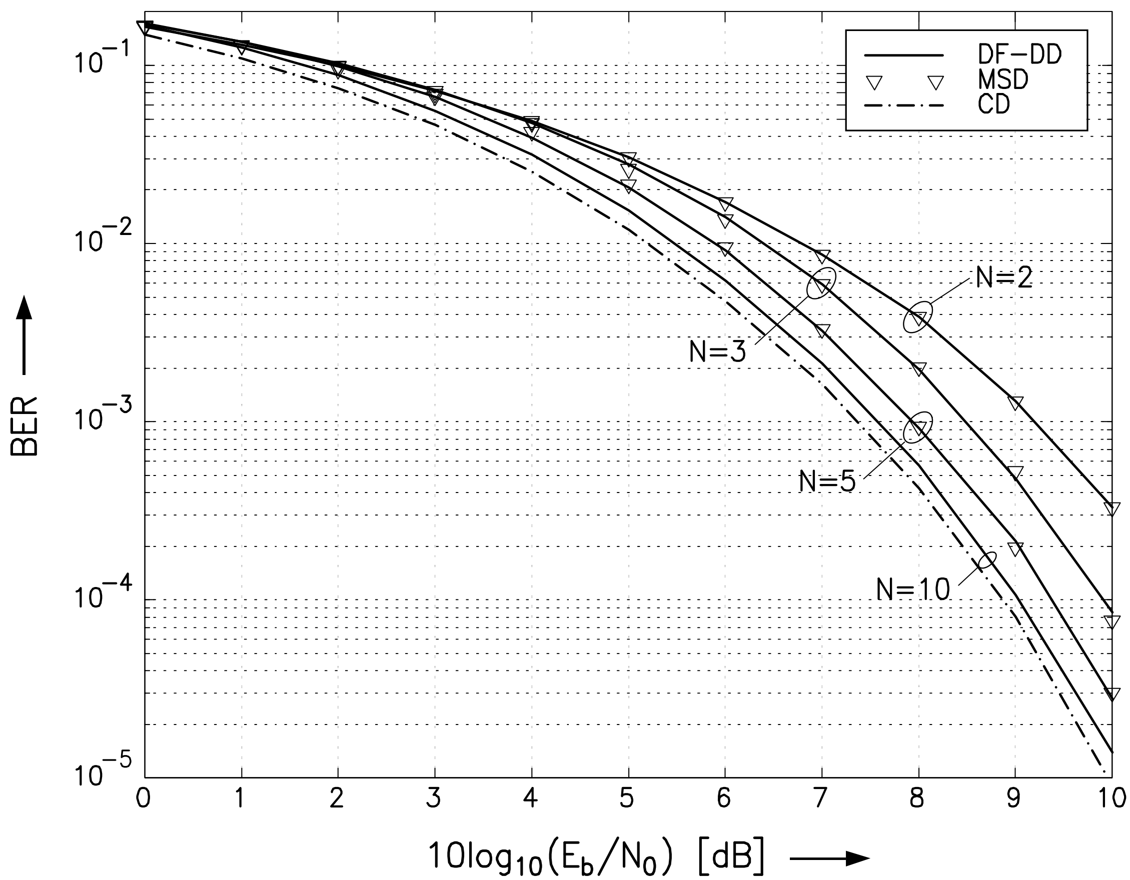
  Obviously, for $N = 2$ the ML (MSDD) decision rule is identical to the heuristically derived conventional differential detection decision rule. For $N > 2$ the gap to coherent detection becomes smaller.

- **Disadvantage of MSDD**
  The trial vector $\tilde{\boldsymbol{a}}$ has $N-1$ elements and each element has M possible values. Therefore, there are M^{N-1} different trial vectors $\tilde{\boldsymbol{a}}$. This means for MSDD we have to make $M^{N-1}/(N-1)$ tests per (scalar) symbol decision. This means the complexity of MSDD grows exponentially with N. For example, for $M=4$ we have to make 4 and 8192 tests for $N=2$ and $N=9$, respectively.

- **Alternatives**
  In order to avoid the exponential complexity of MSDD there are two different approaches:

  1. The first approach is to replace the above *brute–force* search with a smarter approach. In this smarter approach the trial vectors $\tilde{\boldsymbol{a}}$ are sorted first. The resulting fast decoding algorithm still performs optimum MSDD but has only a complexity of $N\log(N)$ (Mackenthun, 1994). For fading channels a similar approach based on *sphere decoding* exists (Pauli et al.).
  2. The second approach is suboptimum but achieves a similar performance as MSDD. Complexity is reduced by using *decision feedback* of previously decided symbols. The resulting scheme is referred to as decision–feedback differential detection (DF–DD). The complexity of this approach is only linear in N (Edbauer, 1992).

- Comparison of BEP (BER) of MSDD (MSD) and DF–DD for 4DPSK



4.4 Optimum Coherent Detection of Continuous Phase Modulation (CPM)

■ CPM Modulation

– *CPM Transmit Signal*
Recall that the CPM transmit signal in complex baseband representation is given by

$$s_b(t,\, \boldsymbol{I}) = \sqrt{\frac{E}{T}}\, \exp(j[\phi(t,\, \boldsymbol{I}) + \phi_0]),$$

where $\boldsymbol{I}$ is the sequence $\{I[k]\}$ of information bearing symbols $I[k] \in \{\pm 1,\, \pm 3,\, \ldots,\, \pm(M-1)\}$, $\phi(t, \boldsymbol{I})$ is the information carrying phase, and ϕ_0 is the initial carrier phase. Without loss of generality, we assume $\phi_0 = 0$ in the following.

– *Information Carrying Phase*
In the interval $kT \leq t \leq (k+1)T$ the phase $\phi(t,\, \boldsymbol{I})$ can be written as

$$\begin{aligned}\phi(t,\, \boldsymbol{I}) \;&=\; \Theta[k] + 2\pi h \sum_{\nu=1}^{L-1} I[k-\nu] q(t - [k-\nu]T) \\ &\qquad +2\pi h I[k] q(t - kT),\end{aligned}$$

where $\Theta[k]$ represents the accumulated phase up to time kT, h is the so–called modulation index, and $q(t)$ is the phase shaping pulse with

$$q(t) = \begin{cases} 0, & t < 0 \\ \text{monotonic}, & 0 \leq t \leq LT \\ 1/2, & t > LT \end{cases}$$

– *Trellis Diagram*
If $h = q/p$ is a rational number with relative prime integers q and p, CPM can be described by a trellis diagram whose number of states S is given by

$$S = \begin{cases} pM^{L-1}, & \text{even}\, q \\ 2pM^{L-1}, & \text{odd}\, q \end{cases}$$

■ Received Signal

The received complex baseband signal is given by

$$r_b(t) = s_b(t, \boldsymbol{I}) + z(t)$$

with complex AWGN $z(t)$, which has a power spectral density of $\Phi_{ZZ}(f) = N_0$.

■ ML Detection

– Since the CPM signal has memory, ideally we have to observe the entire received signal $r_b(t)$, $-\infty \leq t \leq \infty$, in order to make a decision on *any* $I[k]$ in the sequence of transmitted signals.

– The conditional pdf $p(r_b(t)|s_b(t,\, \tilde{\boldsymbol{I}}))$ is given by

$$p(r_b(t)|s_b(t,\, \tilde{\boldsymbol{I}})) \propto \exp\left(-\frac{1}{N_0}\int\limits_{-\infty}^{\infty} |r_b(t) - s_b(t,\, \tilde{\boldsymbol{I}})|^2 \, \mathrm{d}t\right),$$

where $\tilde{\boldsymbol{I}} \in \{\pm 1,\, \pm 3,\, \dots,\, \pm(M-1)\}$ is a trial sequence. For

ML detection we have the decision rule

$$
\begin{aligned}
\hat{\boldsymbol{I}} &= \underset{\tilde{\boldsymbol{I}}}{\operatorname{argmax}} \left\{ p(r_b(t)|s_b(t,\, \tilde{\boldsymbol{I}})) \right\} \\
&= \underset{\tilde{\boldsymbol{I}}}{\operatorname{argmax}} \left\{ \ln[p(r_b(t)|s_b(t,\, \tilde{\boldsymbol{I}}))] \right\} \\
&= \underset{\tilde{\boldsymbol{I}}}{\operatorname{argmax}} \left\{ -\int\limits_{-\infty}^{\infty} |r_b(t) - s_b(t,\, \tilde{\boldsymbol{I}})|^2 \, \mathrm{d}t \right\} \\
&= \underset{\tilde{\boldsymbol{I}}}{\operatorname{argmax}} \left\{ -\int\limits_{-\infty}^{\infty} |r_b(t)|^2 \, \mathrm{d}t - \int\limits_{-\infty}^{\infty} |s_b(t,\, \tilde{\boldsymbol{I}})|^2 \, \mathrm{d}t \right. \\
&\qquad\qquad \left. + \int\limits_{-\infty}^{\infty} 2\mathsf{Re}\{r_b(t) s_b^*(t,\, \tilde{\boldsymbol{I}})\} \, \mathrm{d}t \right\} \\
&= \underset{\tilde{\boldsymbol{I}}}{\operatorname{argmax}} \left\{ \int\limits_{-\infty}^{\infty} \mathsf{Re}\{r_b(t) s_b^*(t,\, \tilde{\boldsymbol{I}})\} \, \mathrm{d}t, \right\}
\end{aligned}
$$

where $\hat{\boldsymbol{I}}$ refers to the ML decision. If $\boldsymbol{I}$ is a sequence of length K, there are M^K different sequences $\tilde{\boldsymbol{I}}$. Since we have to calculate the function $\int_{-\infty}^{\infty} \mathsf{Re}\{r_b(t) s_b^*(t,\, \tilde{\boldsymbol{I}})\} \, \mathrm{d}t$ for each of these sequences, the complexity of ML detection with *brute–force* search grows exponentially with the sequence length K, which is prohibitive for a practical implementation.

Viterbi Algorithm

– The exponential complexity of brute–force search can be avoided using the so–called *Viterbi algorithm (VA)*.

– Introducing the definition

$$\Lambda[k] = \int_{-\infty}^{(k+1)T} \mathsf{Re}\{r_b(t)s_b^*(t,\, \tilde{\boldsymbol{I}})\}\, \mathrm{d}t,$$

we observe that the function to be maximized for ML detection is $\Lambda[\infty]$. On the other hand, $\Lambda[k]$ may be calculated recursively as

$$\begin{aligned}
\Lambda[k] &= \int_{-\infty}^{kT} \mathsf{Re}\{r_b(t)s_b^*(t,\, \tilde{\boldsymbol{I}})\}\, \mathrm{d}t + \int_{kT}^{(k+1)T} \mathsf{Re}\{r_b(t)s_b^*(t,\, \tilde{\boldsymbol{I}})\}\, \mathrm{d}t \\
\Lambda[k] &= \Lambda[k-1] + \int_{kT}^{(k+1)T} \mathsf{Re}\{r_b(t)s_b^*(t,\, \tilde{\boldsymbol{I}})\}\, \mathrm{d}t \\
\Lambda[k] &= \Lambda[k-1] + \lambda[k],
\end{aligned}$$

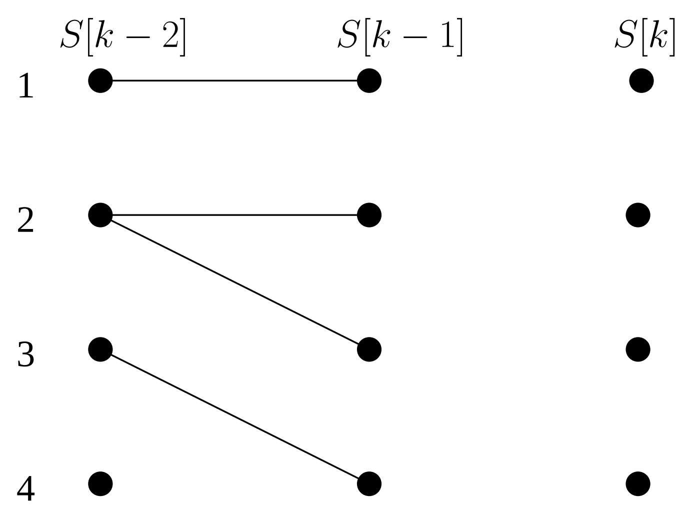
where we use the definition

$$\begin{aligned}
\lambda[k] &= \int_{kT}^{(k+1)T} \mathsf{Re}\{r_b(t)s_b^*(t,\, \tilde{\boldsymbol{I}})\}\, \mathrm{d}t. \\
&= \int_{kT}^{(k+1)T} \mathsf{Re}\Bigg\{ r_b(t) \exp\Bigg(-j\Bigg[\tilde{\Theta}[k] + 2\pi h \sum_{\nu=1}^{L-1} \tilde{I}[k-\nu] \\
&\qquad \cdot q(t-[k-\nu]T) + 2\pi h \tilde{I}[k] q(t-kT) \Bigg]\Bigg)\Bigg\}\, \mathrm{d}t.
\end{aligned}$$

$\Lambda[k]$ and $\lambda[k]$ are referred to as the *accumulated metric* and the *branch metric* of the VA, respectively.

– Since CPM can be described by a trellis with a finite number of states $S = pM^{L-1}$ ($S = 2pM^{L-1}$), at time kT we have to consider only S different $\Lambda[\boldsymbol{S}[k-1], k-1]$. Each $\Lambda[\boldsymbol{S}[k-1], k-1]$ corresponds to exactly one state $\boldsymbol{S}[k-1]$ which is defined by

$$\boldsymbol{S}[k-1] = [\tilde{\Theta}[k-1],\ \tilde{I}[k-(L-1)],\ \ldots,\ \tilde{I}[k-1]].$$



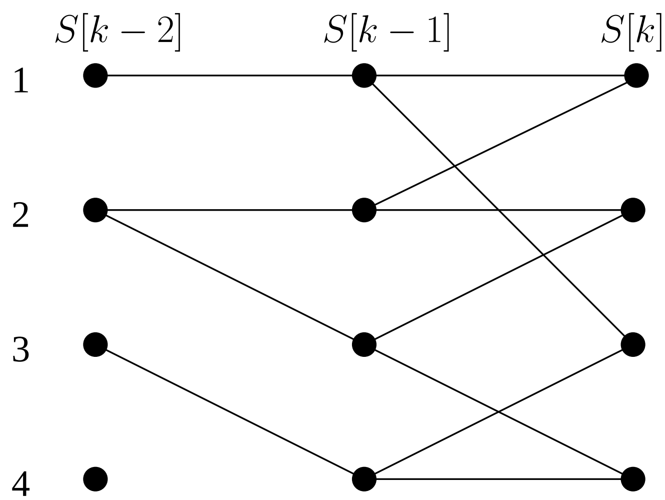
For the interval $kT \leq t \leq (k+1)T$ we have to calculate M branch metrics $\lambda[\boldsymbol{S}[k-1], \tilde{I}[k], k]$ for each state $\boldsymbol{S}[k-1]$ corresponding to the M different $\tilde{I}[k]$. Then at time $(k+1)T$ the new states are defined by

$$\boldsymbol{S}[k] = [\tilde{\Theta}[k],\ \tilde{I}[k-(L-2)],\ \ldots,\ \tilde{I}[k]],$$

with $\tilde{\Theta}[k] = \tilde{\Theta}[k-1] + \pi h \tilde{I}[k-(L-1)]$. M branches defined by $\boldsymbol{S}[k-1]$ and $\tilde{I}[k]$ emanate in each state $\boldsymbol{S}[k]$. We calculate the M accumulated metrics

$$\Lambda[\boldsymbol{S}[k-1],\, \tilde{I}[k],\, k] = \Lambda[\boldsymbol{S}[k-1],\, k-1] + \lambda[\boldsymbol{S}[k-1],\, \tilde{I}[k],\, k]$$

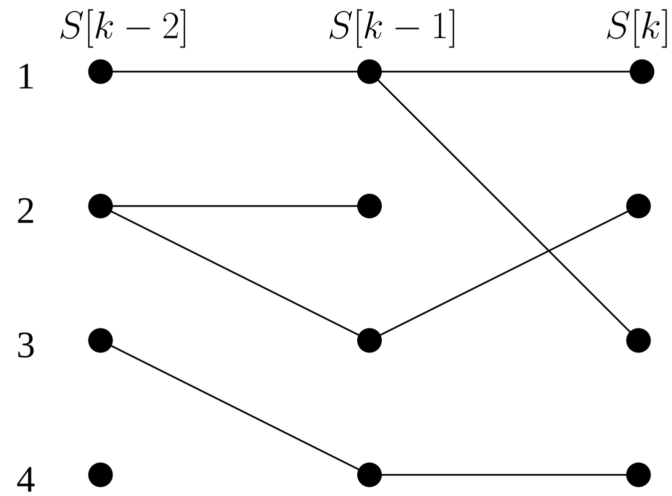
for each state $\boldsymbol{S}[k]$.



From all the paths (partial sequences) that emanate in a state, we have to retain only that one with the largest accumulated metric denoted by

$$\Lambda[\boldsymbol{S}[k],\, k] = \mathsf{max}_{\tilde{I}[k]} \left\{ \Lambda[\boldsymbol{S}[k-1],\, \tilde{I}[k],\, k] \right\},$$

since any other path with a smaller accumulated metric at time $(k+1)T$ cannot have a larger metric $\Lambda[\infty]$. Therefore, at time $(k+1)T$ there will be again only S so–called surviving paths with corresponding accumulated metrics.



– The above steps are carried out for all symbol intervals and at the end of the transmission, a decision is made on the transmitted sequence. Alternatively, at time kT we may use the symbol $\tilde{I}[k-k_0]$ corresponding to the surviving path with the largest accumulated metric as estimate for $I[k-k_0]$. It has been shown that as long as the decision delay k_0 is large enough (e.g. $k_0 \geq 5\log_2(S)$), this method yields the same performance as true ML detection.

– The complexity of the VA is *exponential* in the number of states, but only *linear* in the length of the transmitted sequence.

■ **Remarks:**

– At the expense of a certain loss in performance the complexity of the VA can be further reduced by reducing the number of states.

– Alternative implementations receivers for CPM are based on Laurent's decomposition of the CPM signal into a sum of PAM signals

5 Signal Design for Bandlimited Channels

- So far, we have not imposed any bandwidth constraints on the transmitted passband signal, or equivalently, on the transmitted baseband signal

$$s_b(t) = \sum_{k=-\infty}^{\infty} I[k] g_T(t - kT),$$

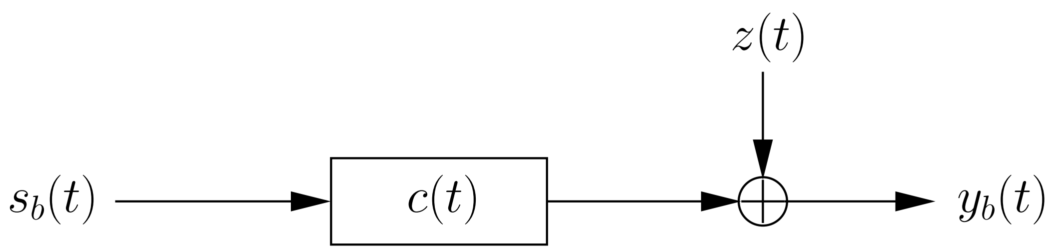
where we assume a linear modulation (PAM, PSK, or QAM), and $I[k]$ and $g_T(t)$ denote the transmit symbols and the transmit pulse, respectively.

- In practice, however, due to the properties of the transmission medium (e.g. cable or multipath propagation in wireless), the underlying transmission channel is bandlimited. If the bandlimited character of the channel is not taken into account in the signal design at the transmitter, the received signal will be (linearly) distorted.

- In this chapter, we design transmit pulse shapes $g_T(t)$ that guarantee that $s_b(t)$ occupies only a certain finite bandwidth W. At the same time, these pulse shapes allow ML symbol–by–symbol detection at the receiver.

5.1 Characterization of Bandlimited Channels

- Many communication channels can be modeled as *linear filters* with (equivalent baseband) impulse response $c(t)$ and frequency response $C(f) = \mathcal{F}\{c(t)\}$.



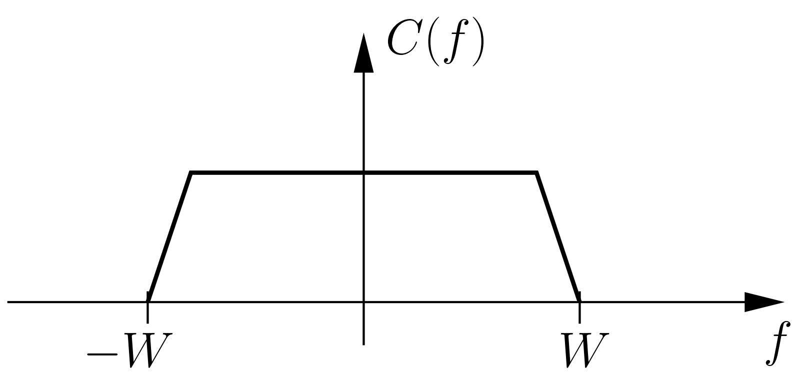
The received signal is given by

$$y_b(t) = \int_{-\infty}^{\infty} s_b(\tau)c(t-\tau)\,\mathrm{d}\tau + z(t),$$

where $z(t)$ denotes complex Gaussian noise with power spectral density N_0.

- We assume that the channel is ideally bandlimited, i.e.,

$$C(f) = 0, \qquad |f| > W.$$



Within the interval $|f| \leq W$, the channel is characterized by

$$C(f) = |C(f)|\, e^{j\Theta(f)}$$

with amplitude response $|C(f)|$ and phase response $\Theta(f)$.

- If $S_b(f) = \mathcal{F}\{s_b(t)\}$ is non–zero only in the interval $|f| \leq W$, then $s_b(t)$ is *not* distorted if and only if

  1. $|C(f)| = A$, i.e., the amplitude response is constant for $|f| \leq W$.
  2. $\Theta(f) = -2\pi f \tau_0$, i.e., the phase response is linear in f, where τ_0 is the delay.

In this case, the received signal is given by

$$y_b(t) = A s_b(t - \tau_0) + z(t).$$

As far as $s_b(t)$ is concerned, the impulse response of the channel can be modeled as

$$c(t) = A\,\delta(t - \tau_0).$$

If the above conditions are not fulfilled, the channel linearly *distorts* the transmitted signal and an equalizer is necessary at the receiver.

5.2 Signal Design for Bandlimited Channels

- We assume an ideal, bandlimited channel, i.e.,

$$C(f) = \begin{cases} 1, & |f| \leq W \\ 0, & |f| > W \end{cases}.$$

- The transmit signal is

$$s_b(t) = \sum_{k=-\infty}^{\infty} I[k] g_T(t - kT).$$

- In order to avoid distortion, we assume

$$G_T(f) = \mathcal{F}\{g_T(t)\} = 0, \qquad |f| > W.$$

  This implies that the received signal is given by

$$\begin{aligned} y_b(t) &= \int_{-\infty}^{\infty} s_b(\tau) c(t - \tau) \, \mathrm{d}\tau + z(t) \\ &= s_b(t) + z(t) \\ &= \sum_{k=-\infty}^{\infty} I[k] g_T(t - kT) + z(t). \end{aligned}$$

- In order to limit the noise power in the *demodulated signal*, $y_b(t)$ is usually filtered with a filter $g_R(t)$. Thereby, $g_R(t)$ can be e.g. a lowpass filter or the optimum matched filter. The filtered received signal is

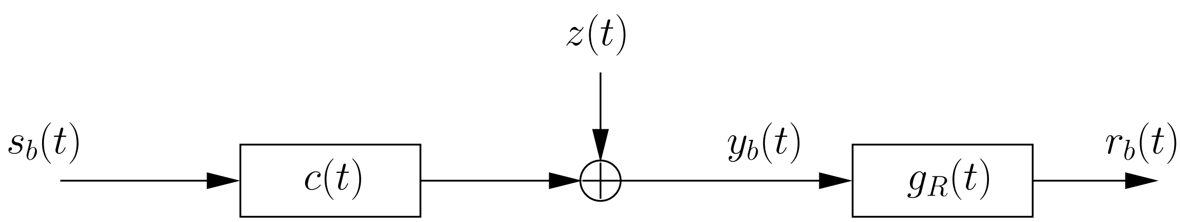
$$\begin{aligned} r_b(t) &= g_R(t) * y_b(t) \\ &= \sum_{k=-\infty}^{\infty} I[k] x(t - kT) + \nu(t) \end{aligned}$$

where "$*$" denotes convolution, and we use the definitions

$$\begin{aligned} x(t) &= g_T(t) * g_R(t) \\ &= \int\limits_{-\infty}^{\infty} g_T(\tau) g_R(t-\tau)\, \mathrm{d}\tau \end{aligned}$$

and

$$\nu(t) = g_R(t) * z(t)$$



- Now, we sample $r_b(t)$ at times $t = kT + t_0$, $k = \ldots, -1, 0, 1, \ldots$, where t_0 is an arbitrary delay. For simplicity, we assume $t_0 = 0$ in the following and obtain

$$\begin{aligned} r_b[k] &= r_b(kT) \\ &= \sum_{\kappa=-\infty}^{\infty} I[\kappa] \underbrace{x(kT - \kappa T)}_{=x[k-\kappa]} + \underbrace{\nu(kT)}_{=z[k]} \\ &= \sum_{\kappa=-\infty}^{\infty} I[\kappa] x[k-\kappa] + z[k] \end{aligned}$$

- We can rewrite $r_b[k]$ as

$$r_b[k] = x[0] \left(I[k] + \frac{1}{x[0]} \sum_{\substack{\kappa=-\infty \\ \kappa \neq k}}^{\infty} I[\kappa]x[k-\kappa] \right) + z[k],$$

where the term

$$\sum_{\substack{\kappa=-\infty \\ \kappa \neq k}}^{\infty} I[\kappa]x[k-\kappa]$$

represents so–called *intersymbol interference (ISI)*. Since $x[0]$ only depends on the amplitude of $g_R(t)$ and $g_T(t)$, respectively, for simplicity we assume $x[0] = 1$ in the following.

- Ideally, we would like to have

$$r_b[k] = I[k] + z[k],$$

which implies

$$\sum_{\substack{\kappa=-\infty \\ \kappa \neq k}}^{\infty} I[\kappa]x[k-\kappa] = 0,$$

i.e., the transmission is ISI free.

- **Problem Statement**
How do we design $g_T(t)$ and $g_R(t)$ to guarantee ISI–free transmission?

■ Solution: The Nyquist Criterion

– Since

$$x(t) = g_R(t) * g_T(t)$$

is valid, the Fourier transform of $x(t)$ can be expressed as

$$X(f) = G_T(f)\, G_R(f),$$

with $G_T(f) = \mathcal{F}\{g_T(t)\}$ and $G_R(f) = \mathcal{F}\{g_R(t)\}$.

– For ISI–free transmission, $x(t)$ has to have the property

$$x(kT) = x[k] = \begin{cases} 1, & k = 0 \\ 0, & k \neq 0 \end{cases} \tag{1}$$

– *The Nyquist Criterion*
According to the Nyquist Criterion, condition (1) is fulfilled if and only if

$$\boxed{\sum_{m=-\infty}^{\infty} X\left(f + \frac{m}{T}\right) = T}$$

is true. This means summing up all spectra that can be obtained by shifting $X(f)$ by multiples of $1/T$ results in a *constant.*

Proof:

$x(t)$ may be expressed as

$$x(t) = \int_{-\infty}^{\infty} X(f) e^{j2\pi f t} \, \mathrm{d}f.$$

Therefore, $x[k] = x(kT)$ can be written as

$$\begin{aligned}
x[k] &= \int_{-\infty}^{\infty} X(f) e^{j2\pi f kT} \, \mathrm{d}f \\
&= \sum_{m=-\infty}^{\infty} \int_{(2m-1)/(2T)}^{(2m+1)/(2T)} X(f) e^{j2\pi f kT} \, \mathrm{d}f \\
&= \sum_{m=-\infty}^{\infty} \int_{-1/(2T)}^{1/(2T)} X(f' + m/T) e^{j2\pi (f'+m/T) kT} \, \mathrm{d}f' \\
&= \int_{-1/(2T)}^{1/(2T)} \underbrace{\sum_{m=-\infty}^{\infty} X(f' + m/T)}_{B(f')} e^{j2\pi f' kT} \, \mathrm{d}f' \\
&= \int_{-1/(2T)}^{1/(2T)} B(f) e^{j2\pi f kT} \, \mathrm{d}f \qquad (2)
\end{aligned}$$

Since

$$B(f) = \sum_{m=-\infty}^{\infty} X(f + m/T)$$

is periodic with period $1/T$, it can be expressed as a *Fourier series*

$$B(f) = \sum_{k=-\infty}^{\infty} b[k] e^{j2\pi f kT} \tag{3}$$

where the Fourier series coefficients $b[k]$ are given by

$$b[k] = T \int\limits_{-1/(2T)}^{1/(2T)} B(f) e^{-j2\pi f kT} \, \mathrm{d}f. \tag{4}$$

From (2) and (4) we observe that

$$b[k] = T x(-kT).$$

Therefore, (1) holds if and only if

$$b[k] = \begin{cases} T, & k = 0 \\ 0, & k \neq 0 \end{cases}.$$

However, in this case (3) yields

$$B(f) = T,$$

which means

$$\sum_{m=-\infty}^{\infty} X(f + m/T) = T.$$

This completes the proof.

■

- The Nyquist criterion specifies the necessary and sufficient condition that has to be fulfilled by the spectrum $X(f)$ of $x(t)$ for ISI–free transmission. Pulse shapes $x(t)$ that satisfy the Nyquist criterion are referred to as *Nyquist pulses*. In the following, we discuss three different cases for the symbol duration T. In particular, we consider $T < 1/(2W)$, $T = 1/(2W)$, and $T > 1/(2W)$, respectively, where W is the bandwidth of the ideally bandlimited channel.

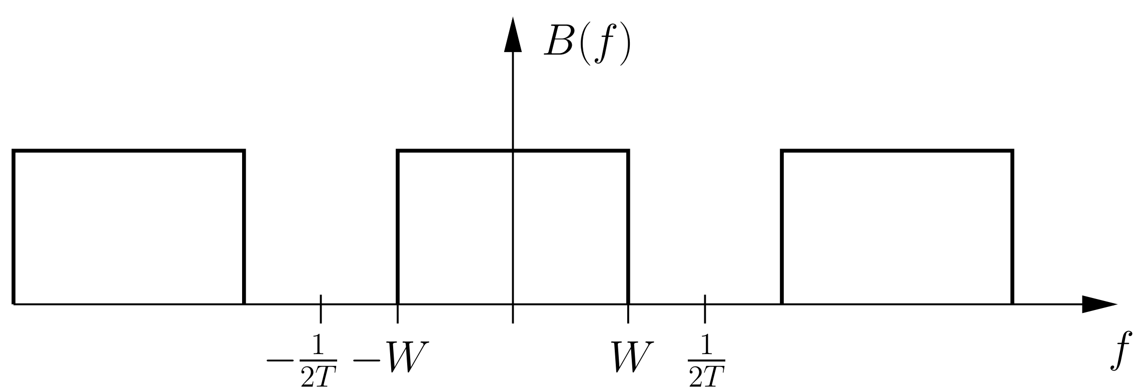
  1. $T < 1/(2W)$
     If the symbol duration T is smaller than $1/(2W)$, we get obviously

     $$\frac{1}{2T} > W$$

     and

     $$B(f) = \sum_{m=-\infty}^{\infty} X(f + m/T)$$

     consists of *non–overlapping* pulses.

     We observe that in this case the condition $B(f) = T$ cannot be fulfilled. Therefore, ISI–free transmission is *not possible*.

2. $T = 1/(2W)$

   If the symbol duration T is equal to $1/(2W)$, we get

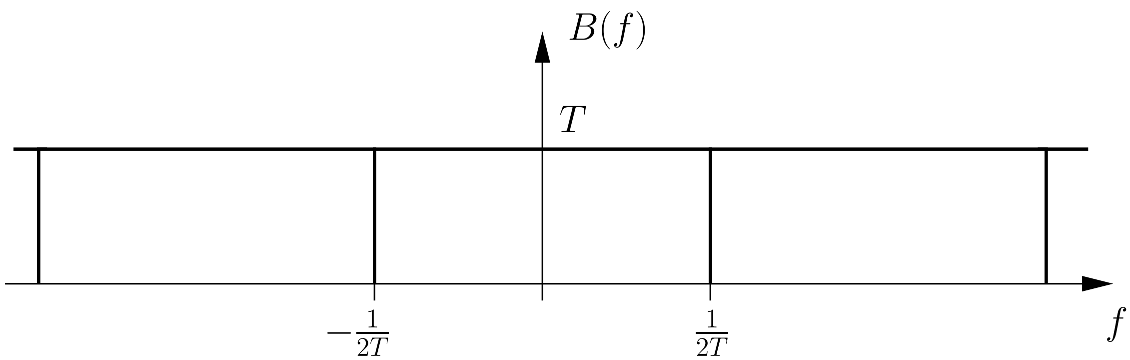
$$\frac{1}{2T} = W$$

   and $B(f) = T$ is achieved for

$$X(f) = \begin{cases} T, & |f| \leq W \\ 0, & |f| > W \end{cases} .$$

   This corresponds to

$$x(t) = \frac{\sin\left(\pi \frac{t}{T}\right)}{\pi \frac{t}{T}}$$



   $T = 1/(2W)$ is also called the *Nyquist rate* and is the fastest rate for which ISI–free transmission is possible.

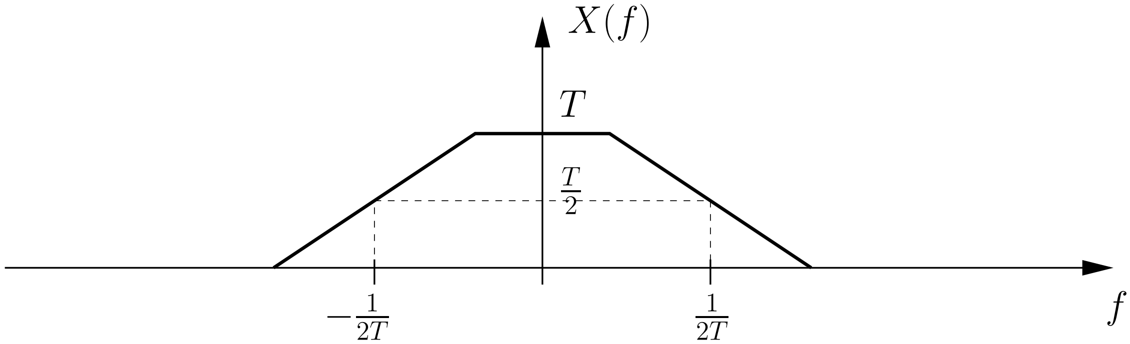
   In practice, however, this choice is usually not preferred since $x(t)$ decays very slowly ($\propto 1/t$) and therefore, time synchronization is problematic.

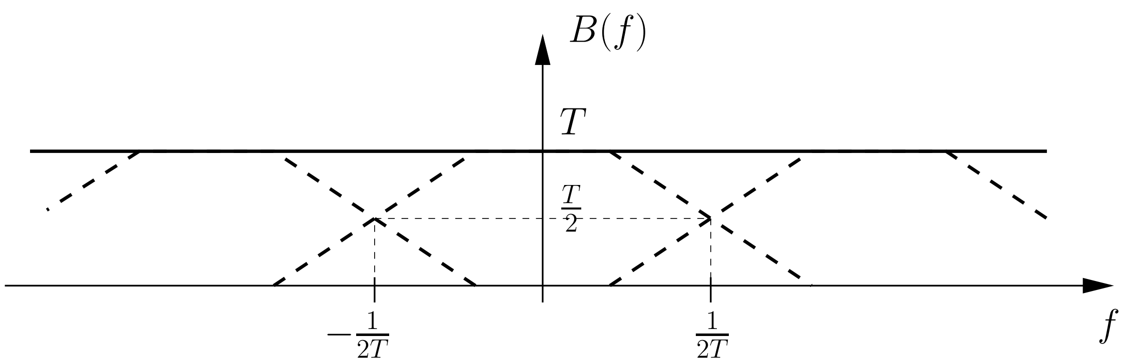
3. $T > 1/(2W)$
   In this case, we have

$$\frac{1}{2T} < W$$

   and $B(f)$ consists of overlapping, shifted replica of $X(f)$. ISI–free transmission is possible if $X(f)$ is chosen properly.

Example:





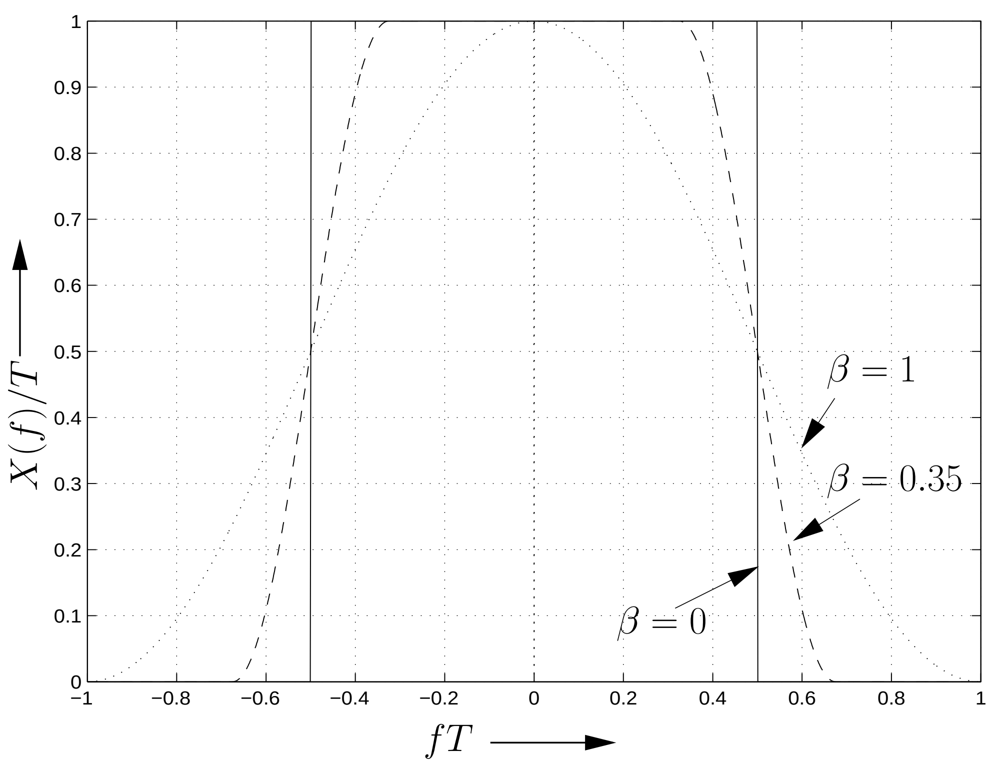
The bandwidth occupied beyond the Nyquist bandwidth $1/(2T)$ is referred to as the *excess bandwidth*. In practice, Nyquist pulses with *raised–cosine* spectra are popular

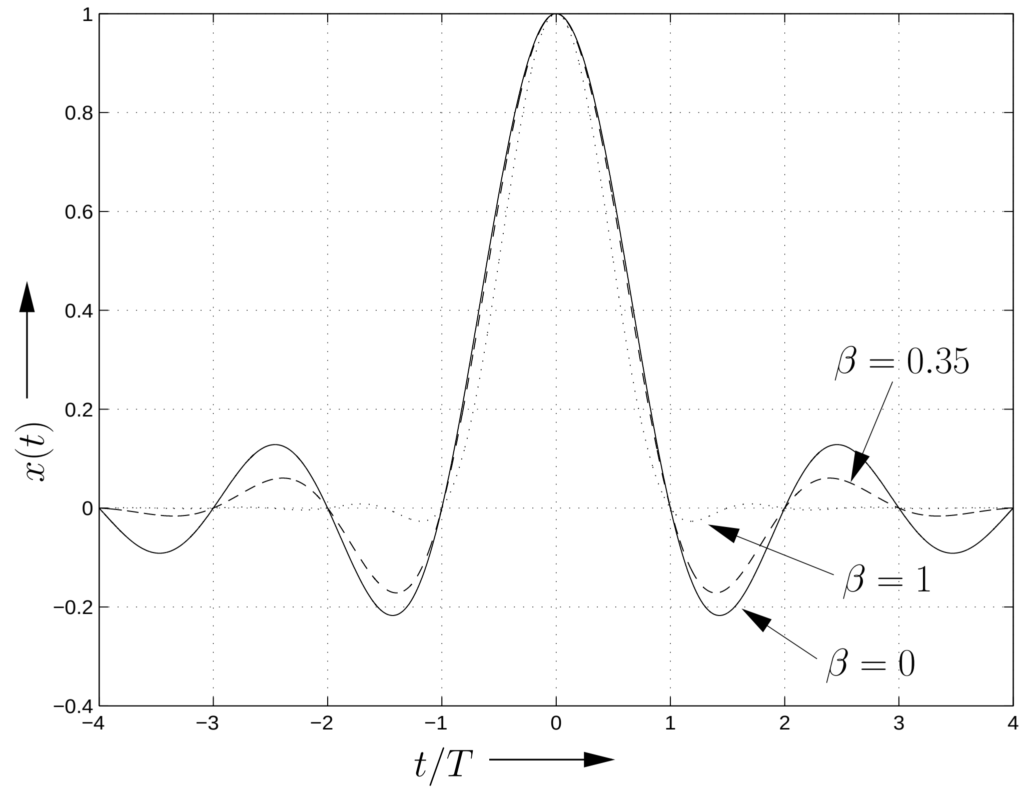
$$X(f) = \begin{cases} T, & 0 \le |f| \le \frac{1-\beta}{2T} \\ \frac{T}{2}\left(1 + \cos\left(\frac{\pi T}{\beta}\left[|f| - \frac{1-\beta}{2T}\right]\right)\right), & \frac{1-\beta}{2T} < |f| \le \frac{1+\beta}{2T} \\ 0, & |f| > \frac{1+\beta}{2T} \end{cases}$$

where β, $0 \le \beta \le 1$, is the roll–off factor. The inverse Fourier transform of $X(f)$ yields

$$x(t) = \frac{\sin(\pi t/T)}{\pi t/T} \frac{\cos(\pi \beta t/T)}{1 - 4\beta^2 t^2/T^2}.$$

For $\beta = 0$, $x(t)$ reduces to $x(t) = \sin(\pi t/T)/(\pi t/T)$. For $\beta > 0$, $x(t)$ decays as $1/t^3$. This fast decay is desirable for time synchronization.




1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−4
−3
−2
−1
0
1
2
3
4
$x(t)$
t/T
$\beta = 0.35$
$\beta = 1$
$\beta = 0$

- $\sqrt{\text{Nyquist}}$–**Filters**
  The spectrum of $x(t)$ is given by

$$G_T(f)\, G_R(f) = X(f)$$

ISI–free transmission is of course possible for any choice of $G_T(f)$ and $G_R(f)$ as long as $X(f)$ fulfills the Nyquist criterion. However, the SNR maximizing optimum choice is

$$\begin{aligned} G_T(f) &= G(f) \\ G_R(f) &= \alpha G^*(f), \end{aligned}$$

i.e., $g_R(t)$ is matched to $g_T(t)$. α is a constant. In that case, $X(f)$ is given by

$$X(f) = \alpha |G(f)|^2$$

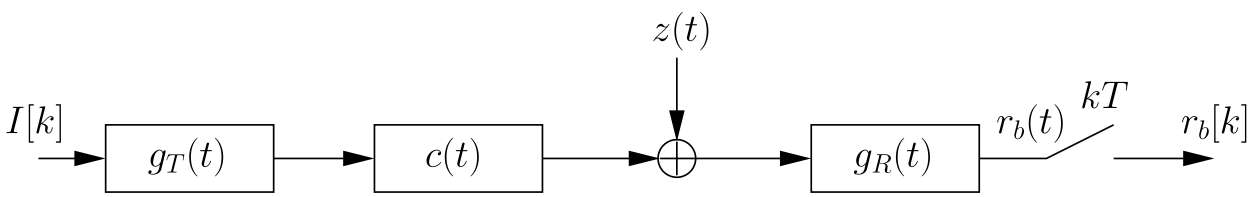
and consequently, $G(f)$ can be expressed as

$$G(f) = \sqrt{\frac{1}{\alpha}\, X(f)}.$$

$G(f)$ is referred to as $\sqrt{\text{Nyquist}}$–*Filter*. In practice, a delay τ_0 may be added to $G_T(f)$ and/or $G_R(f)$ to make them causal filters in order to ensure physical realizability of the filters.

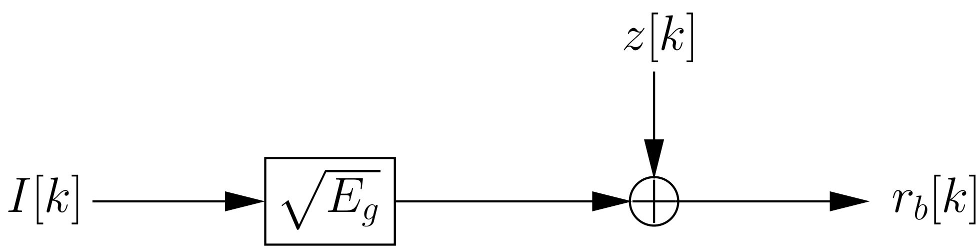
5.3 Discrete–Time Channel Model for ISI–free Transmission

■ **Continuous–Time Channel Model**



$g_T(t)$ and $g_R(t)$ are $\sqrt{\text{Nyquist}}$–Impulses that are matched to each other.

■ **Equivalent Discrete–Time Channel Model**



$z[k]$ is a discrete–time AWGN process with variance $\sigma_Z^2 = N_0$.

Proof:

– *Signal Component*

The overall channel is given by

$$x[k] = g_T(t) * c(t) * g_R(t)\Big|_{t=kT} = g_T(t) * g_R(t)\Big|_{t=kT},$$

where we have used the fact that the channel acts as an ideal lowpass filter. We assume

$$\begin{aligned} g_R(t) &= g(t) \\ g_T(t) &= \frac{1}{\sqrt{E_g}}\, g^*(-t), \end{aligned}$$

where $g(t)$ is a $\sqrt{\text{Nyquist}}$–Impulse, and E_g is given by

$$E_g = \int\limits_{-\infty}^{\infty} |g(t)|^2 \, \mathrm{d}t$$

Consequently, $x[k]$ is given by

$$\begin{aligned} x[k] &= \frac{1}{\sqrt{E_g}}\, g(t) * g^*(-t)\Big|_{t=kT} \\ &= \frac{1}{\sqrt{E_g}}\, g(t) * g^*(-t)\Big|_{t=0} \quad \delta[k] \\ &= \frac{1}{\sqrt{E_g}}\, E_g\, \delta[k] \\ &= \sqrt{E_g}\, \delta[k] \end{aligned}$$

– *Noise Component*
  $z(t)$ is a complex AWGN process with power spectral density

$\Phi_{ZZ}(f) = N_0$. $z[k]$ is given by

$$\begin{aligned} z[k] &= g_R(t) * z(t)\Big|_{t=kT} \\ &= \frac{1}{\sqrt{E_g}}\, g^*(-t) * z(t)\Big|_{t=kT} \\ &= \frac{1}{\sqrt{E_g}} \int\limits_{-\infty}^{\infty} z(\tau) g^*(kT+\tau)\, \mathrm{d}\tau \end{aligned}$$

$*$ *Mean*

$$\begin{aligned} \mathcal{E}\{z[k]\} &= \mathcal{E}\left\{\frac{1}{\sqrt{E_g}} \int\limits_{-\infty}^{\infty} z(\tau) g^*(kT+\tau)\, \mathrm{d}\tau\right\} \\ &= \frac{1}{\sqrt{E_g}} \int\limits_{-\infty}^{\infty} \mathcal{E}\{z(\tau)\} g^*(kT+\tau)\, \mathrm{d}\tau \\ &= 0 \end{aligned}$$

$*$ *ACF*

$$\begin{aligned}
\phi_{ZZ}[\lambda] &= \mathcal{E}\{z[k+\lambda]z^*[k]\} \\
&= \frac{1}{E_g}\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty}\underbrace{\mathcal{E}\{z(\tau_1)z^*(\tau_2)\}}_{N_0\delta(\tau_1-\tau_2)}g^*([k+\lambda]T+\tau_1) \\
&\qquad\qquad\qquad\qquad \cdot g(kT+\tau_2)\,\mathrm{d}\tau_1\mathrm{d}\tau_2 \\
&= \frac{N_0}{E_g}\int\limits_{-\infty}^{\infty} g^*([k+\lambda]T+\tau_1)g(kT+\tau_1)\,\mathrm{d}\tau_1 \\
&= \frac{N_0}{E_g}E_g\,\delta[\lambda] \\
&= N_0\,\delta[\lambda]
\end{aligned}$$

This shows that $z[k]$ is a discrete–time AWGN process with variance $\sigma_Z^2 = N_0$ and the proof is complete.

■

- The discrete–time, demodulated received signal can be modeled as

$$r_b[k] = \sqrt{E_g}\,I[k] + z[k].$$

- This means the demodulated signal is identical to the demodulated signal obtained for *time–limited* transmit pulses (see Chapter 3 and 4). Therefore, the optimum detection strategies developed in Chapter 4 can be also applied for finite bandwidth transmission as long as the Nyquist criterion is fulfilled.

6 Equalization of Channels with ISI

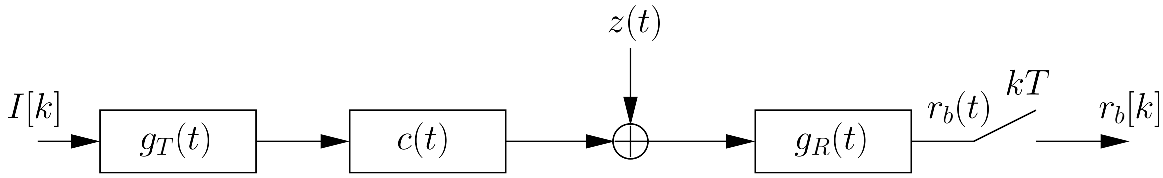
- Many practical channels are bandlimited and linearly distort the transmit signal.
- In this case, the resulting ISI channel has to be equalized for reliable detection.
- There are many different equalization techniques. In this chapter, we will discuss the three most important equalization schemes:
  1. Maximum–Likelihood Sequence Estimation (MLSE)
  2. Linear Equalization (LE)
  3. Decision–Feedback Equalization (DFE)

  Throughout this chapter we assume linear memoryless modulations such as PAM, PSK, and QAM.

6.1 Discrete–Time Channel Model

- **Continuous–Time Channel Model**

  The continuous–time channel is modeled as shown below.



  – *Channel* $c(t)$

    In general, the channel $c(t)$ is not ideal, i.e., $|C(f)|$ is not a constant over the range of frequencies where $G_T(f)$ is non–zero. Therefore, linear distortions are inevitable.

  – *Transmit Filter* $g_T(t)$

    The transmit filter $g_T(t)$ may or may not be a $\sqrt{\text{Nyquist}}$–Filter, e.g. in the North American D–AMPS mobile phone system a square–root raised cosine filter with roll–off factor $\beta = 0.35$ is used, whereas in the European EDGE mobile communication system a linearized Gaussian minimum–shift keying (GMSK) pulse is employed.

  – *Receive Filter* $g_R(t)$

    We assume that the receive filter $g_R(t)$ is a $\sqrt{\text{Nyquist}}$–Filter. Therefore, the filtered, sampled noise $z[k] = g_R(t) * z(t)|_{t=kT}$ is white Gaussian noise (WGN).

    Ideally, $g_R(t)$ consists of a filter matched to $g_T(t) * c(t)$ and a noise whitening filter. The drawback of this approach is that $g_R(t)$ depends on the channel, which may change with time in wireless applications. Therefore, in practice often a fixed but suboptimum $\sqrt{\text{Nyquist}}$–Filter is preferred.

– *Overall Channel* $h(t)$

The overall channel impulse response $h(t)$ is given by

$$h(t) = g_T(t) * c(t) * g_R(t).$$

■ **Discrete–Time Channel Model**

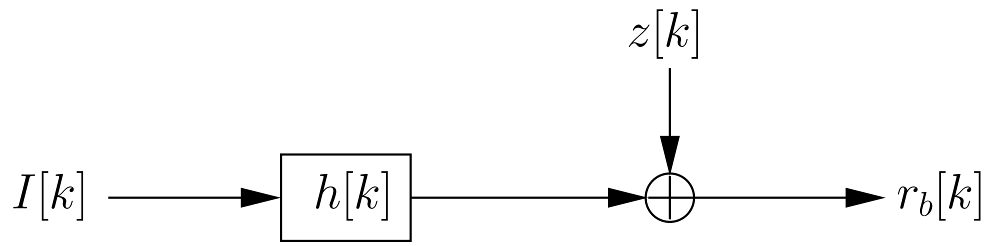
The sampled received signal is given by

$$\begin{aligned} r_b[k] &= r_b(kT) \\ &= \left. \left(\sum_{m=-\infty}^{\infty} I[m] h(t - mT) + g_R(t) * z(t) \right) \right|_{t=kT} \\ &= \sum_{m=-\infty}^{\infty} I[m] \underbrace{h(kT - mT)}_{=h[k-m]} + \underbrace{\left. g_R(t) * z(t) \right|_{t=kT}}_{=z[k]} \\ &= \sum_{l=-\infty}^{\infty} h[l] I[k - l] + z[k], \end{aligned}$$

where $z[k]$ is AWGN with variance $\sigma_Z^2 = N_0$, since $g_R(t)$ is a $\sqrt{\text{Nyquist}}$–Filter.

In practice, $h[l]$ can be truncated to some finite length L. If we assume causality of $g_T(t)$, $g_R(t)$, and $c(t)$, $h[l] = 0$ holds for $l < 0$, and if L is chosen large enough $h[l] \approx 0$ holds also for $l \geq L$. Therefore, $r_b[k]$ can be rewritten as

$$\boxed{r_b[k] = \sum_{l=0}^{L-1} h[l] I[k - l] + z[k]}$$



For all equalization schemes derived in the following, it is assumed that the overall channel impulse response $h[k]$ is perfectly known, and only the transmitted information symbols $I[k]$ have to be estimated. In practice, $h[k]$ is unknown, of course, and has to be estimated first. However, this is not a major problem and can be done e.g. using a training sequence of known symbols.

6.2 Maximum–Likelihood Sequence Estimation (MLSE)

- We consider the transmission of a block of K unknown information symbols $I[k]$, $0 \leq k \leq K-1$, and assume that $I[k]$ is known for $k < 0$ and $k \geq K$, respectively.
- We collect the transmitted information sequence $\{I[k]\}$ in a vector

$$\boldsymbol{I} = [I[0] \ \dots \ I[K-1]]^T$$

and the corresponding vector of discrete–time received signals is given by

$$\boldsymbol{r}_b = [r_b[0] \ \dots \ r_b[K+L-2]]^T.$$

Note that $r_b[K+L-2]$ is the last received signal that contains $I[K-1]$.

- **ML Detection**

  For ML detection, we need the pdf $p(\boldsymbol{r}_b|\boldsymbol{I})$ which is given by

$$p(\boldsymbol{r}_b|\boldsymbol{I}) \propto \exp\left(-\frac{1}{N_0}\sum_{k=0}^{K+L-2}\left|r_b[k]-\sum_{l=0}^{L-1}h[l]I[k-l]\right|^2\right).$$

  Consequently, the ML detection rule is given by

$$\begin{aligned}\hat{\boldsymbol{I}} &= \underset{\tilde{\boldsymbol{I}}}{\operatorname{argmax}}\left\{p(\boldsymbol{r}_b|\tilde{\boldsymbol{I}})\right\} \\ &= \underset{\tilde{\boldsymbol{I}}}{\operatorname{argmax}}\left\{\ln[p(\boldsymbol{r}_b|\tilde{\boldsymbol{I}})]\right\} \\ &= \underset{\tilde{\boldsymbol{I}}}{\operatorname{argmin}}\left\{-\ln[p(\boldsymbol{r}_b|\tilde{\boldsymbol{I}})]\right\} \\ &= \underset{\tilde{\boldsymbol{I}}}{\operatorname{argmin}}\left\{\sum_{k=0}^{K+L-2}\left|r_b[k]-\sum_{l=0}^{L-1}h[l]\tilde{I}[k-l]\right|^2\right\},\end{aligned}$$

  where $\hat{\boldsymbol{I}}$ and $\tilde{\boldsymbol{I}}$ denote the estimated sequence and a trial sequence, respectively. Since the above decision rule suggest that we detect the entire sequence $\boldsymbol{I}$ based on the received sequence $\boldsymbol{r}_b$, this optimal scheme is known as *Maximum–Likelihood Sequence Estimation (MLSE)*.

- Notice that there are M^K different trial sequences/vectors $\tilde{\boldsymbol{I}}$ if M–ary modulation is used. Therefore, the complexity of MLSE with brute–force search is exponential in the sequence length K. This is not acceptable for a practical implementation even for relatively small sequence lengths. Fortunately, the exponential complexity in K can be overcome by application of the *Viterbi Algorithm (VA)*.

■ Viterbi Algorithm (VA)

For application of the VA we need to define a *metric* that can be computed, recursively. Introducing the definition

$$\Lambda[k+1] = \sum_{m=0}^{k} \left| r_b[m] - \sum_{l=0}^{L-1} h[l]\tilde{I}[m-l] \right|^2,$$

we note that the function to be minimized for MLSE is $\Lambda[K+L-1]$. On the other hand,

$$\Lambda[k+1] = \Lambda[k] + \lambda[k]$$

with

$$\lambda[k] = \left| r_b[k] - \sum_{l=0}^{L-1} h[l]\tilde{I}[k-l] \right|^2$$

is valid, i.e., $\Lambda[k+1]$ can be calculated recursively from $\Lambda[k]$, which renders the application of the VA possible.

For M–ary modulation an ISI channel of length L can be described by a trellis diagram with M^{L-1} states since the signal component

$$\sum_{l=0}^{L-1} h[l]\tilde{I}[k-l]$$

can assume M^L different values that are determined by the M^{L-1} states

$$\boldsymbol{S}[k] = [\tilde{I}[k-1],\, \ldots,\, \tilde{I}[k-(L-1)]]$$

and the M possible transitions $\tilde{I}[k]$ to state

$$\boldsymbol{S}[k+1] = [\tilde{I}[k],\, \ldots,\, \tilde{I}[k-(L-2)]].$$

Therefore, the VA operates on a trellis with M^{L-1} states.

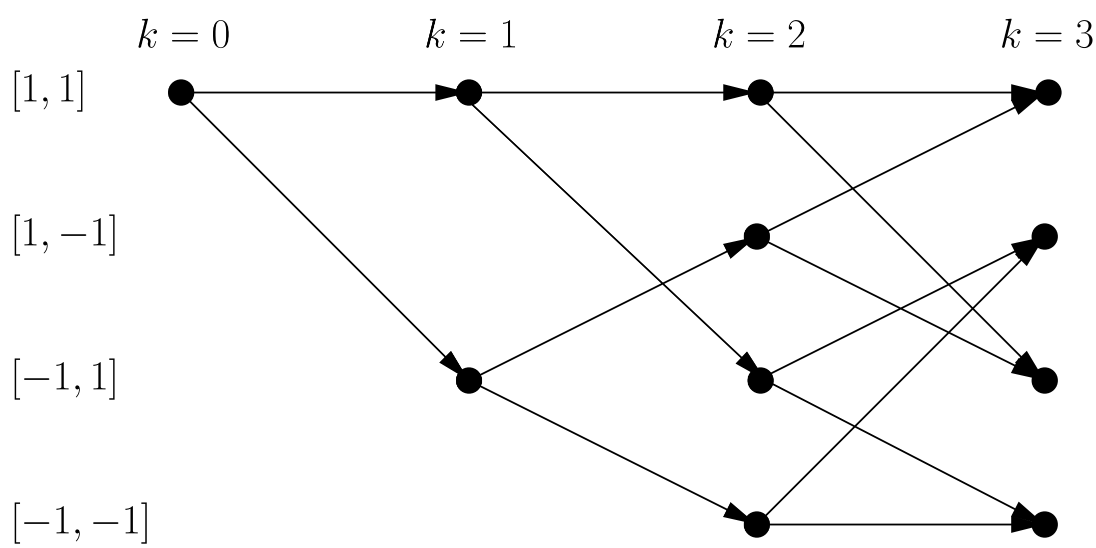
Example:

We explain the VA more in detail using an example. We assume BPSK transmission, i.e., $I[k] \in \{\pm 1\}$, and $L = 3$. For $k < 0$ and $k \geq K$, we assume that $I[k] = 1$ is transmitted.

– There are $M^{L-1} = 2^2 = 4$ states, and $M = 2$ transitions per state. State $\boldsymbol{S}[k]$ is defined as

$$\boldsymbol{S}[k] = [\tilde{I}[k-1],\ \tilde{I}[k-2]]$$

– Since we know that $I[k] = 1$ for $k < 0$, state $\boldsymbol{S}[0] = [1,\ 1]$ holds, whereas $\boldsymbol{S}[1] = [\tilde{I}[0],\ 1]$, and $\boldsymbol{S}[2] = [\tilde{I}[1],\ \tilde{I}[0]]$, and so on. The resulting trellis is shown below.



– $k = 0$

Arbitrarily and without loss of optimality, we may set the *accumulated* metric corresponding to state $\boldsymbol{S}[k]$ at time $k = 0$ equal to zero

$$\Lambda(\boldsymbol{S}[0], 0) = \Lambda([1,\ 1], 0) = 0.$$

Note that there is only one accumulated metric at time $k = 0$ since $\boldsymbol{S}[0]$ is known at the receiver.

– $k = 1$

The accumulated metric corresponding to $\boldsymbol{S}[1] = [\tilde{I}[0],\ 1]$ is given by

$$\begin{aligned} \Lambda(\boldsymbol{S}[1], 1) &= \Lambda(\boldsymbol{S}[0], 0) + \lambda(\boldsymbol{S}[0], \tilde{I}[0], 0) \\ &= \lambda(\boldsymbol{S}[0], \tilde{I}[0], 0) \end{aligned}$$

Since there are two possible states, namely $\boldsymbol{S}[1] = [1,\ 1]$ and $\boldsymbol{S}[1] = [-1,\ 1]$, there are two corresponding accumulated metrics at time $k = 1$.

– $k = 2$

Now, there are 4 possible states $\boldsymbol{S}[2] = [\tilde{I}[1],\ \tilde{I}[0]]$ and for each state a corresponding accumulated metric

$$\Lambda(\boldsymbol{S}[2], 2) = \Lambda(\boldsymbol{S}[1], 1) + \lambda(\boldsymbol{S}[1], \tilde{I}[1], 1)$$

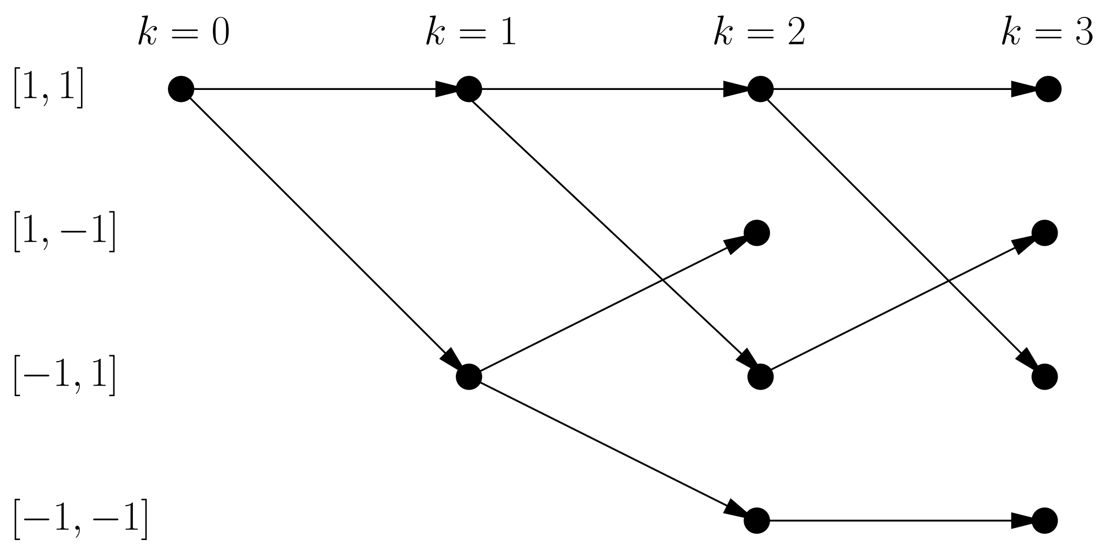
has to be calculated.

– $k = 3$

At $k = 3$ two branches emanate in each state $\boldsymbol{S}[3]$. However, since of the two paths that emanate in the same state $\boldsymbol{S}[3]$ that path which has the smaller accumulated metric $\Lambda(\boldsymbol{S}[3], 3)$ also will have the smaller metric at time $k = K+L-2 = K+1$, we need to retain only the path with the smaller $\Lambda(\boldsymbol{S}[3], 3)$. This path is also referred to as the *surviving path*. In mathematical terms, the accumulated metric for state $\boldsymbol{S}[3]$ is given by

$$\Lambda(\boldsymbol{S}[3], 3) = \underset{\tilde{I}[2]}{\operatorname{argmin}} \{\Lambda(\boldsymbol{S}[2], 2) + \lambda(\boldsymbol{S}[2], \tilde{I}[2], 2)\}$$

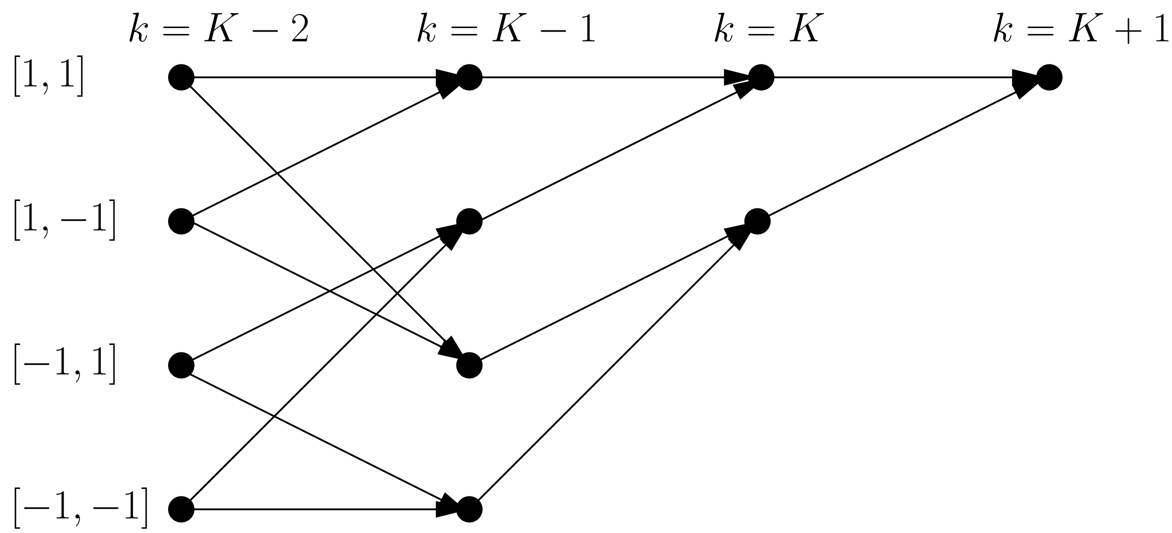
If we retain only the surviving paths, the above trellis at time $k = 3$ may be as shown be below.



– $k \geq 4$

All following steps are similar to that at time $k = 3$. In each step k we retain only $M^{L-1} = 4$ surviving paths and the corresponding accumulated branch metrics.

– *Termination of Trellis*
Since we assume that for $k \geq K$, $I[k] = 1$ is transmitted, the end part of the trellis is as shown below.



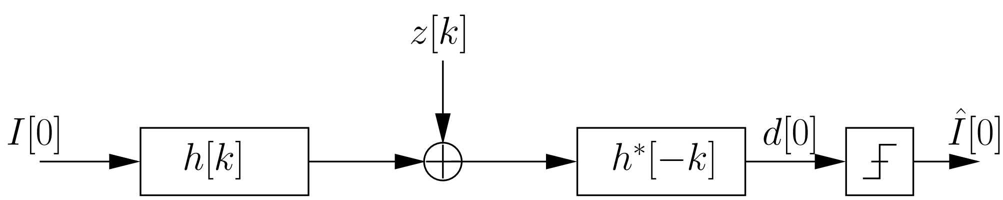
At time $k = K + L - 2 = K + 1$, there is only one surviving path corresponding to the ML sequence.

- Since only M^{L-1} paths are retained at each step of the VA, the complexity of the VA is *linear* in the sequence length K, but exponential in the length L of the overall channel impulse response.
- If the VA is implemented as described above, a decision can be made only at time $k = K+L-2$. However, the related delay may be unacceptable for large sequence lengths K. Fortunately, empirical studies have shown that the surviving paths tend to *merge* relatively quickly, i.e., at time k a decision can be made on the symbol $I[k-k_0]$ if the delay k_0 is chosen large enough. In practice, $k_0 \approx 5(L-1)$ works well and gives almost optimum results.

- **Disadvantage of MLSE with VA**
  In practice, the complexity of MLSE using the VA is often still too high. This is especially true if M is larger than 2. For those cases other, suboptimum equalization strategies have to be used.

- *Historical Note*
  MLSE using the VA in the above form has been introduced by Forney in 1972. Another variation was given later by Ungerböck in 1974. Ungerböck's version uses a matched filter at the receiver but does not require noise whitening.

- *Lower Bound on Performance*
  Exact calculation of the SEP or BEP of MLSE is quite involved and complicated. However, a simple lower bound on the performance of MLSE can be obtained by assuming that just one symbol $I[0]$ is transmitted. In that way, possibly detrimental interference from neighboring symbols is avoided. It can be shown that the optimum ML receiver for that scenario includes a filter matched to $h[k]$ and a decision can be made only based on the matched filter output at time $k = 0$.



  The decision variable $d[0]$ is given by

$$d[0] = \sum_{l=0}^{L-1} |h[l]|^2 \, I[0] + \sum_{l=0}^{L-1} h^*[-l] z[-l].$$

We can model $d[0]$ as

$$d[0] = E_h I[0] + z_0[0],$$

where

$$E_h = \sum_{l=0}^{L-1} |h[l]|^2$$

and $z_0[0]$ is Gaussian noise with variance

$$\begin{aligned}\sigma_0^2 &= \mathcal{E}\left\{ \left| \sum_{l=0}^{L-1} h^*[-l] z[-l] \right|^2 \right\} \\ &= E_h \sigma_Z^2 = E_h N_0\end{aligned}$$

Therefore, this corresponds to the transmission of $I[0]$ over a *non–ISI* channel with E_S/N_0 ratio

$$\frac{E_S}{N_0} = \frac{E_h^2}{E_h N_0} = \frac{E_h}{N_0},$$

and the related SEP or BEP can be calculated easily. For example, for the BEP of BPSK we obtain

$$P_{\mathrm{MF}} = Q\left(\sqrt{2\,\frac{E_h}{N_0}}\right).$$

For the true BEP of MLSE we get

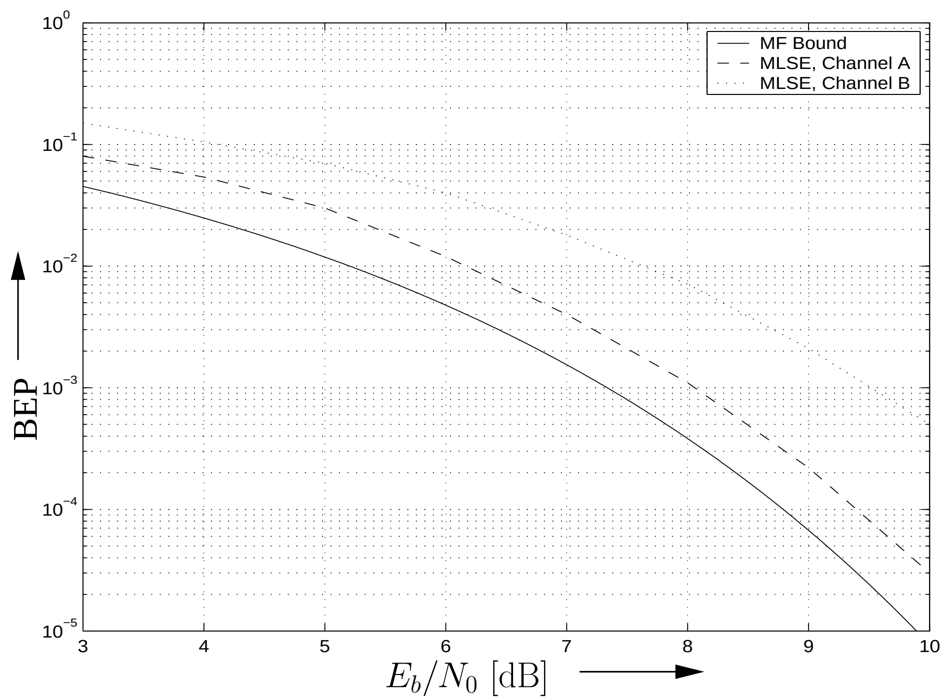
$$P_{\mathrm{MLSE}} \geq P_{\mathrm{MF}}.$$

The above bound is referred to as the *matched–filter (MF)* bound. The tightness of the MF bound largely depends on the underlying channel. For example, for a channel with $L = 2$, $h[0] = h[1] = 1$

and BPSK modulation the loss of MLSE compared to the MF bound is 3 dB. On the other hand, for random channels as typically encountered in wireless communications the MF bound is relatively tight.

Example:

For the following example we define two test channels of length $L = 3$. Channel A has an impulse response of $h[0] = 0.304$, $h[1] = 0.903$, $h[2] = 0.304$, whereas the impulse response of Channel B is given by $h[0] = 1/\sqrt{6}$, $h[1] = 2/\sqrt{6}$, $h[2] = 1/\sqrt{6}$. The received energy per symbol is in both cases $E_S = E_h = 1$. Assuming QPSK transmission, the received energy per bit E_b is $E_b = E_S/2$. The performance of MLSE along with the corresponding MF bound is shown below.



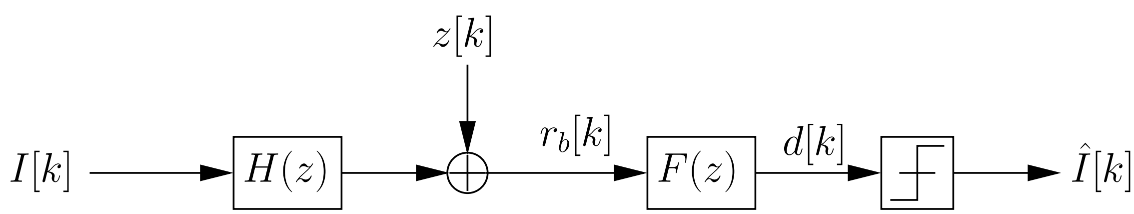
6.3 Linear Equalization (LE)

- Since MLSE becomes too complex for long channel impulse responses, in practice, often suboptimum equalizers with a lower complexity are preferred.

- The most simple suboptimum equalizer is the so–called *linear equalizer*. Roughly speaking, in LE a linear filter

$$
\begin{aligned}
F(z) &= Z\{f[k]\} \\
&= \sum_{k=-\infty}^{\infty} f[k] z^{-k}
\end{aligned}
$$

  is used to *invert* the channel transfer function $H(z) = Z\{h[k]\}$, and symbol–by–symbol decisions are made subsequently. $f[k]$ denotes the equalizer filter coefficients.



  Linear equalizers are categorized with respect to the following two criteria:

  1. Optimization criterion used for calculation of the filter coefficients $f[k]$. Here, we will adopt the so–called *zero–forcing (ZF)* criterion and the *minimum mean–squared error (MMSE)* criterion.

  2. Finite length vs. infinite length equalization filters.

6.3.1 Optimum Linear Zero–Forcing (ZF) Equalization

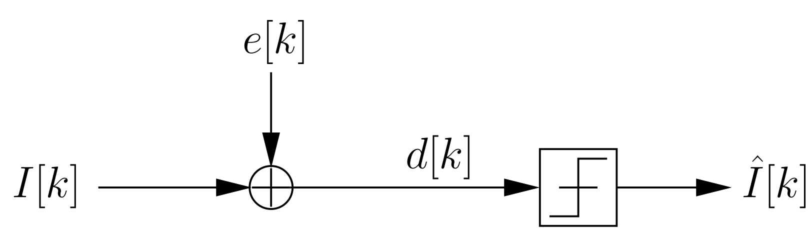
- Optimum ZF equalization implies that we allow for equalizer filters with infinite length impulse response (IIR).
- *Zero–forcing* means that it is our aim to force the residual intersymbol interference in the decision variable $d[k]$ to zero.
- Since we allow for IIR equalizer filters $F(z)$, the above goal can be achieved by

$$\boxed{F(z) = \frac{1}{H(z)}}$$

  where we assume that $H(z)$ has no roots on the unit circle. Since in most practical applications $H(z)$ can be modeled as a filter with finite impulse response (FIR), $F(z)$ will be an IIR filter in general.
- Obviously, the resulting overall channel transfer function is

$$H_{\mathrm{ov}}(z) = H(z)F(z) = 1,$$

  and we arrive at the equivalent channel model shown below.



- The decision variable $d[k]$ is given by

$$d[k] = I[k] + e[k]$$

where $e[k]$ is *colored Gaussian noise* with power spectral density

$$\begin{aligned}\Phi_{ee}(e^{j2\pi fT}) &= N_0\,|F(e^{j2\pi fT})|^2 \\ &= \frac{N_0}{|H(e^{j2\pi fT})|^2}.\end{aligned}$$

The corresponding error variance can be calculated to

$$\begin{aligned}\sigma_e^2 &= \mathcal{E}\{|e[k]|^2\} \\ &= T \int\limits_{-1/(2T)}^{1/(2T)} \Phi_{ee}(e^{j2\pi fT})\,\mathrm{d}f \\ &= T \int\limits_{-1/(2T)}^{1/(2T)} \frac{N_0}{|H(e^{j2\pi fT})|^2}\,\mathrm{d}f.\end{aligned}$$

The signal–to–noise ratio (SNR) is given by

$$\boxed{\mathrm{SNR}_{\mathrm{IIR-ZF}} = \frac{\mathcal{E}\{|I[k]|^2\}}{\sigma_e^2} = \frac{1}{T \displaystyle\int\limits_{-1/(2T)}^{1/(2T)} \frac{N_0}{|H(e^{j2\pi fT})|^2}\,\mathrm{d}f}}$$

- We may consider two extreme cases for $H(z)$:

  1. $|H(e^{j2\pi fT})| = \sqrt{E_h}$
     If $H(z)$ has an allpass characteristic $|H(e^{j2\pi fT})| = \sqrt{E_h}$, we get $\sigma_e^2 = N_0/E_h$ and

     $$\mathrm{SNR}_{\mathrm{IIR-ZF}} = \frac{E_h}{N_0}.$$

     This is the same SNR as for an undistorted AWGN channel, i.e., no performance loss is suffered.

  2. *$H(z)$ has zeros close to the unit circle.*
     In that case $\sigma_e^2 \to \infty$ holds and

     $$\mathrm{SNR}_{\mathrm{IIR-ZF}} \to 0$$

     follows. In this case, ZF equalization leads to a very poor performance. Unfortunately, for wireless channels the probability of zeros close to the unit circle is very high. Therefore, linear ZF equalizers are not employed in wireless receivers.

- **Error Performance**
  Since optimum ZF equalization results in an equivalent channel with additive Gaussian noise, the corresponding BEP and SEP can be easily computed. For example, for BPSK transmission we get

$$\boxed{P_{\mathrm{IIR-ZF}} = Q\left(\sqrt{2\,\mathrm{SNR}_{\mathrm{IIR-ZF}}}\right)}$$

Example:

We consider a channel with two coefficients and energy 1

$$H(z) = \frac{1}{\sqrt{1+|c|^2}}\,(1 - cz^{-1}),$$

where c is complex. The equalizer filter is given by

$$F(z) = \sqrt{1+|c|^2}\,\frac{z}{z-c}$$

In the following, we consider two cases: $|c| < 1$ and $|c| > 1$.

1. $|c| < 1$

   In this case, a stable, *causal* impulse response is obtained.

$$f[k] = \sqrt{1+|c|^2}\,c^k u[k],$$

   where $u[k]$ denotes the unit step function. The corresponding error variance is

$$\begin{aligned}
\sigma_e^2 &= T \int\limits_{-1/(2T)}^{1/(2T)} \frac{N_0}{|H(e^{j2\pi fT})|^2}\,\mathrm{d}f \\
&= N_0 T \int\limits_{-1/(2T)}^{1/(2T)} |F(e^{j2\pi fT})|^2\,\mathrm{d}f \\
&= N_0 \sum_{k=-\infty}^{\infty} |f[k]|^2 \\
&= N_0 (1+|c|^2) \sum_{k=0}^{\infty} |c|^{2k} \\
&= N_0 \frac{1+|c|^2}{1-|c|^2}.
\end{aligned}$$

The SNR becomes

$$\mathrm{SNR}_{\mathrm{IIR-ZF}} = \frac{1}{N_0}\frac{1-|c|^2}{1+|c|^2}.$$

2. $|c| > 1$

Now, we can realize the filter as stable and *anti–causal* with impulse response

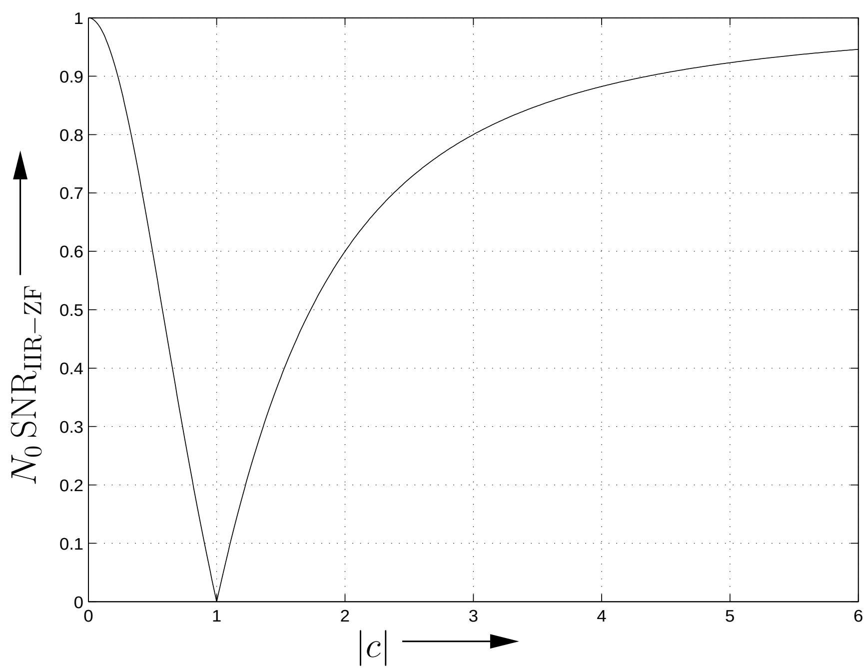
$$f[k] = -\frac{\sqrt{1+|c|^2}}{c}\, c^{k+1} u[-(k+1)].$$

Using similar techniques as above, the error variance becomes

$$\sigma_e^2 \;=\; N_0\,\frac{1+|c|^2}{|c|^2-1},$$

and we get for the SNR

$$\mathrm{SNR}_{\mathrm{IIR-ZF}} = \frac{1}{N_0}\frac{|c|^2-1}{1+|c|^2}.$$



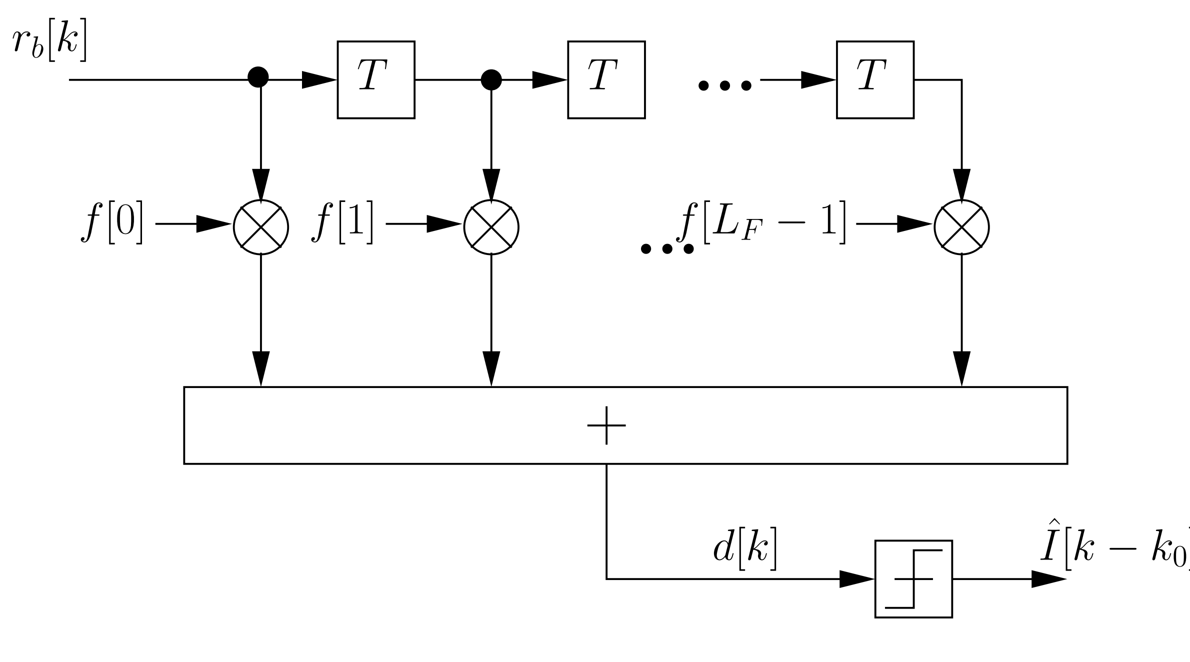
Obviously, the SNR drops to zero as $|c|$ approaches one, i.e., as the root of $H(z)$ approaches the unit circle.

6.3.2 ZF Equalization with FIR Filters

- In this case, we impose a causality and a length constraint on the equalizer filter and the transfer function is given by

$$F(z) = \sum_{k=0}^{L_F-1} f[k]z^{-k}$$

In order to be able to deal with "non–causal components", we introduce a decision delay $k_0 \geq 0$, i.e., at time k, we estimate $I[k-k_0]$. Here, we assume a fixed value for k_0, but in practice, k_0 can be used for optimization.



- Because of the finite filter length, a complete elimination of ISI is in general not possible.

- **Alternative Criterion**: Peak–Distortion Criterion
  Minimize the maximum possible distortion of the signal at the equalizer output due to ISI.

- **Optimization**
  In mathematical terms the above criterion can be formulated as follows.
  Minimize

$$D = \sum_{\substack{k=-\infty \\ k \neq k_0}}^{\infty} |h_{\mathrm{ov}}[k]|$$

subject to

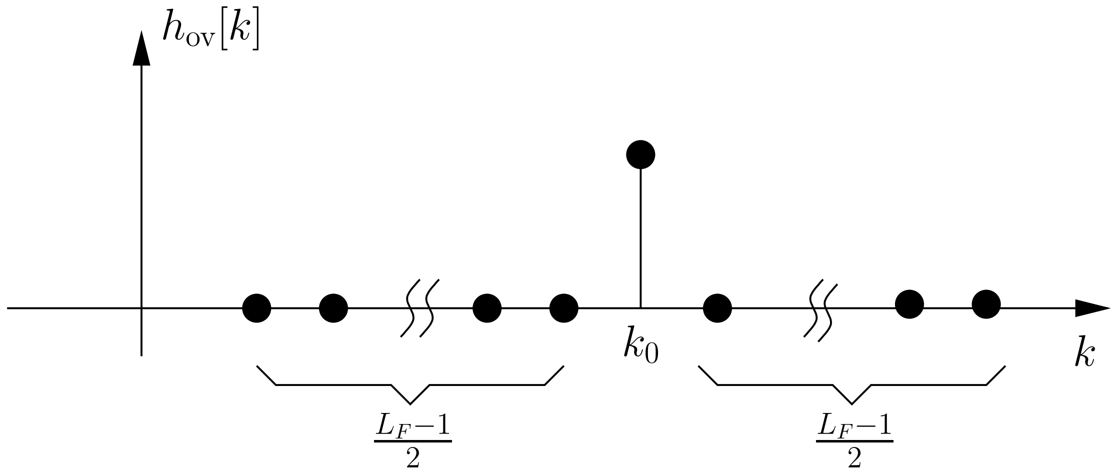
$$h_{\mathrm{ov}}[k_0] = 1,$$

where $h_{\mathrm{ov}}[k]$ denotes the overall impulse response (channel and equalizer filter).

Although D is a convex function of the equalizer coefficients, it is in general difficult to find the optimum filter coefficients. An exception is the special case when the *binary eye* at the equalizer input is open

$$\frac{1}{|h[k_1]|} \sum_{\substack{k=-\infty \\ k \neq k_1}}^{\infty} |h[k]| < 1$$

for some k_1. In this case, if we assume furthermore $k_0 = k_1 + (L_F - 1)/2$ (L_F odd), D is minimized if and only if the overall impulse response $h_{\mathrm{ov}}[k]$ has $(L_F - 1)/2$ consecutive zeros to the left and to the right of $h_{\mathrm{ov}}[k_0] = 1$.



- This shows that in this special case the Peak–Distortion Criterion corresponds to the *ZF criterion* for equalizers with finite order. Note that there is no restriction imposed on the remaining coefficients of $h_{\mathrm{ov}}[k]$ ("don't care positions").

- **Problem**
  If the binary eye at the equalizer input is closed, in general, D is *not* minimized by the ZF solution. In this case, the coefficients at the "don't care positions" may take on large values.

- **Calculation of the ZF Solution**
  The above ZF criterion leads us to the conditions

$$h_{\mathrm{ov}}[k] = \sum_{m=0}^{q_F} f[m]h[k-m] = 0$$

where $k \in \{k_0 - q_F/2, \ldots, k_0 - 1, k_0 + 1, \ldots, k_0 + q_F/2\}$, and

$$h_{\mathrm{ov}}[k_0] = \sum_{m=0}^{q_F} f[m]h[k_0-m] = 1,$$

and $q_F = L_F - 1$. The resulting system of linear equations to be solved can be written as

$$\boldsymbol{H}\boldsymbol{f} = [0\ 0\ \dots\ 0\ 1\ 0\ \dots\ 0]^T,$$

with the $L_F \times L_F$ matrix

$$\boldsymbol{H} = \begin{bmatrix} h[k_0 - q_F/2] & h[k_0 - q_F/2 - 1] & \cdots & h[k_0 - 3q_F/2] \\ h[k_0 - q_F/2 + 1] & h[k_0 - q_F/2] & \cdots & h[k_0 - 3q_F/2 + 1] \\ \vdots & \vdots & \vdots & \vdots \\ h[k_0 - 1] & h[k_0 - 2] & \cdots & h[k_0 - q_F - 1] \\ h[k_0] & h[k_0 - 1] & \cdots & h[k_0 - q_F] \\ h[k_0 + 1] & h[k_0] & \cdots & h[k_0 - q_F + 1] \\ \vdots & \vdots & \vdots & \vdots \\ h[k_0 + q_F/2] & h[k_0 + q_F/2 - 1] & \cdots & h[k_0 - q_F/2] \end{bmatrix}$$

and coefficient vector

$$\boldsymbol{f} = [f[0]\ f[1]\ \dots\ f[q_F]]^T.$$

The ZF solution is given by

$$\boxed{\boldsymbol{f} = \boldsymbol{H}^{-1}\,[0\ 0\ \dots\ 0\ 1\ 0\ \dots\ 0]^T}$$

or in other words, the optimum vector is the $(q_F/2 + 1)$th row of the inverse of $\boldsymbol{H}$.

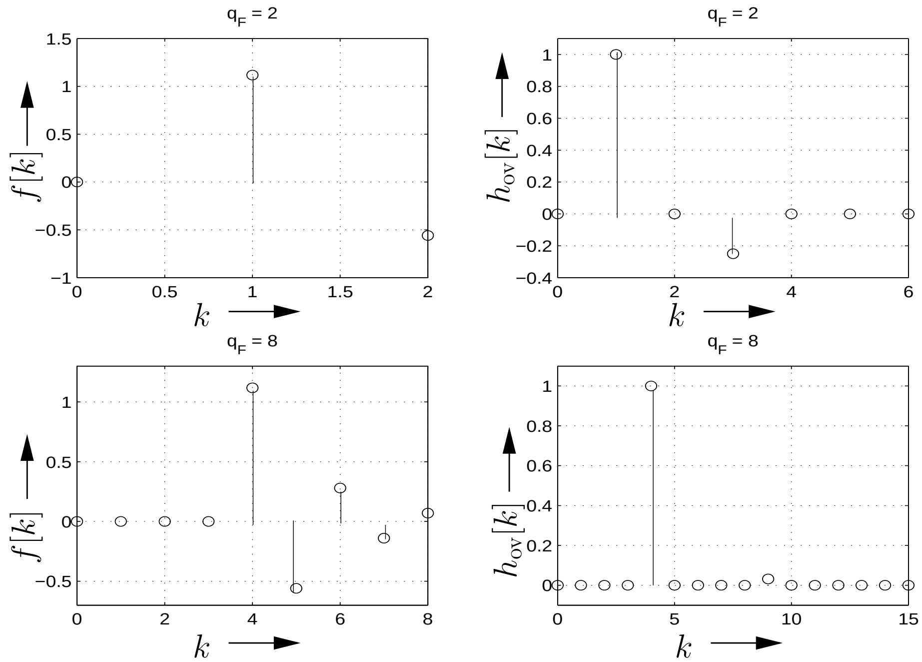
Example:

We assume

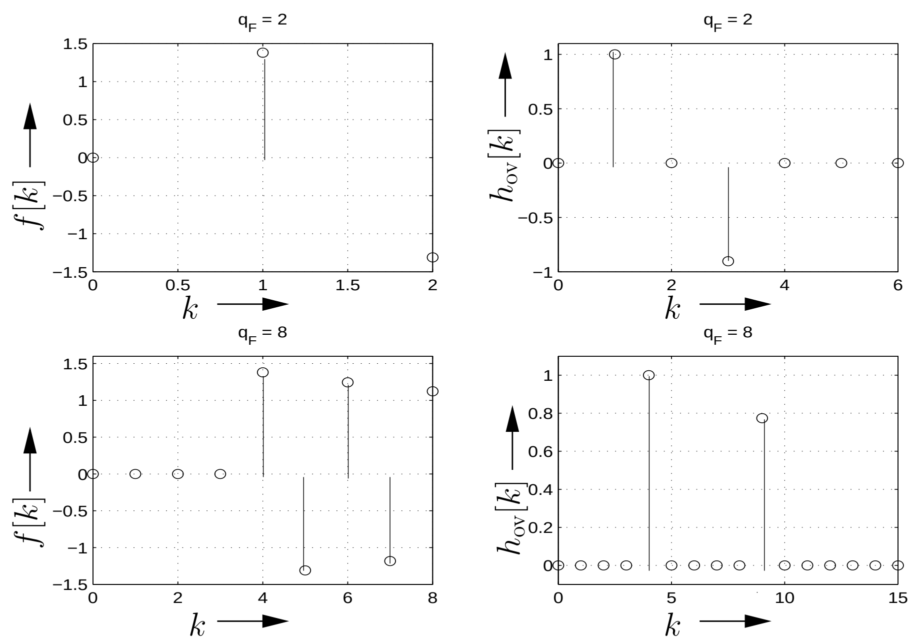
$$H(z) = \frac{1}{\sqrt{1 + |c|^2}}\,(1 - cz^{-1}),$$

and $k_0 = q_F/2$.

1. First we assume $c = 0.5$, i.e., $h[k]$ is given by $h[0] = 2/\sqrt{5}$ and $h[1] = 1/\sqrt{5}$.



2. In our second example we have $c = 0.95$, i.e., $h[k]$ is given by $h[0] = 0.73$ and $h[1] = 0.69$.



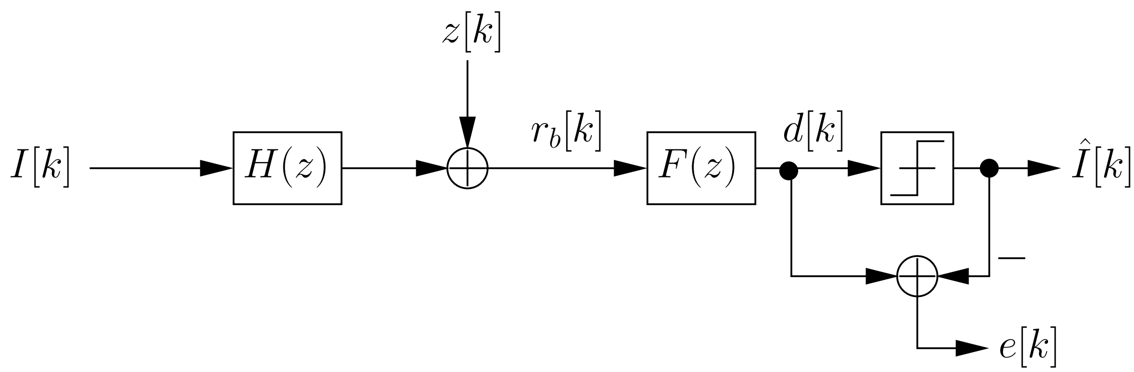
We observe that the residual interference is larger for shorter equalizer filter lengths and increases as the root of $H(z)$ approaches the unit circle.

6.3.3 Optimum Minimum Mean–Squared Error (MMSE) Equalization

- **Objective**

Minimize the variance of the error signal

$$e[k] = d[k] - \hat{I}[k].$$



- **Advantage over ZF Equalization**

The MMSE criterion ensures an optimum trade–off between residual ISI in $d[k]$ and noise enhancement. Therefore, MMSE equalizers achieve a significantly lower BEP compared to ZF equalizers at low–to–moderate SNRs.

Calculation of Optimum Filter $F(z)$

– The error signal $e[k] = d[k] - \hat{I}[k]$ depends on the *estimated* symbols $\hat{I}[k]$. Since it is very difficult to take into account the effect of possibly erroneous decisions, for filter optimization it is usually assumed that $\hat{I}[k] = I[k]$ is valid. The corresponding error signal is

$$e[k] = d[k] - I[k].$$

– **Cost Function**

The cost function for filter optimization is given by

$$\begin{aligned} J &= \mathcal{E}\{|e[k]|^2\} \\ &= \mathcal{E}\left\{\left(\sum_{m=-\infty}^{\infty} f[m]r_b[k-m] - I[k]\right)\right. \\ &\qquad \left.\left(\sum_{m=-\infty}^{\infty} f^*[m]r_b^*[k-m] - I^*[k]\right)\right\}, \end{aligned}$$

which is the error variance.

– **Optimum Filter**

We obtain the optimum filter coefficients from

$$\begin{aligned} \frac{\partial J}{\partial f^*[\kappa]} &= \mathcal{E}\left\{\left(\sum_{m=-\infty}^{\infty} f[m]r_b[k-m] - I[k]\right) r_b^*[k-\kappa]\right\} \\ &= \mathcal{E}\left\{e[k]r_b^*[k-\kappa]\right\} = 0, \qquad \kappa \in \{\ldots, -1, 0, 1, \ldots\}, \end{aligned}$$

where we have used the following rules for *complex differen-*

tiation

$$\begin{aligned}\frac{\partial f^*[\kappa]}{\partial f^*[\kappa]} &= 1 \\ \frac{\partial f[\kappa]}{\partial f^*[\kappa]} &= 0 \\ \frac{\partial |f[\kappa]|^2}{\partial f^*[\kappa]} &= f[\kappa].\end{aligned}$$

We observe that the error signal and the input of the MMSE filter must be *orthogonal*. This is referred to as the *orthogonality principle* of MMSE optimization.

The above condition can be modified to

$$\mathcal{E}\left\{e[k]r_b^*[k-\kappa]\right\} = \mathcal{E}\left\{d[k]r_b^*[k-\kappa]\right\} - \mathcal{E}\left\{I[k]r_b^*[k-\kappa]\right\}$$

The individual terms on the right hand side of the above equation can be further simplified to

$$\mathcal{E}\left\{d[k]r_b^*[k-\kappa]\right\} = \sum_{m=-\infty}^{\infty} f[m]\,\underbrace{\mathcal{E}\left\{r_b[k-m]r_b^*[k-\kappa]\right\}}_{\phi_{rr}[\kappa-m]},$$

and

$$\begin{aligned}\mathcal{E}\left\{I[k]r_b^*[k-\kappa]\right\} &= \sum_{m=-\infty}^{\infty} h^*[m]\,\underbrace{\mathcal{E}\left\{I[k]I^*[k-\kappa-m]\right\}}_{\phi_{II}[\kappa+m]} \\ &= \sum_{\mu=-\infty}^{\infty} h^*[-\mu]\phi_{II}[\kappa-\mu],\end{aligned}$$

respectively. Therefore, we obtain

$$f[k] * \phi_{rr}[k] = h^*[-k] * \phi_{II}[k],$$

and the $\mathcal{Z}$–transform of this equation is

$$F(z)\Phi_{rr}(z) = H^*(1/z^*)\Phi_{II}(z)$$

with

$$\begin{aligned}\Phi_{rr}(z) &= \sum_{k=-\infty}^{\infty} \phi_{rr}[k]z^{-k} \\ \Phi_{II}(z) &= \sum_{k=-\infty}^{\infty} \phi_{II}[k]z^{-k}.\end{aligned}$$

The optimum filter transfer function is given by

$$F(z) = \frac{H^*(1/z^*)\Phi_{II}(z)}{\Phi_{rr}(z)}$$

Usually, we assume that the noise $z[k]$ and the data sequence $I[k]$ are white processes and mutually uncorrelated. We assume furthermore that the variance of $I[k]$ is normalized to 1. In that case, we get

$$\begin{aligned}\phi_{rr}[k] &= h[k] * h^*[-k] * \phi_{II}[k] + \phi_{ZZ}[k] \\ &= h[k] * h^*[-k] + N_0\,\delta[k],\end{aligned}$$

and

$$\begin{aligned}\Phi_{rr}(z) &= H(z)H^*(1/z^*) + N_0 \\ \Phi_{II}(z) &= 1.\end{aligned}$$

The optimum MMSE filter is given by

$$\boxed{F(z) = \frac{H^*(1/z^*)}{H(z)H^*(1/z^*) + N_0}}$$

We may consider two limiting cases.

1. $N_0 \to 0$

   In this case, we obtain

$$F(z) = \frac{1}{H(z)},$$

   i.e., in the high SNR region the MMSE solution approaches the ZF equalizer.

2. $N_0 \to \infty$

   We get

$$F(z) = \frac{1}{N_0} H^*(1/z^*),$$

   i.e., the MMSE filter approaches a discrete–time matched filter.

- **Autocorrelation of Error Sequence**

  The ACF of the error sequence $e[k]$ is given by

$$\begin{aligned}\phi_{ee}[\lambda] &= \mathcal{E}\{e[k]e^*[k-\lambda]\} \\ &= \mathcal{E}\left\{e[k](d[k-\lambda] - I[k-\lambda])^*\right\} \\ &= \phi_{ed}[\lambda] - \phi_{eI}[\lambda]\end{aligned}$$

  $\phi_{ed}[\lambda]$ can be simplified to

$$\begin{aligned}\phi_{ed}[\lambda] &= \sum_{m=-\infty}^{\infty} f^*[m] \underbrace{\mathcal{E}\{e[k]r_b^*[k-\lambda-m]\}}_{=0} \\ &= 0.\end{aligned}$$

  This means that the error signal $e[k]$ is also orthogonal to the equalizer output signal $d[k]$. For the ACF of the error we obtain

$$\begin{aligned}\phi_{ee}[\lambda] &= -\phi_{eI}[\lambda] \\ &= \phi_{II}[\lambda] - \phi_{dI}[\lambda].\end{aligned}$$

■ Error Variance

The error variance σ_e^2 is given by

$$\begin{aligned}\sigma_e^2 &= \phi_{II}[0] - \phi_{dI}[0] \\ &= 1 - \phi_{dI}[0].\end{aligned}$$

σ_e^2 can be calculated most easily from the the power spectral density

$$\begin{aligned}\Phi_{ee}(z) &= \Phi_{II}(z) - \Phi_{dI}(z) \\ &= 1 - F(z)H(z) \\ &= 1 - \frac{H(z)H^*(1/z^*)}{H(z)H^*(1/z^*) + N_0} \\ &= \frac{N_0}{H(z)H^*(1/z^*) + N_0}.\end{aligned}$$

More specifically, σ_e^2 is given by

$$\sigma_e^2 = T \int_{-1/(2T)}^{1/(2T)} \Phi_{ee}(e^{j2\pi fT})\,\mathrm{d}f$$

or

$$\boxed{\sigma_e^2 = T \int_{-1/(2T)}^{1/(2T)} \frac{N_0}{|H(e^{j2\pi fT})|^2 + N_0}\,\mathrm{d}f}$$

- **Overall Transfer Function**

The overall transfer function is given by

$$\begin{aligned} H_{\text{ov}}(z) &= H(z)F(z) \\ &= \frac{H(z)H^*(1/z^*)}{H(z)H^*(1/z^*)+N_0} \\ &= \frac{1}{1+\frac{N_0}{H(z)H^*(1/z^*)}} \\ &= 1-\frac{N_0}{H(z)H^*(1/z^*)+N_0} \end{aligned}$$

Obviously, $H_{\text{ov}}(z)$ is not a constant but depends on z, i.e., there is residual intersymbol interference. The coefficient $h_{\text{ov}}[0]$ is obtained from

$$\begin{aligned} h_{\text{ov}}[0] &= T \int\limits_{-1/(2T)}^{1/(2T)} H_{\text{ov}}(e^{j2\pi fT})\,\mathrm{d}f \\ &= T \int\limits_{-1/(2T)}^{1/(2T)} (1-\Phi_{ee}(e^{j2\pi fT}))\,\mathrm{d}f \\ &= 1-\sigma_e^2 < 1. \end{aligned}$$

Since $h_{\text{ov}}[0] < 1$ is valid, MMSE equalization is said to be *biased*.

- **SNR**

The decision variable $d[k]$ may be rewritten as

$$\begin{aligned} d[k] &= I[k]+e[k] \\ &= h_{\text{ov}}[0]I[k]+\underbrace{e[k]+(1-h_{\text{ov}}[0])I[k]}_{=e'[k]}, \end{aligned}$$

where $e'[k]$ does not contain $I[k]$. Using $\phi_{ee}[\lambda] = -\phi_{eI}[\lambda]$ the variance of $e'[k]$ is given by

$$\begin{aligned}
\sigma_{e'}^2 &= \mathcal{E}\{|e[k]|^2\} \\
&= (1 - h_{\mathrm{ov}}[0])^2 + 2(1 - h_{\mathrm{ov}}[0])\phi_{eI}[0] + \sigma_e^2 \\
&= \sigma_e^4 - 2\sigma_e^4 + \sigma_e^2 \\
&= \sigma_e^2 - \sigma_e^4.
\end{aligned}$$

Therefore, the SNR for MMSE equalization with IIR filters is given by

$$\begin{aligned}
\mathrm{SNR}_{\mathrm{IIR-MMSE}} &= \frac{h_{\mathrm{ov}}^2[0]}{\sigma_{e'}^2} \\
&= \frac{(1 - \sigma_e^2)^2}{\sigma_e^2(1 - \sigma_e^2)}
\end{aligned}$$

which yields

$$\boxed{\mathrm{SNR}_{\mathrm{IIR-MMSE}} = \frac{1 - \sigma_e^2}{\sigma_e^2}}$$

Example:

We consider again the channel with one root and transfer function

$$H(z) = \frac{1}{\sqrt{1+|c|^2}}\,(1 - cz^{-1}),$$

where c is a complex constant. After some straightforward manipulations the error variance is given by

$$\begin{aligned}
\sigma_e^2 &= \mathcal{E}\{|e[k]|^2\} \\
&= T \int\limits_{-1/(2T)}^{1/(2T)} \frac{N_0}{|H(e^{j2\pi fT})|^2 + N_0}\,\mathrm{d}f \\
&= \frac{N_0}{1+N_0}\,\frac{1}{\sqrt{1-\beta^2}},
\end{aligned}$$

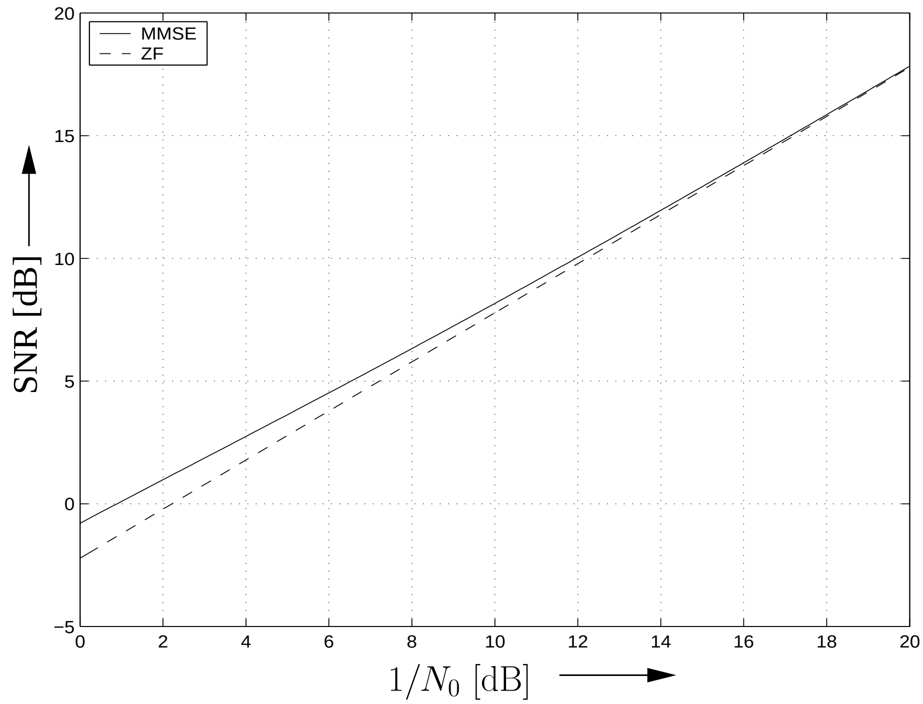
where β is defined as

$$\beta = \frac{2|c|}{(1+N_0)(1+|c|^2)}.$$

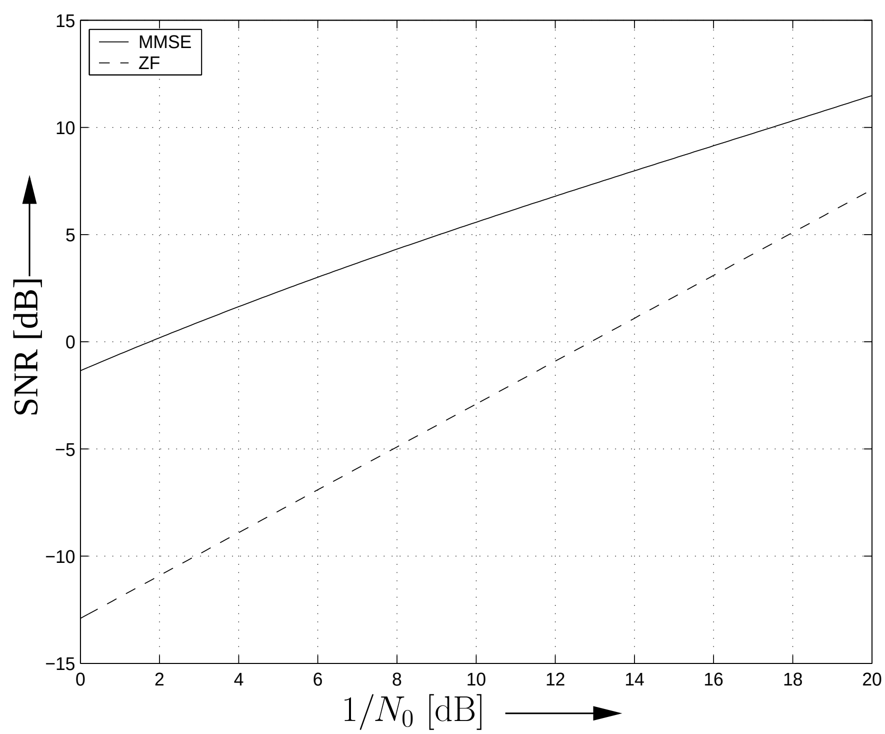
It is easy to check that for $N_0 \to 0$, i.e., for high SNRs σ_e^2 approaches the error variance for linear ZF equalization.

We illustrate the SNR for two different cases.

1. $|c| = 0.5$


MMSE
ZF
SNR [dB]
1/N0 [dB]
20
15
10
5
0
−5
0
2
4
6
8
10
12
14
16
18
20

2. $|c| = 0.95$


MMSE
ZF
SNR [dB]
1/N0 [dB]
15
10
5
0
−5
−10
−15
0
2
4
6
8
10
12
14
16
18
20

As expected, for high input SNRs (small noise variances), the (output) SNR for ZF equalization approaches that for MMSE equalization. For larger noise variances, however, MMSE equalization yields a significantly higher SNR, especially if $H(z)$ has zeros close to the unit circle.

6.3.4 MMSE Equalization with FIR Filters

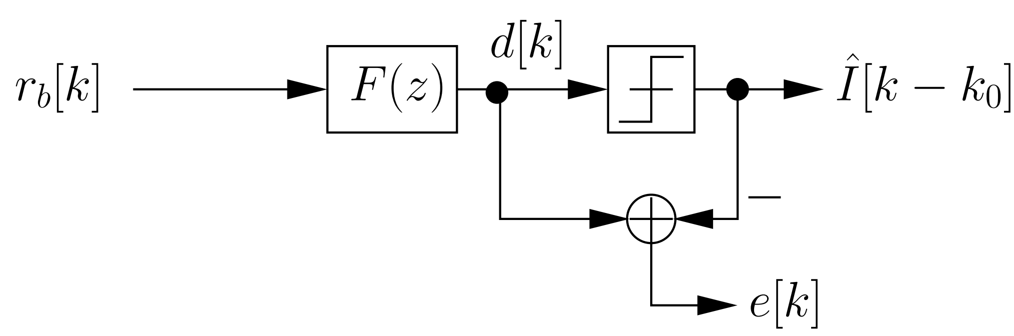
- In practice, FIR filters are employed. The equalizer output signal in that case is given by

$$\begin{aligned} d[k] &= \sum_{m=0}^{q_F} f[m] r_b[k-m] \\ &= \boldsymbol{f}^H \boldsymbol{r}_b, \end{aligned}$$

where $q_F = L_F - 1$ and the definitions

$$\begin{aligned} \boldsymbol{f} &= [f[0] \ \ldots \ f[q_F]]^H \\ \boldsymbol{r}_b &= [r_b[k] \ \ldots \ r_b[k-q_F]]^T \end{aligned}$$

are used. Note that vector $\boldsymbol{f}$ contains the *complex conjugate* filter coefficients. This is customary in the literature and simplifies the derivation of the optimum filter coefficients.



■ Error Signal

The error signal $e[k]$ is given by

$$e[k] = d[k] - I[k-k_0],$$

where we allow for a decision delay $k_0 \geq 0$ to account for non–causal components, and we assume again $\hat{I}[k-k_0] = I[k-k_0]$ for the sake of mathematical tractability.

■ Cost Function

The cost function for filter optimization is given by

$$\begin{aligned} J(\boldsymbol{f}) &= \mathcal{E}\{|e[k]|^2\} \\ &= \mathcal{E}\left\{\left(\boldsymbol{f}^H\boldsymbol{r}_b - I[k-k_0]\right)\left(\boldsymbol{f}^H\boldsymbol{r}_b - I[k-k_0]\right)^H\right\} \\ &= \boldsymbol{f}^H \underbrace{\mathcal{E}\{\boldsymbol{r}_b\boldsymbol{r}_b^H\}}_{\boldsymbol{\Phi}_{rr}} \boldsymbol{f} - \boldsymbol{f}^H \underbrace{\mathcal{E}\{\boldsymbol{r}_b I^*[k-k_0]\}}_{\boldsymbol{\varphi}_{rI}} \\ &\quad -\mathcal{E}\{I[k-k_0]\boldsymbol{r}_b^H\}\boldsymbol{f} + \mathcal{E}\{|I[k-k_0]|^2\} \\ &= \boldsymbol{f}^H\boldsymbol{\Phi}_{rr}\boldsymbol{f} - \boldsymbol{f}^H\boldsymbol{\varphi}_{rI} - \boldsymbol{\varphi}_{rI}^H\boldsymbol{f} + 1, \end{aligned}$$

where $\boldsymbol{\Phi}_{rr}$ denotes the autocorrelation matrix of vector $\boldsymbol{r}_b$, and $\boldsymbol{\varphi}_{rI}$ is the crosscorrelation vector between $\boldsymbol{r}_b$ and $I[k-k_0]$. $\boldsymbol{\Phi}_{rr}$ is given by

$$\boldsymbol{\Phi}_{rr} = \begin{bmatrix} \phi_{rr}[0] & \phi_{rr}[1] & \cdots & \phi_{rr}[q_F] \\ \phi_{rr}[-1] & \phi_{rr}[0] & \cdots & \phi_{rr}[q_F-1] \\ \vdots & \vdots & \ddots & \vdots \\ \phi_{rr}[-q_F] & \phi_{rr}[-q_F+1] & \cdots & \phi_{rr}[0] \end{bmatrix},$$

where $\phi_{rr}[\lambda] = \mathcal{E}\{r_b^*[k]r_b[k+\lambda]\}$. The crosscorrelation vector can be calculated as

$$\boldsymbol{\varphi}_{rI} = [\phi_{rI}[k_0]\ \phi_{rI}[k_0-1]\ \ldots\ \phi_{rI}[k_0-q_F]]^T,$$

where $\phi_{rI}[\lambda] = \mathcal{E}\{r_b[k+\lambda]I^*[k]\}$. Note that for independent, identically distributed input data and AWGN, we get

$$\begin{aligned}\phi_{rr}[\lambda] &= h[\lambda] * h^*[-\lambda] + N_0\,\delta[\lambda] \\ \phi_{rI}[\lambda] &= h[k_0+\lambda].\end{aligned}$$

This completely specifies $\boldsymbol{\Phi}_{rr}$ and $\boldsymbol{\varphi}_{rI}$.

- **Filter Optimization**

The optimum filter coefficient vector can be obtained be setting the gradient of $J(\boldsymbol{f})$ equal to zero

$$\frac{\partial J(\boldsymbol{f})}{\partial \boldsymbol{f}^*} = \boldsymbol{0}.$$

For calculation of this gradient, we use the following rules for differentiation of scalar functions with respect to (complex) vectors:

$$\begin{aligned}\frac{\partial}{\partial \boldsymbol{f}^*}\boldsymbol{f}^H\boldsymbol{X}\boldsymbol{f} &= \boldsymbol{X}\boldsymbol{f} \\ \frac{\partial}{\partial \boldsymbol{f}^*}\boldsymbol{f}^H\boldsymbol{x} &= \boldsymbol{x} \\ \frac{\partial}{\partial \boldsymbol{f}^*}\boldsymbol{x}^H\boldsymbol{f} &= \boldsymbol{0},\end{aligned}$$

where $\boldsymbol{X}$ and $\boldsymbol{x}$ denote a matrix and a vector, respectively.

With these rules we obtain

$$\frac{\partial J(\boldsymbol{f})}{\partial \boldsymbol{f}^*} = \boldsymbol{\Phi}_{rr}\boldsymbol{f} - \boldsymbol{\varphi}_{rI} = \boldsymbol{0}$$

or

$$\boldsymbol{\Phi}_{rr}\boldsymbol{f} = \boldsymbol{\varphi}_{rI}.$$

This equation is often referred to as the *Wiener–Hopf equation*. The MMSE or Wiener solution for the optimum filter coefficients $\boldsymbol{f}_{\mathrm{opt}}$ is given by

$$\boxed{\boldsymbol{f}_{\mathrm{opt}} = \boldsymbol{\Phi}_{rr}^{-1}\boldsymbol{\varphi}_{rI}}$$

- **Error Variance**
  The minimum error variance is given by

$$\begin{aligned}\sigma_e^2 &= J(\boldsymbol{f}_{\mathrm{opt}}) \\ &= 1 - \boldsymbol{\varphi}_{rI}^H \boldsymbol{\Phi}_{rr}^{-1}\boldsymbol{\varphi}_{rI} \\ &= 1 - \boldsymbol{\varphi}_{rI}^H \boldsymbol{f}_{\mathrm{opt}}\end{aligned}$$

- **Overall Channel Coefficient** $h_{\mathrm{ov}}[k_0]$
  The coefficient $h_{\mathrm{ov}}[k_0]$ is given by

$$\begin{aligned}h_{\mathrm{ov}}[k_0] &= \boldsymbol{f}_{\mathrm{opt}}^H \boldsymbol{\varphi}_{rI} \\ &= 1 - \sigma_e^2 < 1,\end{aligned}$$

  i.e., also the optimum FIR MMSE filter is *biased*.

- **SNR**
  Similar to the IIR case, the SNR at the output of the optimum FIR filter can be calculated to

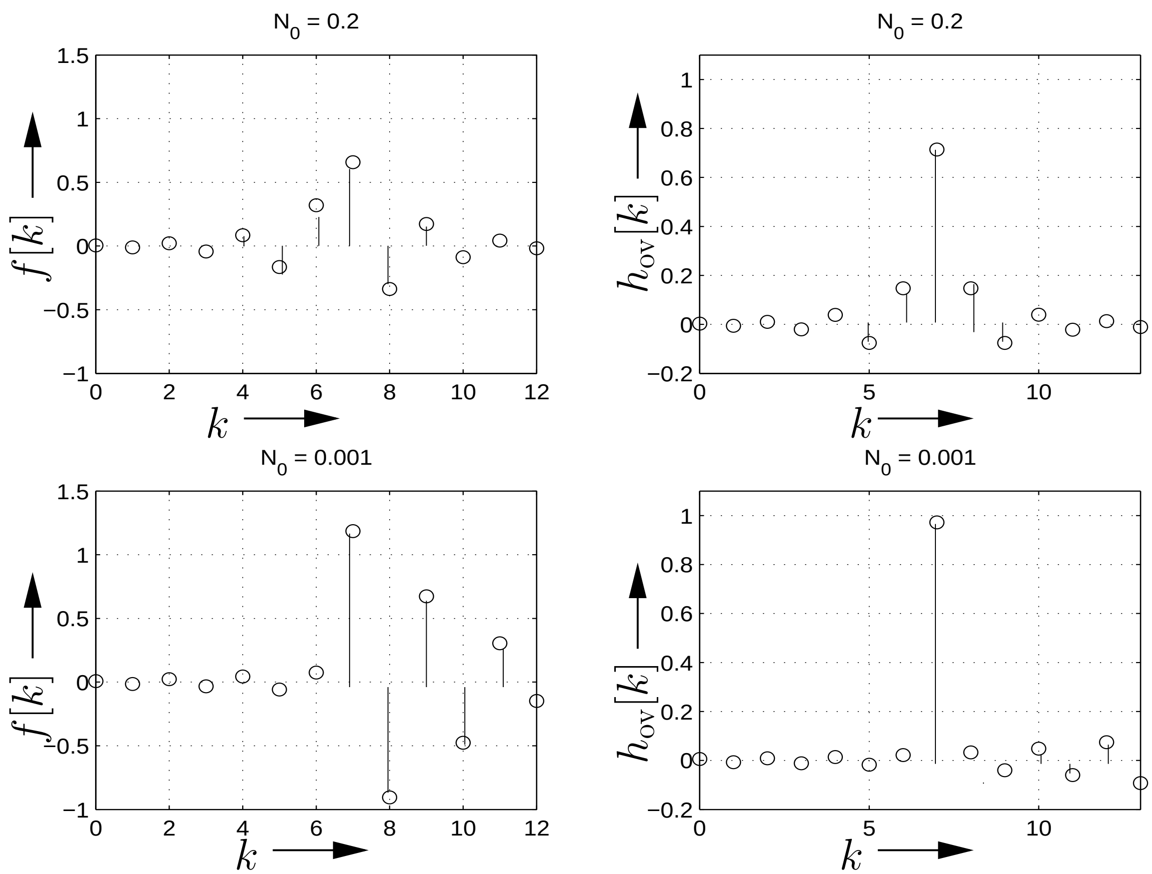
$$\boxed{\mathrm{SNR}_{\mathrm{FIR-MMSE}} = \frac{1-\sigma_e^2}{\sigma_e^2}}$$

Example:

Below, we show $f[k]$ and $h_{\text{ov}}[k]$ for a channel with one root and transfer function

$$H(z) = \frac{1}{\sqrt{1+|c|^2}}(1 - cz^{-1}).$$

We consider the case $c = 0.8$ and different noise variances $\sigma_Z^2 = N_0$. Furthermore, we use $q_F = 12$ and $k_0 = 7$.

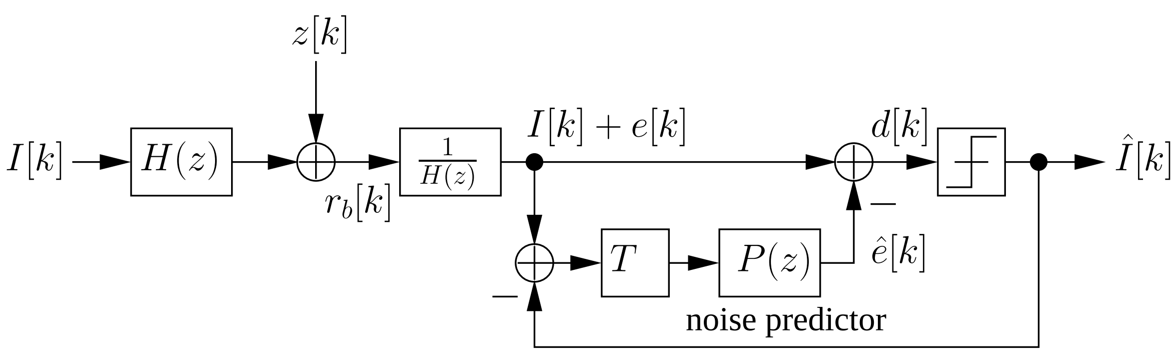


We observe that the residual ISI in $h_{\text{ov}}[k]$ is smaller for the smaller noise variance, since the MMSE filter approaches the ZF filter for $N_0 \to 0$. Also the bias decreases with decreasing noise variance, i.e., $h_{\text{ov}}[k_0]$ approaches 1.

6.4 Decision–Feedback Equalization (DFE)

- **Drawback of Linear Equalization**
  In linear equalization, the equalizer filter enhances the noise, especially in severely distorted channels with roots close to the unit circle. The noise variance at the equalizer output is *increased* and the noise is *colored*. In many cases, this leads to a poor performance.

- **Noise Prediction**
  The above described drawback of LE can be avoided by application of *linear noise prediction*.



The linear FIR noise predictor

$$P(z) = \sum_{m=0}^{L_P-1} p[m] z^{-m}$$

predicts the current noise sample $e[k]$ based on the previous L_P noise samples $e[k-1]$, $e[k-2]$, ..., $e[k-L_P]$. The estimate $\hat{e}[k]$ for $e[k]$ is given by

$$\hat{e}[k] = \sum_{m=0}^{L_P-1} p[m] e[k-1-m].$$

Assuming $\hat{I}[k-m] = I[k-m]$ for $m \geq 1$, the new decision variable is

$$d[k] = I[k] + e[k] - \hat{e}[k].$$

If the predictor coefficients are suitably chosen, we expect that the variance of the new error signal $e[k] - \hat{e}[k]$ is smaller than that of $e[k]$. Therefore, noise prediction improves performance.

- **Predictor Design**

  Usually an MMSE criterion is adopted for optimization of the predictor coefficients, i.e., the design objective is to minimize the error variance

$$\mathcal{E}\{|e[k] - \hat{e}[k]|^2\}.$$

Since this is a typical MMSE problem, the optimum predictor coefficients can be obtained from the Wiener–Hopf equation

$$\boldsymbol{\Phi}_{ee}\boldsymbol{p} = \boldsymbol{\varphi}_e$$

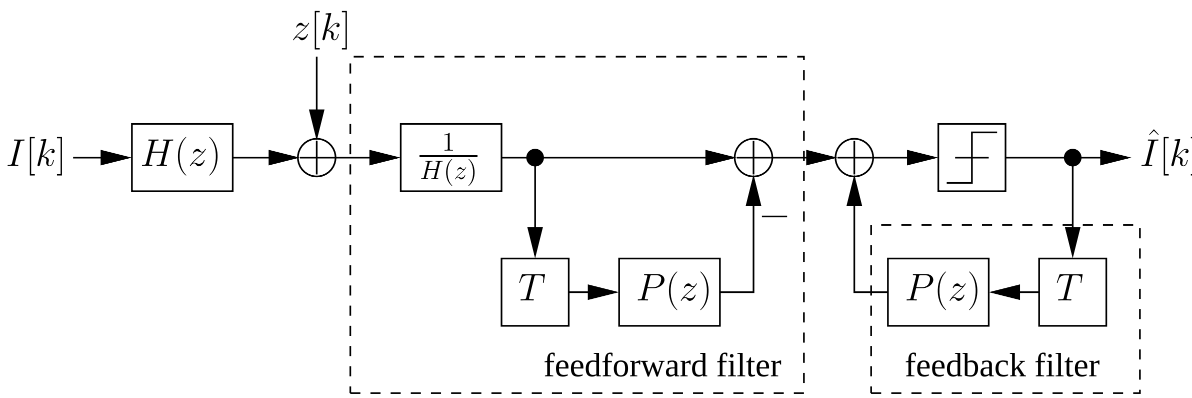
with

$$\begin{aligned}
\boldsymbol{\Phi}_{ee} &= \begin{bmatrix} \phi_{ee}[0] & \phi_{ee}[1] & \cdots & \phi_{ee}[L_P-1] \\ \phi_{ee}[-1] & \phi_{ee}[0] & \cdots & \phi_{ee}[L_P-2] \\ \vdots & \vdots & \ddots & \vdots \\ \phi_{ee}[-(L_P-1)] & \phi_{ee}[-(L_P-2)] & \cdots & \phi_{ee}[0] \end{bmatrix}, \\
\boldsymbol{\varphi}_e &= [\phi_{ee}[-1]\ \phi_{ee}[-2]\ \ldots\ \phi_{ee}[-L_P]]^T \\
\boldsymbol{p} &= [p[0]\ p[1]\ \ldots\ p[L_P-1]]^H,
\end{aligned}$$

where the ACF of $e[k]$ is defined as $\phi_{ee}[\lambda] = \mathcal{E}\{e^*[k]e[k+\lambda]\}$.

■ New Block Diagram

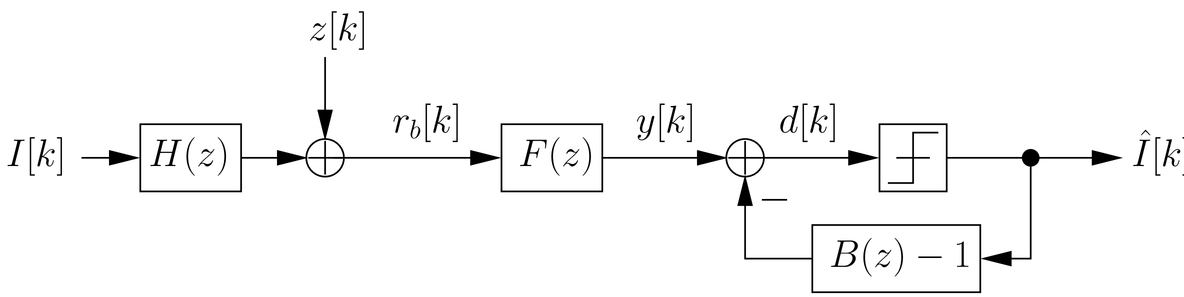
The above block diagram of linear equalization and noise prediction can be rearranged as follows.



The above structure consists of two filters. A *feedforward filter* whose input is the channel output signal $r_b[k]$ and a *feedback filter* that feeds back previous decisions $\hat{I}[k-m]$, $m \geq 1$. An equalization scheme with this structure is referred to as *decision–feedback equalization (DFE)*. We have shown that the DFE structure can be obtained in a natural way from linear equalization and noise prediction.

- **General DFE**

  If the predictor has infinite length, the above DFE scheme corresponds to optimum zero–forcing (ZF) DFE. However, the DFE concept can be generalized of course allowing for different filter optimization criteria. The structure of a general DFE scheme is shown below.



In general, the feedforward filter is given by

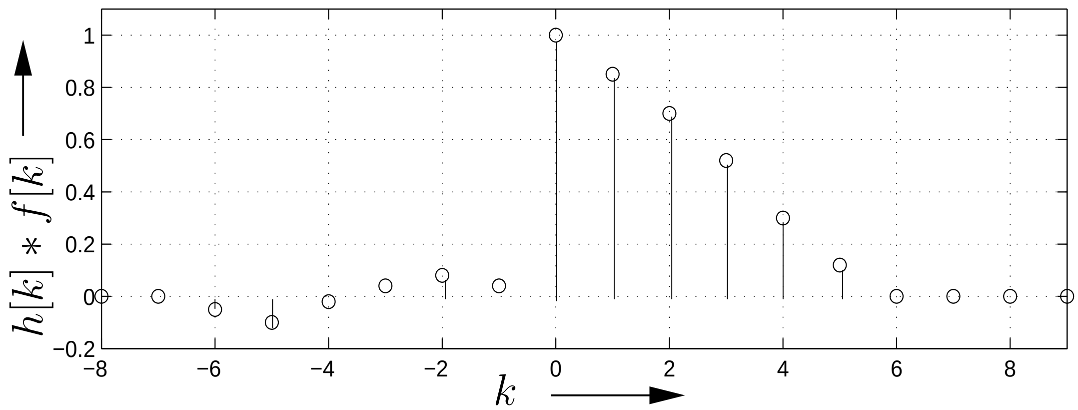
$$F(z) = \sum_{k=-\infty}^{\infty} f[k]z^{-k},$$

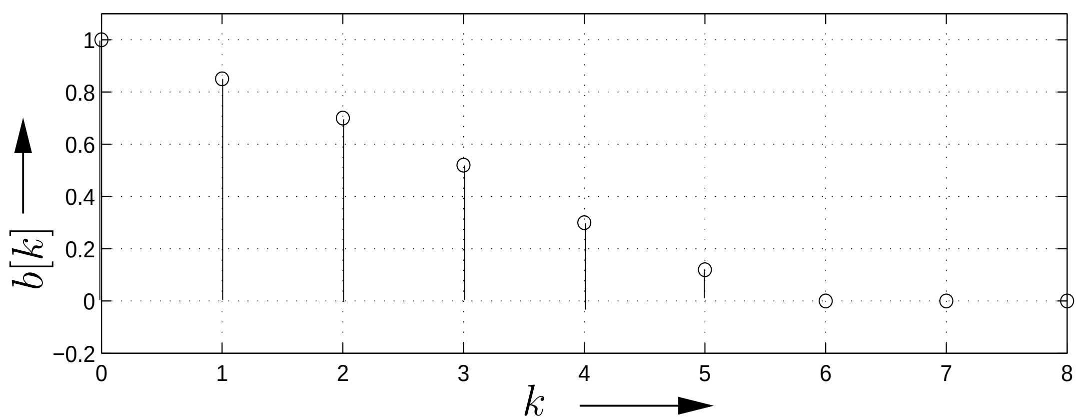
and the feedback filter

$$B(z) = 1 + \sum_{k=1}^{L_B-1} b[k]z^{-k}$$

is *causal* and *monic* ($b[k] = 1$). Note that $F(z)$ may also be an FIR filter. $F(z)$ and $B(z)$ can be optimized according to any suitable criterion, e.g. ZF or MMSE criterion.

Typical Example:





- **Properties of DFE**

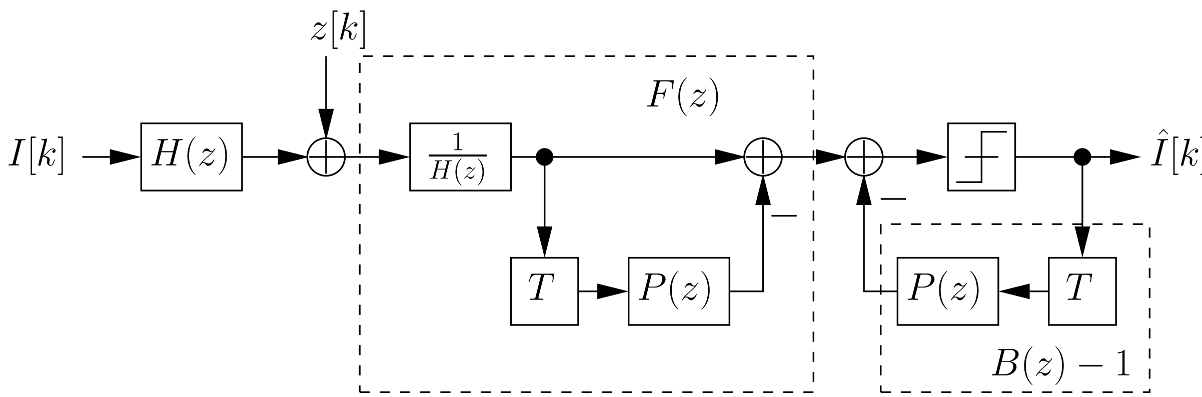
  – The feedforward filter has to suppress only the pre–cursor ISI. This imposes fewer constraints on the feedforward filter and therefore, the noise enhancement for DFE is significantly smaller than for linear equalization.

  – The post–cursors are canceled by the feedback filter. This causes no additional noise enhancement since the slicer eliminates the noise before feedback.

  – Feedback of wrong decisions causes error propagation. Fortunately, this error propagation is usually not catastrophic but causes some performance degradation compared to error free feedback.

6.4.1 Optimum ZF–DFE

- Optimum ZF–DFE may be viewed as a combination of optimum linear ZF equalization and optimum noise prediction.



- **Equalizer Filters (I)**
  The feedforward filter (FFF) is the cascade of the linear equalizer $1/H(z)$ and the *prediction error filter* $P_e(z) = 1 - z^{-1}P(z)$

$$F(z) = \frac{P_e(z)}{H(z)}.$$

  The feedback filter (FBF) is given by

$$B(z) = 1 - z^{-1}P(z).$$

- **Power Spectral Density of Noise**
  The power spectral density of the noise component $e[k]$ is given by

$$\Phi_{ee}(z) = \frac{N_0}{H(z)H^*(1/z^*)}$$

Optimum Noise Prediction

The optimum noise prediction error filter is a *noise whitening filter*, i.e., the power spectrum $\Phi_{\nu\nu}(z)$ of $\nu[k] = e[k] - \hat{e}[k]$ is a constant

$$\Phi_{\nu\nu}(z) = P_e(z)P_e^*(1/z^*)\Phi_{ee}(z) = \sigma_\nu^2,$$

where σ_ν^2 is the variance of ν. A more detailed analysis shows that $P_e(z)$ is given by

$$P_e(z) = \frac{1}{Q(z)}.$$

$Q(z)$ is monic and stable, and is obtained by *spectral factorization* of $\Phi_{ee}(z)$ as

$$\Phi_{ee}(z) = \sigma_\nu^2\, Q(z)Q^*(1/z^*). \tag{5}$$

Furthermore, we have

$$\begin{aligned}\Phi_{ee}(z) &= \frac{N_0}{H(z)H^*(1/z^*)} \\ &= \frac{N_0}{H_{\min}(z)H_{\min}^*(1/z^*)},\end{aligned} \tag{6}$$

where

$$H_{\min}(z) = \sum_{m=0}^{L-1} h_{\min}[m]z^{-m}$$

is the minimum phase equivalent of $H(z)$, i.e., we get $H_{\min}(z)$ from $H(z)$ by mirroring all zeros of $H(z)$ that are outside the unit circle into the unit circle. A comparison of Eqs. (5) and (6) shows that $Q(z)$ is given by

$$Q(z) = \frac{h_{\min}[0]}{H_{\min}(z)},$$

where the multiplication by $h_{\min}[0]$ ensures that $Q(z)$ is monic. Since $H_{\min}(z)$ is minimum phase, all its zeros are inside or on the unit circle, therefore $Q(z)$ is stable. The prediction error variance is given by

$$\sigma_\nu^2 = \frac{N_0}{|h_{\min}[0]|^2}.$$

The optimum noise prediction–error filter is obtained as

$$P_e(z) = \frac{H_{\min}(z)}{h_{\min}[0]}.$$

- **Equalizer Filters (II)**
  With the above result for $P_e(z)$, the optimum ZF–DFE FFF is given by

$$\begin{aligned} F(z) &= \frac{P_e(z)}{H(z)} \\ F(z) &= \frac{1}{h_{\min}[0]} \frac{H_{\min}(z)}{H(z)}, \end{aligned}$$

whereas the FBF is obtained as

$$\begin{aligned} B(z) &= 1 - z^{-1}P(z) = 1 - (1 - P_e(z)) \\ &= P_e(z) \\ &= \frac{H_{\min}(z)}{h_{\min}[0]}. \end{aligned}$$

- **Overall Channel**

The overall forward channel is given by

$$\begin{aligned} H_{\mathrm{ov}}(z) &= H(z)F(z) \\ &= \frac{H_{\mathrm{min}}(z)}{h_{\mathrm{min}}[0]} \\ &= \sum_{m=0}^{L-1} \frac{h_{\mathrm{min}}[m]}{h_{\mathrm{min}}[0]} z^{-m} \end{aligned}$$

This means the FFF filter $F(z)$ transforms the channel $H(z)$ into its (scaled) minimum phase equivalent. The FBF is given by

$$\begin{aligned} B(z) - 1 &= \frac{H_{\mathrm{min}}(z)}{h_{\mathrm{min}}[0]} - 1 \\ &= \sum_{m=1}^{L-1} \frac{h_{\mathrm{min}}[m]}{h_{\mathrm{min}}[0]} z^{-m} \end{aligned}$$

Therefore, assuming error–free feedback the equivalent overall channel including forward and backward part is an ISI–free channel with gain 1.

- **Noise**

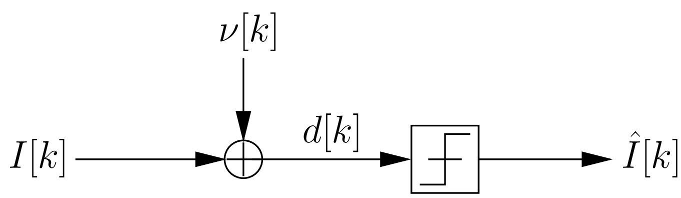
The FFF $F(z)$ is an allpass filter since

$$\begin{aligned} F(z)F^*(1/z^*) &= \frac{1}{|h_{\mathrm{min}}[0]|^2} \frac{H_{\mathrm{min}}(z)H^*_{\mathrm{min}}(1/z^*)}{H(z)H^*(1/z^*)} \\ &= \frac{1}{|h_{\mathrm{min}}[0]|^2}. \end{aligned}$$

Therefore, the noise component $\nu[k]$ is AWGN with variance

$$\sigma_\nu^2 = \frac{N_0}{|h_{\mathrm{min}}[0]|^2}.$$

The equivalent overall channel model assuming error–free feedback is shown below.



- **SNR**

  Obviously, the SNR of optimum ZF–DFE is given by

$$\begin{aligned}\mathrm{SNR}_{\mathrm{ZF-DFE}} &= \frac{1}{\sigma_\nu^2} \\ &= \frac{|h_{\min}[0]|^2}{N_0}.\end{aligned}$$

  Furthermore, it can be shown that $h_{\min}[0]$ can be calculated in closed form as a function of $H(z)$. This leads to

$$\boxed{\mathrm{SNR}_{\mathrm{ZF-DFE}} = \exp\left(T \int\limits_{-1/(2T)}^{1/(2T)} \ln\left(\frac{|H(e^{j2\pi fT})|^2}{N_0}\right)\, \mathrm{d}f \right).}$$

Example:

– *Channel*

We assume again a channel with one root and transfer function

$$H(z) = \frac{1}{\sqrt{1+|c|^2}}\,(1 - cz^{-1}).$$

– *Normalized Minimum–Phase Equivalent*

If $|c| \le 1$, $H(z)$ is already minimum phase and we get

$$\begin{aligned} P_e(z) &= \frac{H_{\min}(z)}{h_{\min}[0]} \\ &= 1 - cz^{-1}, \qquad\qquad |c| \le 1. \end{aligned}$$

If $|c| > 1$, the root of $H(z)$ has to be mirrored into the unit circle. Therefore, $H_{\min}(z)$ will have a zero at $z = 1/c^*$, and we get

$$\begin{aligned} P_e(z) &= \frac{H_{\min}(z)}{h_{\min}[0]} \\ &= 1 - \frac{1}{c^*}\,z^{-1}, \qquad\qquad |c| > 1. \end{aligned}$$

– *Filters*

The FFF is given by

$$F(z) = \begin{cases} \frac{1}{\sqrt{1+|c|^2}}, & |c| \le 1 \\ \frac{1}{\sqrt{1+|c|^2}} \frac{z-1/c^*}{z-c}, & |c| > 1 \end{cases}$$

The corresponding FBF is

$$B(z) - 1 = \begin{cases} -cz^{-1}, & |c| \le 1 \\ -\frac{1}{c^*}\,z^{-1}, & |c| > 1 \end{cases}$$

– *SNR*

The SNR can be calculated to

$$\mathrm{SNR}_{\mathrm{ZF-DFE}} = \exp\left(T \int\limits_{-1/(2T)}^{1/(2T)} \ln\left(\frac{|H(e^{j2\pi fT})|^2}{N_0}\right) \mathrm{d}f\right).$$

$$= \exp\left(T \int\limits_{-1/(2T)}^{1/(2T)} \ln\left(\frac{|1 - ce^{-2\pi fT}|^2}{(1+|c|^2)N_0}\right) \mathrm{d}f\right).$$

After some straightforward manipulations, we obtain

$$\mathrm{SNR}_{\mathrm{ZF-DFE}} = \frac{1}{2N_0}\left(1 + \frac{|1-|c|^2|}{1+|c|^2}\right)$$

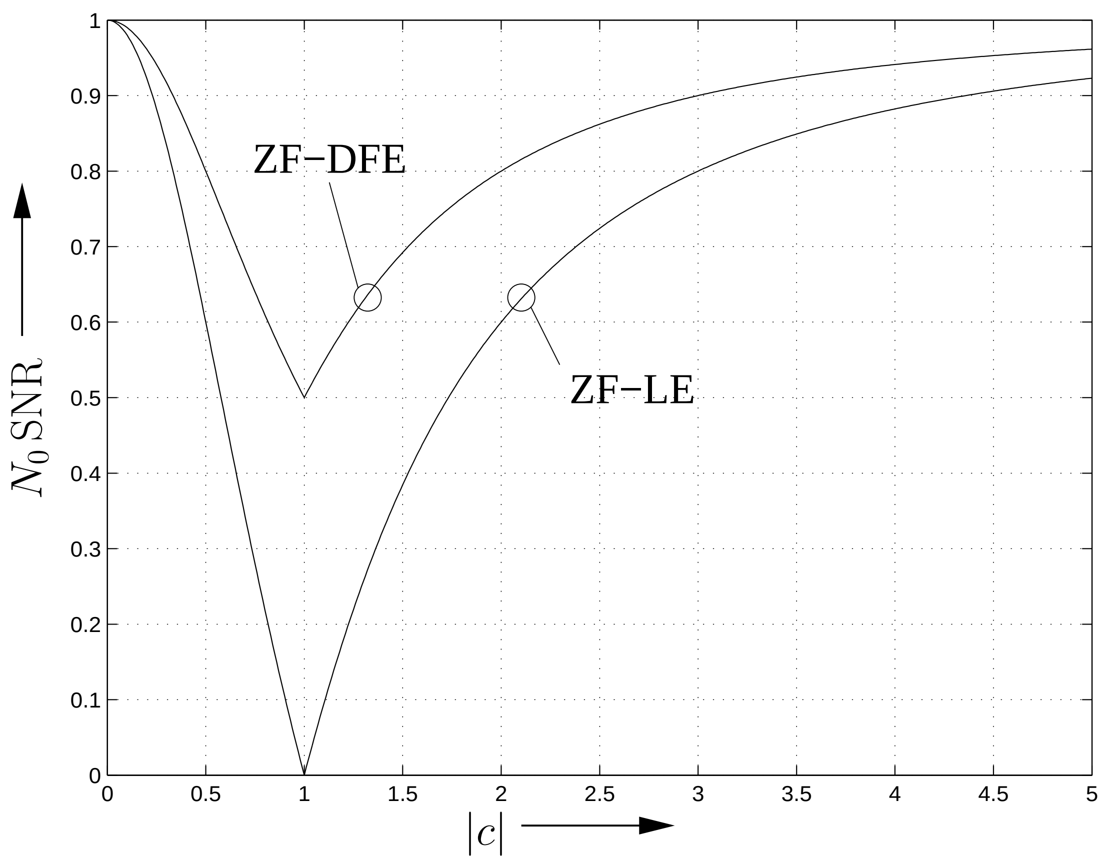
For a given N_0 the SNR is minimized for $|c| = 1$. In that case, we get $\mathrm{SNR}_{\mathrm{ZF-DFE}} = 1/(2N_0)$, i.e., there is a 3 dB loss compared to the pure AWGN channel. For $|c| = 0$ and $|c| \to \infty$, we get $\mathrm{SNR}_{\mathrm{ZF-DFE}} = 1/N_0$, i.e., there is no loss compared to the AWGN channel.

– *Comparison with ZF–LE*

For linear ZF equalization we had

$$\mathrm{SNR}_{\mathrm{ZF-LE}} = \frac{1}{N_0}\,\frac{|1-|c|^2|}{1+|c|^2}.$$

This means in the worst case $|c| = 1$, we get $\mathrm{SNR}_{\mathrm{ZF-LE}} = 0$ and reliable transmission is not possible. For $|c| = 0$ and $|c| \to \infty$ we obtain $\mathrm{SNR}_{\mathrm{ZF-LE}} = 1/N_0$ and no loss in performance is suffered compared to the pure AWGN channel.



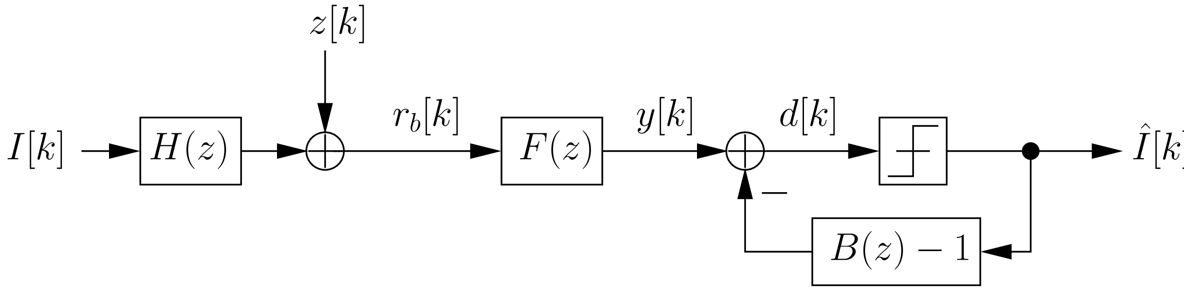
6.4.2 Optimum MMSE–DFE

- We assume a FFF with doubly–infinite response

$$F(z) = \sum_{k=-\infty}^{\infty} f[k]z^{-k}$$

and a causal FBF with

$$B(z) = 1 + \sum_{k=1}^{\infty} b[k]z^{-k}$$



- *Optimization Criterion*
  In optimum MMSE–DFE, we optimize FFF and FBF for minimization of the variance of the error signal

$$e[k] = d[k] - I[k].$$

  This error variance can be expressed as

$$\begin{aligned} J &= \mathcal{E}\left\{|e[k]|^2\right\} \\ &= \mathcal{E}\left\{\left(\sum_{\kappa=-\infty}^{\infty} f[\kappa]r_b[k-\kappa] - \sum_{\kappa=1}^{\infty} b[\kappa]I[k-\kappa] - I[k]\right)\right. \\ &\qquad \left.\left(\sum_{\kappa=-\infty}^{\infty} f^*[\kappa]r_b^*[k-\kappa] - \sum_{\kappa=1}^{\infty} b^*[\kappa]I^*[k-\kappa] - I^*[k]\right)\right\}. \end{aligned}$$

- **FFF Optimization**
  Differentiating J with respect to $f^*[\nu]$, $-\infty < \nu < \infty$, yields

$$\begin{aligned} \frac{\partial J}{\partial f^*[\nu]} &= \sum_{\kappa=-\infty}^{\infty} f[\kappa]\underbrace{\mathcal{E}\{r_b[k-\kappa]r_b^*[k-\nu]\}}_{\phi_{rr}[\nu-\kappa]} \\ &\quad - \sum_{\kappa=1}^{\infty} b[\kappa]\underbrace{\mathcal{E}\{I[k-\kappa]r_b^*[k-\nu]\}}_{\phi_{Ir}[\nu-\kappa]} - \underbrace{\mathcal{E}\{I[k]r_b^*[k-\nu]\}}_{\phi_{Ir}[\nu]} \end{aligned}$$

Letting $\partial J/(\partial f^*[\nu]) = 0$ and taking the $\mathcal{Z}$-transform of the above equation leads to

$$F(z)\Phi_{rr}(z) = B(z)\Phi_{Ir}(z),$$

where $\Phi_{rr}(z)$ and $\Phi_{Ir}(z)$ denote the $\mathcal{Z}$-transforms of $\phi_{rr}[\lambda]$ and $\phi_{Ir}[\lambda]$, respectively.

Assuming i.i.d. sequences $I[\cdot]$ of unit variance, we get

$$\begin{aligned}\Phi_{rr}(z) &= H(z)H^*(1/z^*) + N_0 \\ \Phi_{Ir}(z) &= H^*(1/z^*)\end{aligned}$$

This results in

$$F(z) = \frac{H^*(1/z^*)}{H(z)H^*(1/z^*) + N_0}\, B(z).$$

Recall that

$$F_{\text{LE}}(z) = \frac{H^*(1/z^*)}{H(z)H^*(1/z^*) + N_0}$$

is the optimum filter for linear MMSE equalization. This means the optimum FFF for MMSE–DFE is the cascade of a optimum linear equalizer and the FBF $B(z)$.

- **FBF Optimization**

The Z–transform $E(z)$ of the error signal $e[k]$ is given by

$$E(z) = F(z)Z(z) + (F(z)H(z) - B(z))I(z).$$

Adopting the optimum $F(z)$, we obtain for the $\mathcal{Z}$–transform of the autocorrelation sequence

$$\begin{aligned}
\Phi_{ee}(z) &= \mathcal{E}\{E(z)E^*(1/z^*)\} \\
&= \frac{B(z)B^*(1/z^*)H(z)H^*(1/z^*)}{(H(z)H^*(1/z^*)+N_0)^2}N_0 \\
&\quad + \frac{N_0^2 B(z)B^*(1/z^*)}{(H(z)H^*(1/z^*)+N_0)^2} \\
&= B(z)B^*(1/z^*)\frac{H(z)H^*(1/z^*)+N_0}{(H(z)H^*(1/z^*)+N_0)^2}N_0 \\
&= B(z)B^*(1/z^*)\underbrace{\frac{N_0}{H(z)H^*(1/z^*)+N_0}}_{\Phi_{e_l e_l}(z)},
\end{aligned}$$

where $e_l[k]$ denotes the error signal at the output of the optimum linear MMSE equalizer, and $\Phi_{e_l e_l}(z)$ is the $\mathcal{Z}$–transform of the autocorrelation sequence of $e_l[k]$.

The optimum FBF filter will minimize the variance of $e_l[k]$. Therefore, the optimum prediction–error filter for $e_l[k]$ is the optimum filter $B(z)$. Consequently, the optimum FBF can be defined as

$$B(z) = \frac{1}{Q(z)},$$

where $Q(z)$ is obtained by *spectral factorization* of $\Phi_{e_l e_l}(z)$

$$\begin{aligned}
\Phi_{e_l e_l}(z) &= \frac{N_0}{H(z)H^*(1/z^*)+N_0} \\
&= \sigma_e^2 Q(z)Q^*(1/z^*).
\end{aligned}$$

The coefficients of $q[k]$, $k \geq 1$, can be calculated recursively as

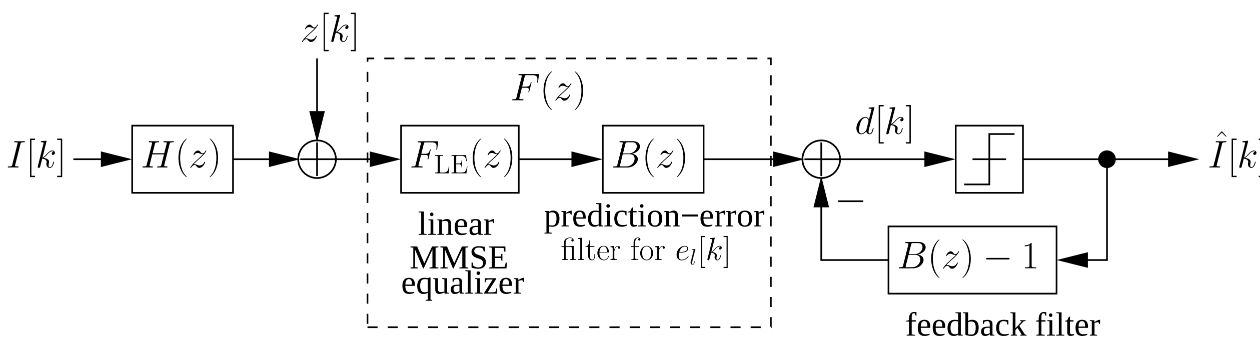
$$q[k] = \sum_{\mu=0}^{k-1} \frac{k-\mu}{k} q[\mu]\beta[k-\mu], \quad k \geq 1$$

with

$$\beta[\mu] = T \int\limits_{-1/(2T)}^{1/(2T)} \ln\left[\Phi_{e_l e_l}(e^{j2\pi fT})\right] e^{j2\pi\mu fT}\,\mathrm{d}f.$$

The error variance σ_e^2 is given by

$$\sigma_e^2 = \exp\left(T \int\limits_{-1/(2T)}^{1/(2T)} \ln\left[\frac{N_0}{|H(e^{j2\pi fT})|^2 + N_0}\right] \mathrm{d}f \right).$$



- **Overall Channel**

The overall forward transfer function is given by

$$\begin{aligned} H_{\mathrm{ov}}(z) &= F(z)H(z) \\ &= \frac{H(z)H^*(1/z^*)}{H(z)H^*(1/z^*) + N_0}\, B(z) \\ &= \left(1 - \frac{N_0}{H(z)H^*(1/z^*) + N_0}\right) B(z) \\ &= B(z) - \frac{\sigma_e^2}{B^*(1/z^*)}. \end{aligned}$$

Therefore, the bias $h_{\mathrm{ov}}[0]$ is given by

$$h_{\mathrm{ov}}[0] = 1 - \sigma_e^2.$$

The part of the transfer function that characterizes the *pre–cursors* is given by

$$-\sigma_e^2 \left(\frac{1}{B^*(1/z^*)} - 1 \right),$$

whereas the part of the transfer function that characterizes the *post–cursors* is given by

$$B(z) - 1$$

Hence, the error signal is composed of bias, pre–cursor ISI, and noise. The post–cursor ISI is perfectly canceled by the FBF.

- **SNR**
  Taking into account the bias, it can be shown that the SNR for optimum MMSE DFE is given by

$$\begin{aligned} \mathrm{SNR}_{\mathrm{MMSE-DFE}} &= \frac{1}{\sigma_e^2} - 1 \\ &= \exp\left(T \int\limits_{-1/(2T)}^{1/(2T)} \ln\left[\frac{|H(e^{j2\pi fT})|^2}{N_0} + 1 \right] \mathrm{d}f \right) - 1. \end{aligned}$$

- **Remark**
  For high SNRs, ZF–DFE and MMSE–DFE become equivalent. In that case, the noise variance is comparatively small and also the MMSE criterion leads to a complete elimination of the ISI.

6.5 MMSE–DFE with FIR Filters

- Since IIR filters cannot be realized, in practice FIR filters have to be employed.
- **Error Signal** $e[k]$
  If we denote FFF length and order by L_F and $q_F = L_F - 1$, respectively, and the FBF length and order by L_B and $q_B = L_B - 1$, respectively, we can write the slicer input signal as

$$d[k] = \sum_{\kappa=0}^{q_F} f[\kappa] r_b[k-\kappa] - \sum_{\kappa=1}^{q_B} b[\kappa] I[k-k_0-\kappa],$$

  where we again allow for a decision delay k_0, $k_0 \geq 0$.

Using vector notation, the error signal can be expressed as

$$\begin{aligned} e[k] &= d[k] - I[k-k_0] \\ &= \boldsymbol{f}^H \boldsymbol{r}_b[k] - \boldsymbol{b}^H \boldsymbol{I}[k-k_0-1] - I[k-k_0] \end{aligned}$$

with

$$\begin{aligned} \boldsymbol{f} &= [f[0] \;\; f[1] \;\; \ldots \;\; f[q_F]]^H \\ \boldsymbol{b} &= [b[1] \;\; f[2] \;\; \ldots \;\; b[q_B]]^H \\ \boldsymbol{r}_b[k] &= [r_b[k] \;\; r_b[k-1] \;\; \ldots \;\; r_b[k-q_F]]^T \\ \boldsymbol{I}[k-k_0-1] &= [I[k-k_0-1] \;\; I[k-k_0-2] \;\; \ldots \;\; I[k-k_0-q_B]]^T. \end{aligned}$$

- **Error Variance** J

The error variance can be obtained as

$$\begin{aligned} J &= \mathcal{E}\{|e[k]|^2\} \\ &= \boldsymbol{f}^H \mathcal{E}\{\boldsymbol{r}_b[k]\boldsymbol{r}_b^H[k]\}\boldsymbol{f} + \boldsymbol{b}^H \mathcal{E}\{\boldsymbol{I}[k-k_0-1]\boldsymbol{I}^H[k-k_0-1]\}\boldsymbol{b} \\ &\quad -\boldsymbol{f}^H \mathcal{E}\{\boldsymbol{r}_b[k]\boldsymbol{I}^H[k-k_0-1]\}\boldsymbol{b} - \boldsymbol{b}^H \mathcal{E}\{\boldsymbol{I}[k-k_0-1]\boldsymbol{r}_b^H[k]\}\boldsymbol{f} \\ &\quad -\boldsymbol{f}^H \mathcal{E}\{\boldsymbol{r}_b[k] I^*[k-k_0]\} - \mathcal{E}\{\boldsymbol{r}_b^H[k] I[k-k_0]\}\boldsymbol{f} \\ &\quad +\boldsymbol{b}^H \mathcal{E}\{\boldsymbol{I}[k-k_0-1] I^*[k-k_0]\} + \mathcal{E}\{\boldsymbol{I}^H[k-k_0-1] I[k-k_0]\}\boldsymbol{b}. \end{aligned}$$

Since $I[k]$ is an i.i.d. sequence and the noise $z[k]$ is white, the following identities can be established:

$$
\begin{aligned}
\mathcal{E}\{\boldsymbol{I}[k-k_0-1]I^*[k-k_0]\} &= \boldsymbol{0} \\
\mathcal{E}\{\boldsymbol{I}[k-k_0-1]\boldsymbol{I}^H[k-k_0-1]\} &= \boldsymbol{I} \\
\mathcal{E}\{\boldsymbol{r}_b[k]\boldsymbol{I}^H[k-k_0-1]\} &= \boldsymbol{H} \\
\boldsymbol{H} = \left[\begin{array}{ccc} h[k_0+1] & \dots & h[k_0+q_B] \\ h[k_0] & \dots & h[k_0+q_B-1] \\ \vdots & \ddots & \vdots \\ h[k_0+1-q_F] & \dots & h[k_0+q_B-q_F] \end{array}\right] & \\
\mathcal{E}\{\boldsymbol{r}_b[k]I^*[k-k_0]\} &= \boldsymbol{h} \\
\boldsymbol{h} &= [h[k_0]\;\; h[k_0-1]\;\; \dots\;\; h[k_0-q_F]]^T \\
\mathcal{E}\{\boldsymbol{r}_b[k]\boldsymbol{r}_b^H[k]\} &= \boldsymbol{\Phi}_{hh} + \sigma_n^2\boldsymbol{I},
\end{aligned}
$$

where $\boldsymbol{\Phi}_{hh}$ denotes the channel autocorrelation matrix. With these definitions, we get

$$
\begin{aligned}
J &= \boldsymbol{f}^H(\boldsymbol{\Phi}_{hh}+\sigma_n^2\boldsymbol{I})\boldsymbol{f} + \boldsymbol{b}^H\boldsymbol{b} + 1 \\
&\quad -\boldsymbol{f}^H\boldsymbol{H}\boldsymbol{b} - \boldsymbol{b}^H\boldsymbol{H}^H\boldsymbol{f} - \boldsymbol{f}^H\boldsymbol{h} - \boldsymbol{h}^H\boldsymbol{f}
\end{aligned}
$$

- **Optimum Filters**

  The optimum filter settings can be obtained by differentiating J with respect to $\boldsymbol{f}^*$ and $\boldsymbol{b}^*$, respectively.

$$
\begin{aligned}
\frac{\partial J}{\partial \boldsymbol{f}^*} &= (\boldsymbol{\Phi}_{hh}+\sigma_n^2\boldsymbol{I})\boldsymbol{f} - \boldsymbol{H}\boldsymbol{b} - \boldsymbol{h} \\
\frac{\partial J}{\partial \boldsymbol{b}^*} &= \boldsymbol{b} - \boldsymbol{H}^H\boldsymbol{f}
\end{aligned}
$$

  Setting the above equations equal to zero and solving for $\boldsymbol{f}$, we get

$$
\boldsymbol{f}_{\mathrm{opt}} = \left(\left(\boldsymbol{\Phi}_{hh} - \boldsymbol{H}\boldsymbol{H}^H\right) + \sigma_n^2\boldsymbol{I}\right)^{-1}\boldsymbol{h}
$$

The optimum FBF is given by

$$\boldsymbol{b}_{\text{opt}} = \boldsymbol{H}^H \boldsymbol{f} = [h_{\text{ov}}[k_0+1] \;\; h_{\text{ov}}[k_0+1] \;\; \ldots \;\; h_{\text{ov}}[k_0+q_B]]^H,$$

where $h_{\text{ov}}[k]$ denotes the overall impulse response comprising channel and FFF. This means the FBF cancels perfectly the postcursor ISI.

- **MMSE**
  The MMSE is given by

$$\begin{aligned} J_{\min} &= 1 - \boldsymbol{h}^H \left(\left(\boldsymbol{\Phi}_{hh} - \boldsymbol{H}\boldsymbol{H}^H \right) + \sigma_n^2 \boldsymbol{I} \right)^{-1} \boldsymbol{h} \\ &= 1 - \boldsymbol{f}_{\text{opt}}^H \boldsymbol{h}. \end{aligned}$$

- **Bias**
  The bias is given by

$$h[k_0] = \boldsymbol{f}_{\text{opt}}^H \boldsymbol{h} = 1 - J_{\min} < 1.$$