Homework 1 Solutions

Problem 2.1: Rather than doing the details of the convolution, we simply sketch the shapes of the waveforms. For a signal $s = s_c + js_s$ and a filter $h = h_c + jh_s$, the convolution

$$y = s * h = (s_c * h_c - s_s * h_s) + j(s_c * h_s + s_s * h_c)$$

For $h(t) = s_{mf}(t) = s^*(-t)$, rough sketches of Re($y$), Im($y$) and $|y|$ are shown in Figure 1. Clearly, the maximum occurs at $t = 0$.

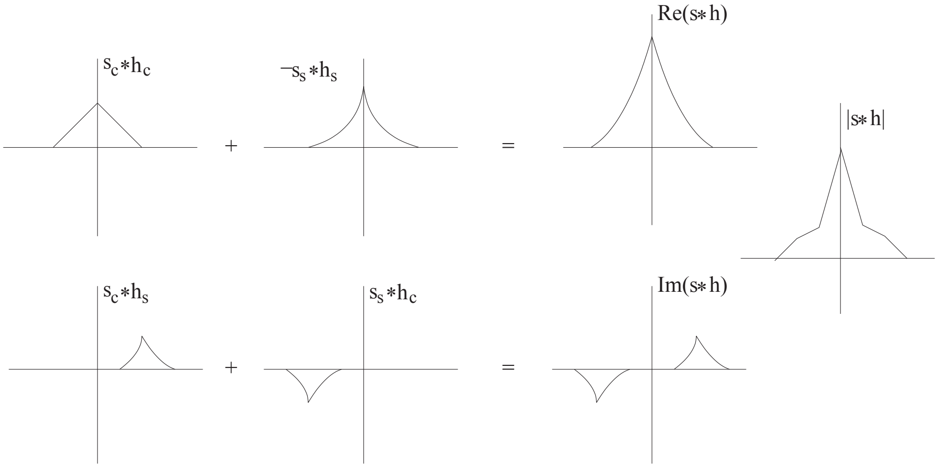

Figure 1: The convolution of a signal with its matched filter yields at peak at the origin.

Problem 2.2:
(a) Multiplication in the time domain corresponds to convolution in the frequency domain. The two sinc functions correspond to boxcars in the frequency domain, convolving which gives that $S(f)$ has a trapezoidal shape, as shown in Figure 2.
(b) We have

$$u(t) = s(t)\cos(100\pi t) = s(t)\frac{e^{j100\pi t} + e^{-j100\pi t}}{2} \leftrightarrow U(f) = \frac{S(f-50) + S(f+50)}{2}$$

The spectrum $U(f)$ is plotted in Figure 2.

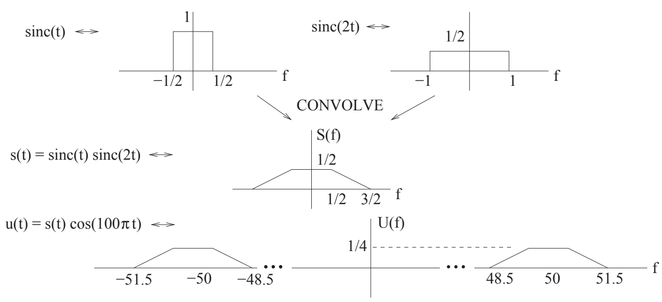

Figure 2: Solution for Problem 2.2.

Problem 2.3: The solution is sketched in Figure 3.
(a) We have $s(t) = I_{[-5,5]} * I_{[-5,5]}$. Since $I_{[-5,5]}(t) \leftrightarrow 10\text{sinc}(10f)$, we have $S(f) = 100\text{sinc}^2(10f)$.
(b) We have

$$u(t) = s(t)\sin(1000\pi t) = s(t)\frac{e^{j1000\pi t} - e^{-j100\pi t}}{2j} \leftrightarrow U(f) = \frac{S(f-50) - S(f+50)}{2j}$$

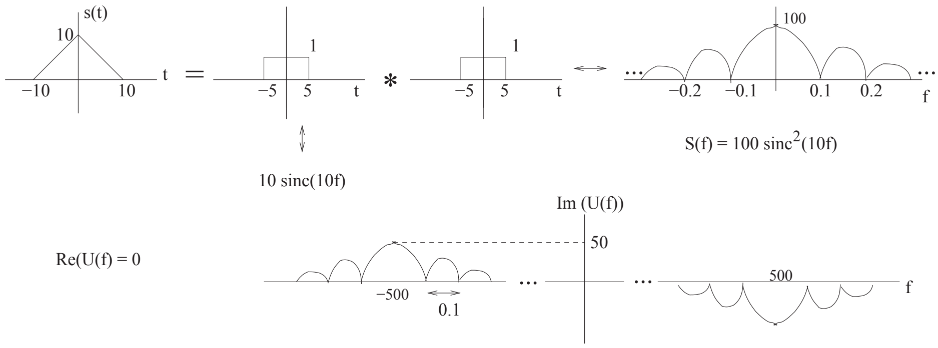

Figure 3: Solution for Problem 2.3.

Problem 2.4: Part (a) is immediate upon expanding $||s - ar||^2$.
(b) The minimizing value of a is easily found to be

$$a_{min} = \frac{\langle s, r\rangle}{||r||^2}$$

Substituting this value into $J(a)$, we obtain upon simplification that

$$J(a_{min}) = ||s||^2 - \frac{\langle s, r\rangle^2}{||r||^2}$$

The condition $J(a_{min}) \geq 0$ is now seen to be equivalent to the Cauchy-Schwartz inequality.
(c) For nonzero s, r, the minimum error $J(a_{min})$ in approximating s by a multiple of r vanishes if and only if s is a multiple of r. This is therefore the condition for equality in the Cauchy-Scwartz inequality. For $s = 0$ or $r = 0$, equality clearly holds. Thus, the condition for equality can be stated in general as: either s is a scalar multiple of r (this includes $s = 0$ as a special case), or r is a scalar multiple of s (this includes $r = 0$ as a special case).
(d) The unit vector in the direction of r is $u = \frac{r}{||r||}$. The best approximation of s as a multiple of r is its projection along u, which is given by

$$\hat{s} = \langle s, u\rangle u = \langle s, \frac{r}{||r||}\rangle \frac{r}{||r||}$$

and the minimum error is $J(a_{min}) = ||s - \hat{s}||^2$.